

MORITA THEORY FOR DERIVED CATEGORIES: A BICATEGORICAL PERSPECTIVE

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ABSTRACT. We present a bicategorical perspective on derived Morita theory for rings, DG algebras, and spectra. This perspective draws a connection between Morita theory and the bicategorical Yoneda Lemma, yielding a conceptual unification of Morita theory in derived and bicategorical contexts. This is motivated by study of Rickard’s theorem for derived equivalences of rings and of Morita theory for ring spectra, which we present in Sections 2 and 4. Along the way, we gain an understanding of the barriers to Morita theory for DG algebras and give a conceptual explanation for the counterexample of Dugger and Shipley.

1. INTRODUCTION

A bicategorical perspective on Morita theory is rooted in the observation that Morita theory is a theory of *bimodules*, not simply left modules or right modules. To give an incomplete survey, this perspective has yielded extensions to the theory of distributors over enriched categories by Fisher-Palmquist and Palmquist [FP75], to subfactor theory by Müger [Müg03], to bialgebroids by Szlachányi [Szl04], and to von Neumann algebras by Brouwer [Bro03].

A largely disjoint body of work has studied Morita theory in derived contexts. This began with the work of Rickard, studying derived categories of rings in [Ric89] and [Ric91]. Rickard’s results were re-treated by Schwede in [Sch04], following [KZ98]. Dugger, Schwede and Shipley give partial extensions to ring spectra and differential graded algebras in [SS03], [DS07], and related work. Derived Morita theory for differential graded categories has been studied by Keller in [Kel94] and Toën in [Toë07]. The development of derived Morita theory has required more delicacy than its bicategorical counterpart, with the counterexample of [DS07] being a barrier to expected generalizations of Rickard’s theorem. The work of Dugger, Shipley, and Toën is motivated in part by this unsatisfactory situation.

We present a conceptual unification of bicategorical Morita theory with Morita theory for derived categories by developing Morita theory in triangulated bicategories. In Section 3, we introduce bicategorical language for those to whom it is unfamiliar, and in Subsection 3.11 we describe what is meant by a triangulated bicategory. From this vantage, we are able to give a conceptual explanation of derived Morita theory for which the results (and counterexample) of [DS07] and [Toë07] become verifying examples.

This is achieved in three stages. First, in Section 5 we remind the reader of the bicategorical Yoneda Lemma (5.5) and explain that what is often called Morita theory is a corollary (5.6) of this Yoneda Lemma. We encourage the intuition that bicategorical Morita theory is as elementary as the bicategorical Yoneda Lemma; this gives one possible reason that bicategorical perspectives have yielded such an abundance of generalizations for classical Morita theory. Second, we modify a standard observation from the context of enriched category theory to explain that, in bicategories with left and right internal homs, Morita theory must necessarily focus on equivalences which are enriched over the internal homs (5.7). This gives a reason for the results and examples mentioned above. Finally, in Section 6 we apply our understanding of the Yoneda Lemma. Our perspective allows us to re-frame the issue of derived Morita theory and sheds some new light on the subtleties there. In Subsection 6.5 we discuss the relationship of Morita theory to ambient enrichments. In classical Morita theory the ambient abelian enrichment is automatically preserved (Remark 2.3), but this is not the case in all other contexts. This provides, for example, a reason why the development of Morita theory has met unexpected barriers in the DG case.

We foreshadow the bicategorical perspective by outlining a proof of Rickard’s theorem in Section 2. After establishing bicategorical terminology in Section 3, Section 4 gives the details of this proof. In Subsection 4.6 we give a generalization of this theory to ring spectra. The last two sections cover some basic model-theoretic results for our bicategory of DG-algebras and their bimodules; they are verifications that expected results from the theory of monoidal stable model categories generalize to our context in straightforward ways. Section 7 gives a bicategorical development of the standard model structure for DG algebras, and Section 8 describes the formal structure arising from change of base algebra.

2. OUTLINE

In this section, we demonstrate our perspective by giving Rickard’s theorem for derived Morita theory of rings, together with an outline of its proof. For reference, we give a statement of the classical Morita theorem together with its proof, and we follow these with some remarks about derived Morita theory for DGAs. The counterexample of [DS07] is given as Example 2.8, and shows that Rickard’s theorem does not generalize to DGAs as stated. We give some hints about where this breakdown occurs, to be described more fully after we have developed the appropriate bicategorical perspective.

Note. In the following, we implicitly understand “module” to mean “right-module”, unless it is otherwise qualified. The most frequent instance of this opposite qualification will be that the endomorphism ring of a right-module acts on its *left*, and vice-versa.

Theorem 2.1 (Rickard). *Let k be a commutative ring, and let R and S be k -algebras. The derived categories $\mathcal{D}_k(R)$ and $\mathcal{D}_k(S)$ are equivalent as triangulated categories if and only if there is an object T of $\mathcal{D}_k(S)$ with the following three properties.*

- (i) T is (quasi-isomorphic to) a bounded complex of finitely-generated projective S -modules.
- (ii) T generates the triangulated category $\mathcal{D}_k(S)$.
- (iii) The graded endomorphism algebra $\mathcal{D}_k(S)[T, T]_*$ is concentrated in degree zero and isomorphic to R as a k -algebra.

Theorem 2.2 (Morita). *Let R and S be rings. The categories Mod_R and Mod_S are equivalent if and only if there is an object P of Mod_S with the following three properties.*

- (i) P is a finitely-generated projective S -module.
- (ii) P generates the abelian category Mod_S .
- (iii) The endomorphism ring $\text{Hom}_S(P, P)$ is isomorphic to R .

For the classical theorem of Morita, we make use of the bicategory \mathcal{M} of rings and their bimodules. The dual basis lemma gives that condition (i) is equivalent to the canonical coevaluation map

$$\nu : P \otimes_S \text{Hom}_S(P, S) \rightarrow \text{Hom}_S(P, P)$$

being an isomorphism. With condition (iii), this can be phrased in the bicategorical context by saying that $(P, \text{Hom}_S(P, S))$ form a dual pair over S and R , which means that the functors $- \otimes_R P$ and $- \otimes_S \text{Hom}(P, S)$ are an adjoint pair. The generating condition, (ii), is equivalent to the canonical evaluation map $\varepsilon : \text{Hom}_S(P, S) \otimes_R P \rightarrow S$ being an isomorphism, and hence this dual pair is an invertible pair, giving an adjoint equivalence of categories. The converse is also easy to see classically, since an equivalence of categories $F : \text{Mod}_R \rightarrow \text{Mod}_S$ induces an isomorphism on the morphisms between modules, and therefore a ring isomorphism

$$R \cong \text{Hom}_R(R, R) \xrightarrow{F} \text{Hom}_S(FR, FR).$$

Moreover, the other two properties are enjoyed by R and preserved by equivalences, so taking $P = FR$ gives the converse.

Remark 2.3. For our future discussion, it is worth noting that this argument takes advantage of the elementary fact that the abelian group structure on $\text{Hom}_R(R, R)$ is necessarily preserved by F . In fact any left adjoint functor between abelian categories is automatically enriched over abelian groups. This is neither expected nor true of more general enrichments.

This point of view on the classical theorem is readily generalized to the proof of Rickard’s theorem. In order to clarify the proof, we separate Rickard’s theorem into a well-known lemma and two propositions.

Lemma 2.4. *Let E be a DG k -algebra whose homology is concentrated in degree zero. Then E is quasi-isomorphic to its homology, and hence there is a triangulated equivalence $\mathcal{D}_k(E) \simeq \mathcal{D}_k(H_*E)$.*

Proof. Let

$$(E_+)_n = \begin{cases} E_n, & n > 0 \\ Z_0(E) = \ker(d_0), & n = 0 \\ 0, & n < 0 \end{cases}$$

Then the projection and inclusion define a zig-zag of quasi-isomorphisms

$$H_0(E) \xleftarrow{\cong} E_+ \xrightarrow{\cong} E,$$

and base change along these maps gives equivalences of derived categories. □

Definition 2.5 (formality). DG algebras which are quasi-isomorphic to their homology are called formal.

Proposition 2.6. *Let R and S be k -algebras. If $F : \mathcal{D}_k(R) \simeq \mathcal{D}_k(S)$ is an equivalence of triangulated categories, then there is an object $T \in \mathcal{D}_k(S)$ with the following two properties:*

- (i) T is (quasi-isomorphic to) a bounded complex of finitely-generated projective S -modules.
- (ii) T generates the triangulated category $\mathcal{D}_k(S)$.

Moreover, the DG endomorphism algebra $\text{End}_S(T)$ is quasi-isomorphic to R as a DG k -algebra.

Proof. A common proof of this proposition (see [Sch04], for example) is to remark that the two conditions are preserved by exact equivalences and are enjoyed by R regarded as a module over itself, hence also $T = FR$ has these properties.

Equivalences induce isomorphisms on homology of endomorphism DG k -algebras, and so $\text{End}_S(T)$ has homology which is concentrated in degree 0 and isomorphic to the homology of R (that is, R itself). Lemma 2.4 shows that $\text{End}_S(T)$ is therefore formal, and hence $\text{End}_S(T)$ and R are quasi-isomorphic DG k -algebras. \square

Since $H_*\text{End}_S(T) = \mathcal{D}_k(S)[T, T]_*$, this proves one implication in Rickard's theorem. The other implication is proved by again applying Lemma 2.4 in the case $E = \text{End}_s(T)$. If R is isomorphic to H_*E , then formality ensures that R and E are quasi-isomorphic and hence $\mathcal{D}_k(E) \simeq \mathcal{D}_k(R)$. The following proposition then proves this direction of Rickard's theorem, by specializing to the case that S is a DG k -algebra concentrated in degree 0.

Note. Dualizable modules over a DG k -algebra are defined in Subsection 3.5, but for the current argument it is enough to observe that when S is a DG k -algebra concentrated in degree 0, then a right-dualizable S -module is simply a bounded complex of finitely generated and projective S -modules.

Proposition 2.7. *Let S be a DG k -algebra, and let $T \in \text{DG}_k(S)$. If T has the following two properties, then $\mathcal{D}_k(S)$ and $\mathcal{D}_k(\text{End}_S(T))$ are equivalent as triangulated categories.*

- (i) T is a right-dualizable S -module.
- (ii) T generates the triangulated category $\mathcal{D}_k(S)$.

The proof is given in Section 4 below. It shows that T has a dual and the desired equivalence is given by the derived tensor product with T ; its inverse is derived tensor with the dual of T . Such equivalences are called *standard derived equivalences*.

Since DG k -algebras are, in general, not formal, we do not expect Rickard's theorem to generalize to DG k -algebras as stated. However, if the third condition for T is strengthened to a requirement that $\text{End}_S(T)$ be quasi-isomorphic to R , then Proposition 2.7 is a proof for one direction. Proposition 2.6 is not generally true when R and S are taken to be DG k -algebras, and this is a further barrier to generalizing Rickard's theorem. The difficulty is that one does not have formality for DG k -algebras in general. More precisely, formality is used in the proof of Proposition 2.6 to show that an equivalence of derived categories (of rings) is sufficient to guarantee a quasi-isomorphism of DG k -algebras between a ring and the endomorphism DG k -algebra of its image under the equivalence. Such a quasi-isomorphism is neither expected nor present in greater generality. We investigate this in Section 5, but for now we give an example to illustrate how Proposition 2.6 can fail in the DG situation.

Example 2.8. In [DS07], an example of two DG rings is given: $C = \mathbb{Z}[e]/(e^4)$ with $|e| = 1$ and $d(e) = 2$, and $A = H_*C$. The model categories of C -modules and A -modules are Quillen equivalent, but there is no possible bimodule with the properties listed in Rickard's theorem. That there can be no such bimodule is proven by noting that A is a DGA over $\mathbb{Z}/2$, but C is not quasi-isomorphic to any DGA over $\mathbb{Z}/2$. Since the endomorphism algebra of any A -module would also be a DGA over $\mathbb{Z}/2$, there cannot be a C - A -bimodule with endomorphism DGA quasi-isomorphic to C . These DGAs do have Quillen equivalent categories of modules. This is shown by a THH (topological Hochschild homology) argument which produces an equivalence of \mathbb{S} -algebras between their Eilenberg-Mac Lane spectra.

To understand the force of this example better, we note that the equivalences arising in Lemma 2.4 and Proposition 2.7 are *standard derived equivalences*; they are given by derived tensor with a DG-bimodule. These are manifestly induced by Quillen equivalences of model categories, namely the underived tensor on the categories of DG-modules. The example above shows, however, that the property of being induced by a Quillen equivalence is not sufficient to characterize the standard derived equivalences. To reiterate, the DGAs in the example do have Quillen equivalent module categories, but the induced equivalence of derived categories *cannot* be a standard derived equivalence.

A key to fully characterizing standard derived equivalences is an observation about the organized way in which standard derived equivalences preserve bimodule structures. If M is an R - S DG-bimodule, then $- \otimes_R M$ preserves *left*-module structure for all right R -modules. By neglect, this can be regarded as a functor from right DG R -modules to right DG S -modules, but to do so forgets too much. In

the example above, the Quillen equivalence of right-module categories does not preserve *left*-module structure, so it cannot induce a standard derived equivalence.

An alternative perspective might point out that the standard derived equivalences also preserve categorical enrichment. That is, with M as above, $- \otimes_R M$ induces morphisms of hom *objects*, and is compatible with the enriched composition in the expected way. The Quillen pair of functors produced in the example of [DS07] is not a pair of DG-enriched functors.

The point of Proposition 5.7 is that these two perspectives are in fact equivalent. Moreover, Corollary 5.6 interprets the Yoneda Lemma (5.5) as a statement that these (equivalent) properties do characterize the standard derived equivalences. These observations are unlikely to be surprising to an enriched category theorist, as they are the apparent generalizations (or specializations) of standard results to our context, but they have been included for the algebraist or topologist who may be unfamiliar with this perspective.

In [KZ98, Ch. 8], Keller remarks that there are no known examples of *non-standard* derived equivalences for rings. Our characterization of Morita theory via the Yoneda Lemma yields the following proposition. The notation $\mathcal{D}_k(X, Y)$ denotes derived categories of bimodules, described in more detail below. Also note that $\text{End}_k(A)$ is taken to mean the derived endomorphism ring.

Proposition (See 6.8). *Let k be a commutative ring, let A be a DG k -algebra and let $f : \mathcal{D}_k(A) \rightarrow \mathcal{D}_k(\text{End}_k(A))$ be an equivalence of triangulated categories. Then f is a standard derived equivalence if and only if the following conditions hold.*

- (i) *The equivalence given by f is an enriched equivalence.*
- (ii) *There is an enriched equivalence $f' : \mathcal{D}_k(A, A) \rightarrow \mathcal{D}_k(\text{End}_k(A), A)$.*
- (iii) *The two equivalences, f and f' are compatible in the following sense: If $T', U' \in \mathcal{D}_k(A, A)$ and $T, U \in \mathcal{D}_k(A) = \mathcal{D}_k(A, k)$, then there are natural maps*

$$\begin{aligned} \text{Ext}_A(T, U') &\rightarrow \text{Ext}_{\text{End}_k(A)}(fT, f'U') && \text{in } \mathcal{D}_k(k, A) \\ \text{Ext}_A(T', U) &\rightarrow \text{Ext}_{\text{End}_k(A)}(f'T', fU) && \text{in } \mathcal{D}_k(A, k) \end{aligned}$$

which commute with the pairing induced by composition. (That is, the squares in Remark 6.4 commute.)

Specializing to the case that A and $\text{Ext}_k(A, A)$ are concentrated in degree 0, this proposition implies that a derived equivalence of rings is standard if and only if it preserves bimodule structure as described above; see Proposition 5.7.

To make the characterization of standard derived equivalences clear, we cannot avoid introducing bicategorical language. In particular, we must describe the notion of *pseudofunctor*, especially *represented pseudofunctor*, and *strong transformation* of (represented) pseudofunctor. This language is relevant because a *component* of a transformation between pseudofunctors is a functor between certain categories, and the question of whether a given derived equivalence is a standard derived equivalence is precisely the same as whether the given functor is a component of a strong transformation between two specific pseudofunctors. We address this fully in Section 5, but we begin in Section 3 by introducing our bicategorical context. In Section 4 we illustrate the bicategorical language with a proof of Proposition 2.7, and in Section 7 and Section 8 we give a further development of the structure present in our bicategorical framework. This is the foundation for our applications of the Yoneda Lemma in Section 5.

3. BICATEGORICAL CONTEXT

We make use of a bicategorical context to organize and clarify our understanding of Morita theory. In this section, we introduce this organizational tool for those to whom it is unfamiliar. For the classical Morita theorem, we consider \mathcal{M} , the bicategory of rings, bimodules, and bimodule maps. For Rickard's theorem, we consider DG_k , the bicategory of DG k -algebras, DG -bimodules and their maps. Associated to this bicategory, we have a derived bicategory, \mathcal{D}_k . We define these bicategories below, and in the remainder of this section we discuss bicategories with a triangulated structure, taking \mathcal{D}_k as a motivating example. Precise and concise definitions can be found in [Lei98], while [Lac07] provides a more expanded guide.

3.1 Rings and Modules. The 0-cells of \mathcal{M} are rings, and for any rings A and B , $\mathcal{M}(A, B)$ is the category of (B, A) -bimodules. So a (B, A) -bimodule ${}_B M_A$ is a 1-cell $M : A \rightarrow B$. The 2-cells between two 1-cells $M : A \rightarrow B$ and $N : A \rightarrow B$ are the bimodule maps $f : {}_B M_A \rightarrow {}_B N_A$. Given three 0-cells, A , B , and C , and two 1-cells, $M : A \rightarrow B$ and $L : B \rightarrow C$, the horizontal composite of L with M is written $L \odot M : A \rightarrow C$. Since M is a (B, A) -bimodule, and L is a (C, B) -bimodule, $L \odot M$ is defined using the tensor product over B ; this produces a (C, A) -bimodule, as desired:

$$L \odot M = L \otimes_B M.$$

A bicategory has, for each 0-cell, A , a unit 1-cell $A \rightarrow A$ satisfying usual unit conditions. We denote this 1-cell also by A ; in the case of rings, this is A regarded as an (A, A) -bimodule.

3.2 Closed structure for \mathcal{M} . A *closed structure* for a bicategory defines right adjoints for \odot . For the bicategory \mathcal{M} the right adjoints for $- \odot M$ and $M \odot -$ are well known. The right adjoint to $- \otimes_B M$ is $\text{Hom}_A(M_A, -)$, homomorphisms of right A -modules, while the right adjoint to $M \otimes_A -$ is $\text{Hom}_B({}_B M, -)$, homomorphisms of left B -modules. To extend these notions to more general bicategories, the adjoint to $- \otimes_B M$ is called “target-hom”, or “right-hom”, and denoted $M \triangleright -$. The adjoint to $M \otimes_A -$ is called “source-hom”, or “left-hom”, and denoted $- \triangleleft M$. The adjunctions are written as

$$\begin{aligned} \mathcal{M}(V \odot M, W) &\cong \mathcal{M}(V, M \triangleright W) \\ \mathcal{M}(M \odot T, U) &\cong \mathcal{M}(T, U \triangleleft M) \end{aligned}$$

The existence of these adjoints is a closed structure for a general bicategory, and we will use \triangleright and \triangleleft to denote the right-hom and left-hom functors in general. The orientation of the triangles is intended to help the reader remember the source and target of the 1-cells $M \triangleright W$ and $U \triangleleft M$. Here, W and M have common source, A , and if C denotes the target of W , then $M \triangleright W$ is a 1-cell $B \rightarrow C$. Likewise, M and U have common target, B , and if D denotes the source of U , then $U \triangleleft M$ is a 1-cell $D \rightarrow A$. A technically complete description of closed structures can be found in [MS06].

3.3 Differential Graded k -algebras. The bicategory DG_k is similar to \mathcal{M} , but here the 0-cells are differential graded k -algebras, the 1-cells are DG bimodules, and the 2-cells are maps of DG bimodules. Like \mathcal{M} , DG_k also has a closed structure. For two 1-cells with common source, P and Q , the target-hom $P \triangleright Q$ denotes the differential graded hom over their common source, and likewise \triangleleft denotes the differential graded hom over common targets.

3.4 The derived bicategory, \mathcal{D}_k . For each pair of DG k -algebras, A and B , there is a model structure for $DG_k(A, B)$ which is a direct generalization of the standard model structure for chain complexes over a ring, and which Section 7 describes in more detail. We use this model structure to understand and work with the derived category of (B, A) -bimodules, which we denote by $\mathcal{D}_k(A, B)$. There is a canonical functor from $DG_k(A, B)$ to $\mathcal{D}_k(A, B)$ and where it adds clarity to our exposition we let $\gamma : DG_k(A, B) \rightarrow \mathcal{D}_k(A, B)$ denote this functor. Note that, for a k -algebra S , $\mathcal{D}_k(S, k) = \mathcal{D}_k(S)$ is the usual derived category of (right) S -modules. The model structure on each 1-cell category satisfies the pushout product condition for \odot -composition (Proposition 8.2), so DG_k is a model bicategory. The derived tensor and hom give a closed bicategory structure for the categories $\mathcal{D}_k(A, B)$, so we regard \mathcal{D}_k as the derived bicategory of DG_k .

3.5 Duality in bicategories. Throughout this subsection we consider fixed 1-cells $X : B \rightarrow A$ and $Y : A \rightarrow B$ in a closed bicategory \mathcal{B} .

Definition 3.6 (Dual pair). We say (X, Y) is a dual pair, or ‘ X is left-dual to Y ’ (‘ Y is right-dual to X ’), or ‘ X is right-dualizable’ (‘ Y is left-dualizable’) to mean that we have 2-cells

$$\eta : A \rightarrow X \odot Y \quad \text{and} \quad \varepsilon : Y \odot X \rightarrow B$$

such that the following composites are the respective identity 2-cells.

$$X \cong A \odot X \xrightarrow{\eta \odot \text{id}} X \odot Y \odot X \xrightarrow{\text{id} \odot \varepsilon} X \odot B \cong X$$

$$Y \cong Y \odot A \xrightarrow{\text{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \text{id}} B \odot Y \cong Y$$

Definition 3.7 (Base and cobase for a dual pair). When (X, Y) is a dual pair in a bicategory \mathcal{B} , we term the source of X (the target of Y) the *base* of the dual pair, and we term the source of Y (the target of X) the *cobase* of the dual pair. Thus, the evaluation map of the dual pair is a two-cell from $Y \odot X$ to the base 1-cell, and the coevaluation (unit) is a two-cell from the cobase 1-cell to $X \odot Y$.

Definition 3.8 (Invertible pair). A dual pair (X, Y) is called invertible if the maps η and ε are isomorphisms. Equivalently, the adjoint pairs described above are adjoint equivalences.

Duality for monoidal categories has been studied at length, and duality in a bicategorical context has been introduced in [MS06, §16.4]. The definition of duality does not require \mathcal{B} to be closed, but we will make use of the following basic facts about duality, some of which do require a closed structure on \mathcal{B} .

Proposition 3.9. *A 1-cell $X \in \mathcal{B}(A, B)$ is right-dualizable if and only if the coevaluation*

$$\nu : X \odot (X \triangleright A) \rightarrow X \triangleright X$$

is an isomorphism. Moreover, this is the case if and only if the map

$$\nu_Z : X \odot (X \triangleright Z) \rightarrow X \triangleright (X \odot Z)$$

is an isomorphism for all 1-cells Z with target A .

Proposition 3.10. *Let (X, Y) be a dual pair in \mathcal{B} , with $X : B \rightarrow A$ and $Y : A \rightarrow B$.*

(1) *For any 0-cell C , we have two adjoint pairs of functors, with left adjoints written on top:*

$$\mathcal{B}(A, C) \begin{array}{c} \xrightarrow{-\odot X} \\ \xleftarrow{-\odot Y} \end{array} \mathcal{B}(B, C)$$

$$\mathcal{B}(C, A) \begin{array}{c} \xrightarrow{Y \odot -} \\ \xleftarrow{X \odot -} \end{array} \mathcal{B}(C, B)$$

The structure maps for the dual pair give the triangle identities necessary to show that the displayed functors are adjoint pairs.

(2) *If \mathcal{B} is closed, then Y is canonically isomorphic to $X \triangleright B$, and for any 1-cell $Z : B \rightarrow D$, the natural map $Z \odot (X \triangleright B) \rightarrow X \triangleright Z$ is an isomorphism.*

The right-dualizable 1-cells in the bicategory \mathcal{M} are the finitely-generated projective bimodules. In fact, they are finitely-generated projective as right-modules over their source (the base of the duality). Lemma 8.10 shows that the retracts of *finite cell* bimodules (Definition 7.1) are right-dualizable in \mathcal{D}_k , and Lemma 8.11 shows that the converse is also true.

3.11 Triangulated bicategories. We recall first the definitions of localizing subcategory and generator for a triangulated category, and then give a definition (3.15) of triangulated bicategory suitable for our purposes. In particular, under this definition \mathcal{D}_k is a triangulated bicategory.

Definition 3.12 (Localizing subcategory).

If \mathcal{T} is a triangulated category with infinite coproducts, a *localizing* subcategory, \mathcal{S} , is a full triangulated subcategory of \mathcal{T} which is closed under coproducts from \mathcal{T} .

Remark 3.13. This is equivalent to the definition for arbitrary triangulated categories of [Hov99], (which requires that a localizing subcategory be thick) because a triangulated subcategory automatically satisfies the 2-out-of-3 property and because in any triangulated category with countable coproducts, idempotents have splittings. See [Nee01, 1.5.2, 1.6.8, and 3.2.7] for details.

Definition 3.14 (Triangulated generator).

A set, \mathcal{P} , of objects in \mathcal{T} (triangulated category with infinite coproducts, as above) is a set of *triangulated generators* (or simply *generators*) if the only localizing subcategory containing \mathcal{P} is \mathcal{T} itself.

Definition 3.15 (Triangulated bicategory [MS06, §16.7]).

A closed bicategory \mathcal{B} will be called a *triangulated bicategory* if for each pair of 0-cells, A and B , $\mathcal{B}(A, B)$ is a triangulated category with infinite coproducts, and if the suspension, Σ , is a pseudofunctor (Subsection 5.1) on \mathcal{B} , and furthermore the local triangulations on \mathcal{B} are compatible as described in the following two axioms.

(TC1) For a 1-cell $X : A \rightarrow B$, there is a natural isomorphism

$$\alpha : X \odot \Sigma A \rightarrow \Sigma X$$

such that the composite below is multiplication by -1 .

$$\Sigma^2 A = \Sigma(\Sigma A) \xrightarrow{\alpha^{-1}} \Sigma A \odot \Sigma A \xrightarrow{\gamma} \Sigma A \odot \Sigma A \xrightarrow{\alpha} \Sigma(\Sigma A) = \Sigma^2 A$$

(TC2) For any 1-cell, W , the functors $W \odot -$, $- \odot W$, $W \triangleright -$, and $- \triangleright W$ are exact.

If \mathcal{B} is a triangulated bicategory and P, Q are 1-cells in $\mathcal{B}(A, B)$, we emphasize that \mathcal{B} is triangulated by writing the abelian group of 2-cells $P \rightarrow Q$ as $\mathcal{B}[P, Q]$ and by writing the graded abelian group obtained by taking shifts of Q as $\mathcal{B}[P, Q]_*$. To emphasize the source and target of P and Q , we may also write $\mathcal{B}(A, B)[P, Q]_*$, as in Theorem 2.1 (where we write $\mathcal{D}_k(S)$ instead of $\mathcal{D}_k(S, k)$).

4. PROOF OF PROPOSITION 2.7

In this section we prove Proposition 2.7, which generalizes one direction of Rickard's theorem to the case of DG k -algebras. We work in the closed triangulated bicategory \mathcal{D}_k , beginning with a few general statements.

Definition 4.1 (\odot -detecting 1-cells).

In any locally additive bicategory, \mathcal{B} , a 1-cell $W : A \rightarrow B$ is called \odot -detecting if triviality for any 1-cell $Z : C \rightarrow A$ is detected by triviality of the composite $W \odot Z$. That is, $Z : C \rightarrow A$ is zero if and only if $W \odot Z = 0$. A collection of 1-cells, \mathcal{E} , in $\mathcal{B}(A, B)$ is called *jointly \odot -detecting* if the objects have this property jointly; that is, $Z = 0$ if and only if $W \odot Z = 0$ for all $W \in \mathcal{E}$.

Remark 4.2. If \mathcal{B} is a monoidal additive category with monoidal product \odot , the unit object is \odot -detecting. In arbitrary locally additive bicategories, if $A \neq B$ then $\mathcal{B}(A, B)$ may not have a single object with this property, but in relevant examples the collection of all 1-cells, $\text{ob}\mathcal{B}(A, B)$, does have this property jointly. As a counter-point to this remark, we have the following lemma.

Lemma 4.3. *Let \mathcal{B} be a triangulated bicategory, and let $P : A \rightarrow B$ be a generator for $\mathcal{B}(A, B)$. If the collection of all 1-cells, $\mathcal{B}(A, B)$, is jointly \odot -detecting, then P is \odot -detecting.*

Proof. Given any 1-cell $Z : C \rightarrow A$ with $P \odot Z = 0$, let \mathcal{S} be the full subcategory of 1-cells, $W : A \rightarrow B$ for which $W \odot Z = 0$. This is a localizing subcategory of $\mathcal{B}(A, B)$, and by assumption $P \in \mathcal{S}$, so $\mathcal{S} = \mathcal{B}(A, B)$, and hence $Z = 0$. \square

Remark 4.4. Since the functors $P \odot -$ are exact, the property of $P \odot -$ detecting trivial objects is equivalent to $P \odot -$ detecting isomorphisms (a 2-cell f is an isomorphism if and only if $P \odot f$ is so).

Now we turn to the proof. Suppose T is a chain complex of (right) S -modules satisfying the dualizability and generator conditions of Proposition 2.7, and let E denote the DG k -algebra $\text{Hom}_S(T, T)$. One might call our first step ‘cobase extension’, as we describe how to extend the dualizable object T to a dual pair with base S and cobase E . The chain complex T is a right DG-module over S , and can be considered as a left module over the DG k -algebra E . We let \tilde{T} denote T regarded as a 1-cell $S \rightarrow E$, and let T denote the 1-cell $S \rightarrow k$. These 1-cells are related by base change along the unit map (of DG k -algebras) $k \rightarrow E$. Restricting scalars on either the left or right of the DG k -algebra E gives rise to a dual pair $({}_k E, E_k)$, and the 1-cell $T : S \rightarrow k$ is recovered from $\tilde{T} : S \rightarrow E$ as the 1-cell composite ${}_k E \odot \tilde{T}$. Moreover, $T \triangleright S$ is recovered as the composite $(\tilde{T} \triangleright S) \odot E_k$, and the coevaluation map of $(T, T \triangleright S)$ is recovered from that of $(\tilde{T}, \tilde{T} \triangleright S)$. In more common language, one might say that $T \triangleright S$ is a right E -module, and the coevaluation map of $(T, T \triangleright S)$ is a map of E - E bimodules.

Because T is (right-)dualizable in $DG_k(S, k)$, it follows that \tilde{T} is (right-)dualizable in $DG_k(S, E)$. Lemma 4.5 below shows that, therefore, \tilde{T} is dualizable in $\mathcal{D}_k(S, E)$. To finish the proof, we use this fact to show that, because T is a generator, \tilde{T} is invertible. Then the invertible pair $(\tilde{T}, \tilde{T} \triangleright S)$ establishes an equivalence of categories, as described in Definition 3.8.

Lemma 4.5. *If X is dualizable in $DG_k(A, B)$, then $\gamma(X)$ is dualizable in $\mathcal{D}_k(A, B)$.*

Proof. The dualizable objects in $DG_k(A, B)$ are retracts of finitely-generated projective (right-)modules over $A \otimes_k B^{\text{op}}$, and hence cofibrant. This is shown by a standard dual-basis-type argument, and the interested reader will find the details in Lemma 8.9. Because X is cofibrant, the functor $X \odot -$ preserves weak equivalences. This also is a standard result, and a proof can be found in [KM95, III.4.1].

Recall that γ denotes the canonical functor $DG_k(A, B) \rightarrow \mathcal{D}_k(A, B)$. Let $Q(X \triangleright A)$ be a cofibrant replacement for $X \triangleright A$. Then $\gamma(X) \odot \gamma(X \triangleright A) = X \odot Q(X \triangleright A)$, and $\gamma(X \odot (X \triangleright A)) = X \odot (X \triangleright A)$, and even though γ is not a strong monoidal functor, we nevertheless have an isomorphism in $\mathcal{D}_k(B, B)$: $\gamma(X) \odot \gamma(X \triangleright A) \xrightarrow{\cong} \gamma(X \odot (X \triangleright A))$. It is a formality now to check that the duality relations hold for $\gamma(X)$ and $\gamma(X \triangleright A)$, and therefore $\gamma(X)$ is dualizable in $\mathcal{D}_k(A, B)$. For those who wish to see it, this formal argument is given explicitly for monoidal categories in [LMS86, III.1.9]. \square

Applying the lemma to \tilde{T} we have, for any C , the adjoint pair of functors induced by a dual pair (Proposition 3.10) shown below. Because $E = \tilde{T} \triangleright \tilde{T}$, the unit of this adjunction is an isomorphism—the inverse to the coevaluation map.

$$\mathcal{D}_k(E, C) \begin{array}{c} \xleftarrow{-\odot \tilde{T}} \\ \xrightarrow{-\odot (\tilde{T} \triangleright S)} \end{array} \mathcal{D}_k(S, C).$$

We finish the proof of Proposition 2.7 by showing that the counit $\text{eval} : (\tilde{T} \triangleright S) \odot \tilde{T} \rightarrow S$ is an isomorphism in $\mathcal{D}_k(S, S)$. Since k is the ground ring for our bicategory \mathcal{D}_k , the 1-cells of $\mathcal{D}_k(S, k)$ are

jointly \odot -detecting (Definition 4.1), and so the generator condition of Proposition 2.7 means that T itself is \odot -detecting (Lemma 4.3). Thus, evaluation $(\tilde{T} \triangleright S) \odot \tilde{T} \rightarrow S$ is an isomorphism in $\mathcal{D}_k(S, S)$ if and only if the map $({}_k E \odot \tilde{T}) \odot (\tilde{T} \triangleright S) \odot \tilde{T} \xrightarrow{1_{\odot \text{eval}}} {}_k E \odot \tilde{T}$ is so (Remark 4.4). The duality of \tilde{T} and $\tilde{T} \triangleright S$ implies that this second map is an isomorphism, because the composite below is the identity and the first map, induced by the unit of the adjunction, is an isomorphism.

$${}_k E \odot \tilde{T} \xrightarrow{\cong} {}_k E \odot \tilde{T} \odot (\tilde{T} \triangleright S) \odot \tilde{T} \xrightarrow{1_{\odot \text{eval}}} {}_k E \odot \tilde{T}$$

Hence the second map in this composite is an isomorphism, and so the counit for the dual pair $(\tilde{T}, \tilde{T} \triangleright S)$ is an isomorphism in $\mathcal{D}_k(S, S)$, giving an equivalence of triangulated categories, as we wished to show.

$$\mathcal{D}_k(E, C) \begin{array}{c} \xrightarrow{-\odot \tilde{T}} \\ \xleftarrow[\cong]{-\odot (\tilde{T} \triangleright S)} \end{array} \mathcal{D}_k(S, C)$$

This equivalence is suitably natural in C , making it a strong transformation of the represented pseudo-functors $\mathcal{D}_k(E, -)$ and $\mathcal{D}_k(S, -)$. A more complete picture of strong transformations and their connection to the Yoneda Lemma for bicategories is described in Section 5.

4.6 Rickard’s theorem for spectra. In this subsection we prove a result analogous to Proposition 2.7, but working instead with a commutative S -algebra, k , and the bicategory \mathcal{S}_k of k -algebras and their bimodules. We extend our previous notation to let \mathcal{D}_k denote the bicategory of derived categories for spectra. One major difference is that instead of working with dualizability on the level of model categories, as we have for algebraic derived Morita theory, we shift to the notion of dualizability in the bicategory of derived categories. The principles and general approach are the same, but the details must be modified slightly. For example, the ‘cobase extension’ step in the algebraic case is nearly transparent, but requires a lemma in the context of spectra. One can prove results about spectra which are dualizable on the model-categorical level but, unlike the DG case, it is difficult to find examples of such spectra. With a relative abundance of spectra which are dualizable in the derived bicategory, we shift our focus in that direction. For the remainder of this section, we use the term ‘dualizable’ to mean dualizable in the bicategory \mathcal{D}_k .

Proposition 4.7. *Let A be a k -algebra, and let T be a fibrant and cofibrant A -module, with endomorphism k -algebra $E = F_A(T, T)$. If T has the following two properties, then $\mathcal{D}_k(A)$ and $\mathcal{D}_k(E)$ are equivalent categories.*

- (i) T is (right-)dualizable as an A -module.
- (ii) T generates the triangulated category $\mathcal{D}_k(A)$.

As with the algebraic version, our proof proceeds in two parts. First (‘cobase extension’) we show that a dual pair in \mathcal{D}_k between two k -algebras can be extended to the endomorphism algebra of the left dual, and that in so doing we produce a new dual pair whose unit is an isomorphism. This rough description is made precise in the statement of Lemma 4.9, after introducing notation for the restriction of scalars functors. In the second part of our proof we use the unit isomorphism of this new dual pair, together with a generating condition, to detect that the evaluation map is also an isomorphism (in \mathcal{D}_k). Hence the new dual pair is an invertible pair, giving an equivalence of categories.

Notation 4.8. Given a map of k -algebras $\iota : B \rightarrow E$, we have two restriction-of-scalars functors: one for restriction of left modules, and another for restriction of right modules. For any k -algebra A , We let $\iota_L^* : \mathcal{S}_k(A, E) \rightarrow \mathcal{S}_k(A, B)$ denote restriction on the left (target), and $\iota_R^* : \mathcal{S}_k(E, A) \rightarrow \mathcal{S}_k(B, A)$ denote restriction on the right (source). Both functors create weak-equivalences and fibrations.

Lemma 4.9. *Let A and B be k -algebras, and let T be fibrant and cofibrant in $\mathcal{S}_k(A, B)$, with endomorphism k -algebra $E = F_A(T, T)$. If T is (right-)dualizable in \mathcal{D}_k , then there is a homotopy dual pair (\tilde{T}, \tilde{D}) with base A and cobase E whose unit is an isomorphism. This dual pair extends T in the sense that $\iota_L^* \tilde{T} \simeq T$, where $\iota : B \rightarrow E$ is the k -algebra map adjoint to the action of B on T .*

Proof. Because T is cofibrant and fibrant in $\mathcal{S}_k(A, B)$, no replacements are necessary and E is the derived endomorphism monoid of T . The unit map $k \rightarrow E$ is obtained as the composite of algebra maps $k \rightarrow B \rightarrow E$. Let \tilde{T} be a cofibrant replacement for T in $\mathcal{S}_k(A, E)$. Recall that T is cofibrant in $\mathcal{S}_k(A, B)$, and hence has the LLP with respect to acyclic fibrations. We construct \tilde{T} by the usual factorization of the map from the initial object, and the forgetful functor ι_L^* creates weak equivalences and fibrations, so the lifting property for T gives a weak equivalence $T \xrightarrow{\cong} \iota_L^* \tilde{T}$.

The canonical dual of T is $F_A(T, A) = T \triangleright A \in \mathcal{S}_k(B, A)$, and we let D denote a cofibrant replacement for $F_A(T, A)$ in $\mathcal{S}_k(B, A)$, so that we have a weak equivalence $D \xrightarrow{\cong} F_A(T, A)$. The

canonical dual of T has a right-action of the endomorphism k -algebra, E , and we let \tilde{D} be a cofibrant replacement for $F_A(T, A)$ in $\mathcal{S}_k(E, A)$, constructed again by the usual factorization. Since the forgetful functor ι_R^* creates weak equivalences and fibrations, we have an acyclic fibration $\iota_R^* \tilde{D} \xrightarrow{\cong} F_A(T, A)$ in $\mathcal{S}_k(B, A)$. Because D is cofibrant, the weak equivalence $D \xrightarrow{\cong} F_A(T, A)$ lifts with respect to acyclic fibrations and hence we have a weak equivalence $D \xrightarrow{\cong} \iota_R^* \tilde{D}$.

Now we show that (\tilde{T}, \tilde{D}) is a dual pair in \mathcal{D}_k . The weak equivalences $\tilde{T} \rightarrow T$ and $\tilde{D} \rightarrow F_A(T, A)$ in $\mathcal{S}_k(A, E)$ and $\mathcal{S}_k(E, A)$, respectively, give maps

$$\tilde{T} \odot \tilde{D} \rightarrow T \odot F_A(T, A) \rightarrow E \text{ and } \tilde{D} \odot \tilde{T} \rightarrow F_A(T, A) \odot T \rightarrow A$$

in $\mathcal{S}_k(E, E)$ and $\mathcal{S}_k(A, A)$, respectively. Moreover, the first map is an isomorphism in $\mathcal{D}_k(E, E)$ because its image under $\iota_L^* \iota_R^*$ is a composite of two isomorphisms in $\mathcal{D}_k(B, B)$:

$$\iota_L^* \tilde{T} \odot \iota_R^* \tilde{D} \cong T \odot D \cong \iota_L^* \iota_R^* E.$$

The inverse to this map gives the unit for the dual pair, and the duality diagrams commute because the corresponding diagrams for T and $F_A(T, A)$ do. Hence the functors $-\odot \tilde{T}$ and $-\odot \tilde{D}$ induce an adjunction

$$\mathcal{D}_k(A, C) \begin{array}{c} \xrightarrow{-\odot \tilde{T}} \\ \xleftarrow{-\odot \tilde{D}} \end{array} \mathcal{D}_k(E, C)$$

and the unit of this adjunction is an isomorphism. \square

Lemma 4.10. *Let T, \tilde{T}, \tilde{D} be as in Lemma 4.9, with $B = k$. If T generates $\mathcal{D}_k(A, k)$, then \tilde{T} is \odot -detecting.*

Proof. As in the algebraic case, this follows because k is the ground object, and hence the collection of all 1-cells is jointly \odot -detecting (Definition 4.1). The generator condition therefore implies that T itself is \odot -detecting (Lemma 4.3). Now ι_L^* creates weak equivalences, and $\iota_L^*(-) = \iota_L^*(E) \odot -$, so \tilde{T} is also \odot -detecting. \square

Using Lemmas 4.9 and 4.10, we finish the proof of Proposition 4.7 as in the algebraic case. Both the composite and the first map displayed below are isomorphisms, and hence the second map is also an isomorphism. But the second map is $\tilde{T} \odot -$ applied to the counit, and since \tilde{T} is \odot -detecting, the counit of the dual pair is therefore an isomorphism in $\mathcal{D}_k(A, A)$.

$$E \odot \tilde{T} \rightarrow \tilde{T} \odot \tilde{D} \odot \tilde{T} \rightarrow \tilde{T} \odot A$$

5. THE BICATEGORICAL YONEDA LEMMA

This section describes the Yoneda Lemma for bicategories. Following [Str80], we avoid giving the detailed definitions, and instead give some general description followed by examples, which will be our main interest. As in Section 3, we suggest [Lac07] or [Lei98] for further background.

5.1 Pseudofunctors. If \mathcal{A} and \mathcal{B} are bicategories, a pseudofunctor $P : \mathcal{A} \rightarrow \mathcal{B}$ (also called a *morphism*) is the bicategorical version of a functor. It is a function on 0-cells and for each pair of 0-cells a functor

$$\mathcal{A}(A, B) \xrightarrow{P_{AB}} \mathcal{B}(PA, PB).$$

These functors are compatible with \odot -composition in that there are 2-cell isomorphisms

$$P_{BC} X' \odot P_{AB} X \xrightarrow{\cong} P_{AC} (X' \odot X)$$

satisfying the natural associativity and unit compatibility conditions.

Our focus is on the represented pseudofunctors. These are a bicategorical version of represented functors for categories, and they take values in the bicategory Cat . In this bicategory, the 0-cells are categories, 1-cells are functors, and 2-cells are natural transformations of functors. For any bicategory \mathcal{B} with 0-cell A , we have the represented pseudofunctor $\mathcal{B}(A, -) : \mathcal{B} \rightarrow Cat$. For a 0-cell $E \in \mathcal{B}$, this pseudofunctor gives a category, $\mathcal{B}(A, E)$. For a 1-cell $M : E \rightarrow E'$, we have the functor $M \odot - : \mathcal{B}(A, E) \rightarrow \mathcal{B}(A, E')$, and 2-cells $M \rightarrow M'$ give natural transformations of such functors. The compatibility isomorphisms which make $\mathcal{B}(A, -)$ a pseudofunctor are precisely the associativity isomorphisms $(M_2 \odot (M_1 \odot -)) \cong (M_2 \odot M_1) \odot -$.

In this context, our introductory remark ‘Morita theory is about bimodules’ can be rephrased as the comment that Morita theory is about represented pseudofunctors. Our remark near the end of

Section 2 that ‘standard derived equivalences preserve bimodule structure’ can be understood as the observation that standard derived equivalences are transformations of represented pseudofunctors.

5.2 (Strong) transformations. A transformation is a kind of bicategorical natural transformation of functors. A transformation of two represented pseudofunctors, $\mathcal{B}(B, -)$ and $\mathcal{B}(A, -)$ is given by

- (1) A family of functors $F_C : \mathcal{B}(B, C) \rightarrow \mathcal{B}(A, C)$. These are the *components* of F .
- (2) For each 1-cell $C \xrightarrow{K} C'$, a natural transformation which, for 1-cells $X \in \mathcal{B}(B, C)$, has component 2-cells

$$K \odot F_C(X) \rightarrow F_{C'}(K \odot X)$$

natural in K and X , with standard associativity and unit compatibilities; namely that the following diagrams commute, with K and X as above, and $L \in \mathcal{B}(C', C'')$.

$$\begin{array}{ccc} L \odot K \odot F_C(X) & \longrightarrow & L \odot F_{C'}(K \odot X) \\ & \searrow & \downarrow \\ & & F_{C''}(L \odot K \odot X) \end{array} \qquad \begin{array}{ccc} C \odot F_C(X) & \longrightarrow & F_c(C \odot X) \\ & \searrow \cong & \downarrow \cong \\ & & F_c(X) \end{array}$$

Notation 5.3. In the following, we will frequently drop the subscripts on the components of our transformations since they may always be determined from context and they tend to make the text less readable.

For developing Morita theory, our interest will be in *strong* transformations; these are transformations for which the component 2-cells shown above are natural isomorphisms. Restricting attention to strong transformations is equivalent to restricting to transformations which have object-wise adjoints (Lemma 5.8). Since the equivalences Morita theory seeks to understand are, in particular, adjoint pairs, this restriction of scope is necessary. Similar ideas are considered for distributors in [FPP75] and [BV02] and for bialgebroids in [Sz104].

The appropriate morphisms of transformations are called *modifications*, but we will not make any explicit reference to them beyond the following definition.

Definition 5.4. For two bicategories \mathcal{A} and \mathcal{B} , $\Psi F[\mathcal{A}, \mathcal{B}]$ denotes the bicategory whose 0-cells are pseudofunctors $\mathcal{A} \rightarrow \mathcal{B}$, 1-cells are strong transformations, and 2-cells are modifications.

Lemma 5.5 (Yoneda [Str80]). *For a pseudofunctor of bicategories $\mathcal{P} : \mathcal{A} \rightarrow \text{Cat}$, evaluation at the unit 1-cell for each 0-cell, A , of \mathcal{A} provides the components for an equivalence of categories*

$$\Psi F[\mathcal{A}, \text{Cat}](\mathcal{A}(A, -), \mathcal{P}) \xrightarrow{\cong} \mathcal{P}A.$$

Corollary 5.6 (Morita II).

$$\Psi F[\mathcal{A}, \text{Cat}](\mathcal{A}(A, -), \mathcal{A}(B, -)) \xrightarrow{\cong} \mathcal{A}(B, A)$$

That is, strong transformations $\mathcal{A}(A, -) \rightarrow \mathcal{A}(B, -)$ are given (precisely) by \odot -composition with a 1-cell $B \rightarrow A$. In particular, strong transformations which induce equivalences $\mathcal{A}(A, C) \simeq \mathcal{A}(B, C)$ for all 0-cells C are given by invertible 1-cells $B \rightarrow A$.

The essential point of the proof, as in the 1-categorical case, is the observation that for a strong transformation, S , and a 1-cell $Z : A \rightarrow C$,

$$S_C(Z) \cong S_C(Z \odot A) \cong Z \odot S_A(A)$$

so that, for any C , the functor S_C is determined by $S_A(A)$, an object in the category $\mathcal{A}(B, A)$. Natural transformations of these functors are determined by morphisms in $\mathcal{A}(B, A)$.

This equivalence can be read with various emphases, yielding various interpretations. One possible interpretation would take strong transformations or strong equivalences as objects of interest and take the equivalence as a characterization of these objects—they can be only those transformations arising as \odot -composition with a 1-cell. A complementary interpretation takes the transformations given by \odot -composition as the basic objects of interest, as in the case of the standard derived equivalences for derived categories of DG k -algebras. From this point of view, the equivalence is an assurance that functors arising in this way are no less, and no more, than the strong transformations.

In the presence of a closed structure for our bicategory, we have a further interpretation. Functors given by \odot -composition with a 1-cell are naturally enriched over an ambient closed structure; in the following proposition, we formalize what is meant by a family of functors enriched over the internal hom, and show that such functors are necessarily the family of components of a transformation.

Proposition 5.7. *Let A and B be 0-cells of a closed bicategory \mathcal{B} , and let F be a family of functors*

$$F_C : \mathcal{B}(A, C) \rightarrow \mathcal{B}(B, C)$$

for 0-cells C . The following are equivalent.

- (1) *For any C , and any 1-cells $T \in \mathcal{B}(A, C_1)$, $U \in \mathcal{B}(A, C_2)$, $V \in \mathcal{B}(A, C_3)$ there are 2-cells*

$$T \triangleright U \rightarrow FT \triangleright FU \text{ in } \mathcal{B}(C_1, C_2)$$

and

$$U \triangleright V \rightarrow FU \triangleright FV \text{ in } \mathcal{B}(C_2, C_3).$$

These 2-cells are natural in T , U , and V , preserve units, and commute with composition, in the sense described by the following.

$$\begin{array}{ccc} (U \triangleright V) \odot (T \triangleright U) & \xrightarrow{\text{comp}} & T \triangleright V \\ \downarrow & & \downarrow \\ (FU \triangleright FV) \odot (FT \triangleright FU) & \xrightarrow{\text{comp}} & FT \triangleright FV \end{array} \quad \begin{array}{ccc} T \triangleright T & \longrightarrow & FT \triangleright FT \\ \uparrow & \nearrow \text{adj. to unit} & \\ C_1 & & \end{array}$$

- (2) *The family F is the family of components for a transformation of represented pseudofunctors. That is, for any 1-cells $X \in \mathcal{B}(A, C_1)$ and $K \in \mathcal{B}(C_1, C_2)$, there are 2-cells*

$$K \odot F(X) \rightarrow F(K \odot X) \text{ in } \mathcal{B}(B, C_2).$$

These 2-cells are natural in K and X , and associative and unital.

Proof. Given maps as in (1), and 1-cells K and X as in (2), we describe the structure 2-cell

$$K \odot F(X) \rightarrow F(K \odot X)$$

as the following composite:

$$K \odot F(X) \xrightarrow{\text{adj. to id}_{K \odot X}} [X \triangleright (K \odot X)] \odot F(X) \xrightarrow{F} [F(X) \triangleright F(K \odot X)] \odot F(X) \xrightarrow{\text{eval}} F(K \odot X).$$

Using the definition of the map, associativity for the structure 2-cell is reduced to the given commutativity with composition. Unitality follows from the unit condition above.

The situation is exactly reversed for the converse. Given maps as in (2) and 1-cells T and U as in (1), we describe

$$T \triangleright U \xrightarrow{F} FT \triangleright FU$$

as adjoint to the map

$$(T \triangleright U) \odot FT \rightarrow F((T \triangleright U) \odot T) \xrightarrow{F(\text{eval})} FU. \quad \square$$

Proposition 5.7 shows that a transformation of represented pseudofunctors can be interpreted as what one might call “a natural family of enriched functors”. The following lemma is proved similarly, and gives a specialized interpretation for the strong transformations: families of left adjoints.

Lemma 5.8. *For F as above, the maps $K \odot F(X) \rightarrow F(K \odot X)$ are isomorphisms for all K and X if and only if F has a family of right adjoints,*

$$\mathcal{B}(A, C) \leftarrow \mathcal{B}(B, C) : G_C$$

and these adjoints have maps $K' \odot G(X') \rightarrow G(K' \odot X')$ making G into a transformation of represented pseudofunctors $\mathcal{B}(B, -) \rightarrow \mathcal{B}(A, -)$. In other words, a transformation F is a strong transformation if and only if it has a right adjoint transformation.

6. PRACTICAL INTERPRETATION

In this section, we return our focus to Morita theory. The question of when a derived equivalence is a *standard* derived equivalence is raised, but not answered, by Rickard’s work. Example 2.8 shows that a short answer to this question is “not always”, and more subtle answers have been explored in the literature of derived Morita theory (see [KZ98], for example). Our inspections of the Yoneda Lemma yield a reinterpretation of this question, and another approach to determining when one might give an affirmative answer; we discuss this in Subsection 6.1. In Subsection 6.5, we turn to the ambient enrichments which are present in both classical and derived Morita theory. We again use the ideas of the previous section, this time to emphasize the relevance of enrichments to Morita theory. We follow this explanation with some examples, illustrating how one might apply these interpretations in practice.

6.1 Components of (strong) transformations. In this subsection we let \mathcal{B} denote a closed bicategory, with the example $\mathcal{B} = \mathcal{D}_k$ as our primary motivation. Let A , B , and I be three fixed 0-cells of \mathcal{B} ; in

I am not the first to use ‘unitality’ in this sense

our motivating example, we take $I = k$. Furthermore, suppose f is simply a functor $\mathcal{B}(A, I) \rightarrow \mathcal{B}(B, I)$. Recalling Corollary 5.6 (Morita II), the question of when f is a *standard* functor is equivalent to the question of whether f is a component of a transformation between pseudofunctors, that is, whether there is a transformation $F : \mathcal{B}(A, -) \rightarrow \mathcal{B}(B, -)$ with $F_I = f$. In particular, we seek to understand the case when f is an equivalence, and characterize when f is a component of a strong transformation. As Lemma 5.8 points out, f being a component of a strong transformation implies that its adjoint is itself a component of a transformation.

In this situation, let \mathcal{B}_A denote the full sub-bicategory of \mathcal{B} whose 0-cells are A and I . There are four 1-cell categories in \mathcal{B}_A ; these are the categories of 1-cells in \mathcal{B} from A to A , from A to I , from I to A , and from I to I . That is, $\mathcal{B}_A(x, y) = \mathcal{B}(x, y)$ for $x, y \in \{A, I\}$. We have the represented pseudofunctor $\mathcal{B}_A(A, -) : \mathcal{B}_A \rightarrow \mathit{Cat}$, and we have also the pseudofunctor $\mathcal{B}(B, -) : \mathcal{B}_A \rightarrow \mathit{Cat}$. Since B is not a 0-cell of \mathcal{B}_A , $\mathcal{B}(B, -)$ is un-represented, but it is nevertheless a pseudofunctor on \mathcal{B}_A and the Yoneda Lemma applies to describe strong transformations $\mathcal{B}_A(A, -) \rightarrow \mathcal{B}(B, -)$. Using the Yoneda lemma twice, we have the following two equivalences:

$$\Psi F[\mathcal{B}_A, \mathit{Cat}](\mathcal{B}_A(A, -), \mathcal{B}(B, -)) \xrightarrow{\cong} \mathcal{B}(B, A) \xleftarrow{\cong} \Psi F[\mathcal{B}, \mathit{Cat}](\mathcal{B}(A, -), \mathcal{B}(B, -)).$$

These two equivalences say in bicategorical language what is apparent to one who considers the proof of the Yoneda Lemma: that strong transformations are determined by their values on the unit 1-cell. One could make this clearer by restricting \mathcal{B}_A further to a single 0-cell, A , since for the equivalence above I is irrelevant. We have chosen to include I so that the equivalences above provide a proof for the following.

Corollary 6.2. *A functor $f : \mathcal{B}(A, I) \rightarrow \mathcal{B}(B, I)$ is a component of a strong transformation $\mathcal{B}(A, -) \rightarrow \mathcal{B}(B, -)$ if and only if it is a component of a strong transformation $F : \mathcal{B}_A(A, -) \rightarrow \mathcal{B}(B, -)$. That is, f is a component of a strong transformation if and only if there is a functor $f' : \mathcal{B}(A, A) \rightarrow \mathcal{B}(B, A)$ and natural 2-cell isomorphisms $K \odot f'(X) \xrightarrow{\cong} f(K \odot X)$, with the apparent associativity requirement, for any 1-cells $X : A \rightarrow A$ and $K : A \rightarrow I$. In this case, the strong transformation F is determined by its two components, $F_I = f$ and $F_A = f'$.*

Dropping the condition that transformation be strong, we can use Proposition 5.7 in the preceding context to achieve a description in terms of the internal hom.

Corollary 6.3. *A functor $f : \mathcal{B}(A, I) \rightarrow \mathcal{B}(B, I)$ is a component of a transformation $\mathcal{B}(A, -) \rightarrow \mathcal{B}(B, -)$ if and only if there is a functor $f' : \mathcal{B}(A, A) \rightarrow \mathcal{B}(B, A)$ and, for 1-cells $T, U \in \mathcal{B}(A, I)$ and $T', U' \in \mathcal{B}(A, A)$, there are 2-cells*

$$\begin{array}{ll} T \triangleright U \rightarrow fT \triangleright fU & \text{in } \mathcal{B}(I, I) \\ T' \triangleright U' \rightarrow f'T' \triangleright f'U' & \text{in } \mathcal{B}(A, A) \\ T \triangleright U' \rightarrow fT \triangleright f'U' & \text{in } \mathcal{B}(I, A) \\ T' \triangleright U \rightarrow f'T' \triangleright fU & \text{in } \mathcal{B}(A, I) \end{array}$$

subject to the compatibility with composition and units described in Proposition 5.7(1).

Remark 6.4. In this context, the compatibility means precisely that all possible diagrams of the form below commute:

$$\begin{array}{ccc} (U \triangleright V) \odot (T \triangleright U) & \longrightarrow & T \triangleright V & & T \triangleright T & \longrightarrow & FT \triangleright FT \\ \downarrow & & \downarrow & & \uparrow & \nearrow & \\ (FU \triangleright FV) \odot (FT \triangleright FU) & \longrightarrow & FT \triangleright FV & & \text{unit} & & \end{array}$$

Where each of T, U , and V is taken to be either in $\mathcal{B}(A, A)$ or $\mathcal{B}(A, I)$, ‘unit’ is taken to be either I , or A , and F is taken to be either f or f' , as appropriate. Note that if all three of the objects, T, U, V are taken from the same category, this condition is precisely the condition which makes f and f' enriched over \triangleright . The first diagram says that the enrichment must commute with the composition pairing, and the second diagram says that the enrichment must preserve units.

We now turn to some more practical interpretations of Section 5. Combining these results with our previous interpretations of Morita theory enables us to give a description of the concepts at work in both classical and derived Morita theory. We follow this description with a few examples of Corollary 6.3 which show that, at least in algebraic contexts, the four-part necessary condition can often be verified formally.

6.5. Proposition 5.7 shows that the standard Morita equivalences in a closed bicategory must be equivalences which are enriched over the internal hom, and likewise that families of equivalences which are

enriched over the internal hom fit together to form standard Morita equivalences. As noted in Remark 2.3, left-adjoint functors (e.g. equivalences) between abelian categories are automatically enriched in abelian groups, and this may be one reason that enrichment has been under-appreciated in these contexts. The topological Morita theorem of Schwede and Shipley [SS03] addresses *spectral* Quillen equivalences—Quillen equivalences enriched in spectra. Likewise, Toën [Toë07] works with DG categories, the morphisms of which are enriched functors. Proposition 5.7 shows that focusing on enriched equivalences is inevitable.

In these papers the authors also address the important question of what model-theoretic assumptions could be verified in practice and would guarantee standard Morita equivalences on the derived level. One possible lesson taught by Example 2.8 is that certainly some assumptions are necessary in general. The results above, however, are independent of model theory, applying to any closed bicategory. They indicate that Quillen equivalences which induce *enriched* transformations on the derived level are necessarily the appropriate equivalences for the development of Morita theory. This perspective can offer an explanation for the results of [DS07] in particular. There, and in related work, the technical notion of *additive* model category is introduced, and it is observed that Quillen equivalences between additive model categories are necessarily additive functors, just as in the classical situation. The result, therefore, is that a zig-zag of Quillen equivalences for which each intermediate model category is *additive* provides a well-behaved notion of Morita equivalence for additive model categories. Our perspective would suggest that this can be extended to more general enriched model categories, with the appropriate notion of Morita equivalence in those settings being enriched Quillen functors. From this point of view, Example 2.8 is an expected example, and others like it will be expected in applications for which Quillen functors are not necessarily enriched.

Corollary 6.3 shows that the property of enrichment may be identified by considering only a specific special case, arising through our restriction from the bicategory \mathcal{B} to the full sub-bicategory, \mathcal{B}_A generated by two 0-cells, A and I . In algebraic examples, this four-part condition can be simplified even further. Before discussing this possibility in more generality, we remind the reader of two algebraic results and their proofs, for reference. These results demonstrate that, in the case of classical Morita theory, the condition in Corollary 6.3 is automatic.

Theorem 6.6 (Morita II [Lam99, 7.18.26]). *Let R and S be two rings, and let*

$$f : \mathcal{M}(R, \mathbb{Z}) \xrightleftharpoons{\quad} \mathcal{M}(S, \mathbb{Z}) : g$$

be an equivalence between the categories of right R -modules and right S -modules. Let $Q = f(R)$ and let $P = g(S)$. Then there are natural bimodule structures making $P \in \mathcal{M}(R, S)$ and $Q \in \mathcal{M}(S, R)$. Using these bimodule structures, there are natural isomorphisms of functors

$$f \cong - \odot Q \quad \text{and} \quad g \cong - \odot P.$$

Proof. The bimodule structures are recognized by the ring isomorphisms

$$R \cong \text{Hom}_R(R, R) \cong \text{Hom}_S(fR, fR) \quad \text{and} \quad S \cong \text{Hom}_S(S, S) \cong \text{Hom}_R(gS, gS).$$

The identification of f is obtained by the following computation, using the fact that g is an adjoint for f and that P is dualizable; the identification of g is similar. For $M \in \mathcal{M}(R, \mathbb{Z})$,

$$f(M) \cong \text{Hom}_S(S, fM) \cong \text{Hom}_R(gS, M) \cong M \otimes_R \text{Hom}_R(P, R) \cong M \otimes_R Q = M \odot Q. \quad \square$$

Recall that equivalences of abelian categories are, in particular, exact and coproduct-preserving.

Theorem 6.7 (Watts [Wat60]). *Let R and S be rings, and let $f : \mathcal{M}(R, \mathbb{Z}) \rightarrow \mathcal{M}(S, \mathbb{Z})$ be a functor from the category of right R -modules to the category of right S -modules. If f is right-exact and preserves direct sums, then there is a bimodule $C \in \mathcal{M}(S, R)$ and a natural isomorphism $f \cong - \odot C$.*

Proof. Since f preserves direct sums, it is automatically enriched over Hom_R . If T' is an (S, R) -bimodule, we observe that $f({}_{\mathbb{Z}}T')$ has a natural S -module structure given by the map of abelian groups

$$\text{Hom}_S({}_{\mathbb{Z}}S, {}_{\mathbb{Z}}S) \rightarrow \text{Hom}_R({}_{\mathbb{Z}}T', {}_{\mathbb{Z}}T') \rightarrow \text{Hom}_S(f({}_{\mathbb{Z}}T'), f({}_{\mathbb{Z}}T'))$$

and we define $f'(T')$ to be the (S, S) -bimodule whose underlying (\mathbb{Z}, S) bimodule is $f({}_{\mathbb{Z}}T')$. The categories of bimodules $\mathcal{M}(R, R)$ and $\mathcal{M}(S, R)$ are defined to be the subcategories of left R -modules in $\mathcal{M}(R, \mathbb{Z})$ and $\mathcal{M}(S, \mathbb{Z})$, respectively, and hence the compatibility conditions of Corollary 6.3 follow formally. This shows that f and f' are components of a transformation $\mathcal{M}(R, -) \rightarrow \mathcal{M}(S, -)$. The theorem is proven once we show that this is a strong transformation. By Lemma 5.8, it suffices to show that this transformation has a right-adjoint transformation. This also follows formally, by the special adjoint functor theorem: f is coproduct-preserving, and right-exact, and hence has a right-adjoint; f' likewise has a right adjoint, and these form an adjoint transformation. \square

In derived contexts, the condition of Corollary 6.3 is no longer automatic, but we can still apply the result to obtain the following explicit description.

Proposition 6.8. *Let k be a commutative ring, let A be a DG k -algebra and let $f : \mathcal{D}_k(A) \rightarrow \mathcal{D}_k(\text{End}_k(A))$ be an equivalence of triangulated categories. Then f is a standard derived equivalence if and only if we have the following:*

- (i) *The equivalence given by f is an enriched equivalence.*
- (ii) *There is an enriched equivalence $f' : \mathcal{D}_k(A, A) \rightarrow \mathcal{D}_k(\text{End}_k(A), A)$.*
- (iii) *The two equivalences, f and f' are compatible in the following sense: If $T', U' \in \mathcal{D}_k(A, A)$ and $T, U \in \mathcal{D}_k(A) = \mathcal{D}_k(A, k)$, then there are natural maps*

$$\begin{aligned} \text{Ext}_A(T, U') &\rightarrow \text{Ext}_{\text{End}_k(A)}(fT, f'U') && \text{in } \mathcal{D}_k(k, A) \\ \text{Ext}_A(T', U) &\rightarrow \text{Ext}_{\text{End}_k(A)}(f'T', fU) && \text{in } \mathcal{D}_k(A, k) \end{aligned}$$

which commute with the pairing induced by composition. (That is, the squares in Remark 6.4 commute.)

7. MODEL STRUCTURE FOR DG ALGEBRAS

We begin with some definitions and reminders from [KM95], and give a model structure for the category of DG-modules over a DG algebra. We make use of the homotopy extension and lifting property (HELP) to streamline the model-theoretic arguments, and emphasize the analogy with topology. When the DG algebra is concentrated in degree 0 (a ring), this is the standard model structure for chain complexes over the ring (Remark 7.12). Let k be a commutative ring, and A a fixed DG k -algebra. We let $S^n = A \otimes_k (k[n])$, where $k[n]$ is a free DG k -module on a single generator in degree n , so S^n is a free A - A -bimodule on a single generator in degree n . We let D^n be a free A - A -bimodule with one generator in degree n and one in degree $n - 1$; the differential on D^n takes the generator in degree n to that in degree $n - 1$. Finally, we let I denote a free A - A -bimodule which has one generator, $\langle I \rangle$, in degree 1, and two generators, $\langle 0 \rangle$ and $\langle 1 \rangle$ in degree 0; on generators, the differential in I is given by $\langle I \rangle \mapsto \langle 0 \rangle - \langle 1 \rangle$. We let \otimes denote \otimes_A , and for any A -module M we let i_0 and i_1 denote the inclusions $M \rightarrow M \otimes I$ corresponding to $M \otimes \langle 0 \rangle$ and $M \otimes \langle 1 \rangle$, respectively.

Definition 7.1 (Relative cell module). A map of A -modules $C_0 \rightarrow C$ is called a relative cell A -module if C is the colimit of a sequence of maps $C_r \rightarrow C_{r+1}$, with each map obtained as a pushout

$$\begin{array}{ccc} \bigoplus_{q_i} S^{q_i} & \longrightarrow & C_r \\ \downarrow & & \downarrow \\ \bigoplus_{q_i} D^{q_i+1} & \longrightarrow & C_{r+1} \end{array}$$

The maps $S^{q_i} \rightarrow C_r$ above are called the *attaching maps* for C_r . If $0 \rightarrow C$ is a relative cell A -module, C is called a cell A -module. If there are only finitely many cells, then C is called a finite cell A -module.

This is generalized by the following definition.

Definition 7.2 ([MS06, 4.5.1]). Let \mathcal{S} be a set of maps in a category \mathcal{C} with coproducts \oplus .

- (a) A *relative \mathcal{S} -cell module* is a map $C_0 \rightarrow C$, with C obtained as a colimit of maps, $C_r \rightarrow C_{r+1}$, formed by pushouts

$$\begin{array}{ccc} \bigoplus_{q \in \mathcal{S}} X_q & \longrightarrow & C_r \\ \downarrow & & \downarrow \\ \bigoplus_{q \in \mathcal{S}} Y_q & \longrightarrow & C_{r+1} \end{array}$$

where each $X_q \rightarrow Y_q$ is a map in \mathcal{S} .

- (b) The set \mathcal{S} is *compact* if, for every map $X \rightarrow Y$ in \mathcal{S} , the source object, X , is small with respect to countable colimits. That is, for every relative \mathcal{S} -cell module $C_0 \rightarrow C$ as above, the natural map below is an isomorphism.

$$\text{colim Hom}_A(X, C_r) \xrightarrow{\cong} \text{Hom}_A(X, \text{colim } C_r)$$

- (c) An *\mathcal{S} -cofibration* is a map which satisfies the LLP with respect to any map satisfying the RLP with respect to all maps in \mathcal{S} .

Definition 7.3 (Cell submodule). If $M = \text{colim } M_r$ and $L = \text{colim } L_r$ are cell A -modules for which each L_r is a submodule of M_r and, for each attaching map $S^q \rightarrow L_r$, the composite $S^q \rightarrow L_r \subset M_r$ is one of the attaching maps for M_r , then L is called a *cell submodule* of M .

Theorem 7.4 (HELP [KM95, III.2.2]). *Let L be a cell submodule of a cell A -module, M , and let $e : N \rightarrow P$ be a quasi-isomorphism of A -modules. Then, given maps which make the solid arrow diagram below commute, there are dashed lifts which commute with the rest of the diagram.*

$$\begin{array}{ccccc}
L & \xrightarrow{i_0} & L \otimes I & \xleftarrow{i_1} & L \\
\downarrow & & \downarrow h & & \downarrow g \\
P & \xleftarrow{e} & N & & P \\
\downarrow f & & \downarrow e & & \downarrow \\
M & \xrightarrow{i_0} & M \otimes I & \xleftarrow{i_1} & M
\end{array}$$

The following lemma clarifies the relationship between HELP and quasi-isomorphisms. It is obvious from [KM95, III.2.1], although they state and prove only one direction. Theorem 7.4 is proven by using the relative cell structure $L \rightarrow M$ to reduce to this lemma.

Note. For one who compares this lemma with [KM95], it may be helpful to point out that the grading is cohomological there, so they use s and $s - 1$ where we use n and $n + 1$.

Lemma 7.5. *For any integer n , a map $e : N \rightarrow P$ of DG-modules over A satisfies HELP with respect to the inclusion $S^n \rightarrow D^{n+1}$ if and only if $e_* : H_*(N) \rightarrow H_*(P)$ is a monomorphism in degree n and an epimorphism in degree $n + 1$.*

Proof. Having HELP with respect to $S^n \rightarrow D^{n+1}$ means having the dotted lifts in any solid-arrow diagram of the type shown below.

$$\begin{array}{ccccc}
S^n & \xrightarrow{i_0} & S^n \otimes I & \xleftarrow{i_1} & S^n \\
\downarrow & & \downarrow \theta & & \downarrow z \\
P & \xleftarrow{e} & N & & P \\
\downarrow w' & & \downarrow \eta & & \downarrow w \\
D^{n+1} & \xrightarrow{i_0} & D^{n+1} \otimes I & \xleftarrow{i_1} & D^{n+1}
\end{array}$$

In words (using subscripts to denote degrees of elements, and including factors of $(-1)^n$ implicitly where appropriate in our correspondence between letters above and letters below), this says that given any cycle, z_n , in N whose image in P is homologous to a boundary, z'_n :

$$z'_n = dw'_{n+1} \text{ and } ez_n - z'_n = d\theta_{n+1},$$

then z_n is a boundary in N of some w_{n+1} and, moreover, the image of that bounding element in P is homologous to the difference between the bounding element for z'_n and the bounding element for $ez_n - z'_n$:

$$z_n = dw_{n+1} \text{ and } ew_{n+1} - w'_{n+1} + \theta_{n+1} = d\eta_{n+2}.$$

If e has this lifting property, then taking $\theta_{n+1} = 0$ shows that e_* is a monomorphism in degree n , and taking $z_n, z'_n = 0$ (so θ_{n+1} is a cycle in P) shows that e_* is an epimorphism in degree $n + 1$. For the converse, e_n being a monomorphism gives the existence of an element, \tilde{w}_{n+1} whose boundary is z_n , since ez_n is homologous to a boundary in P . The existence of η_{n+2} follows because e_{n+1} is an epimorphism and $e\tilde{w} - w'_{n+1} + \theta_{n+1}$ is a cycle, so there is some cycle \hat{w} in N for which $(e\tilde{w} - w'_{n+1} + \theta_{n+1}) - e\hat{w} = d\eta_{n+2}$. The element w_{n+1} is taken to be the difference $\tilde{w} - \hat{w}$. \square

As HELP indicates, we use the inclusions $S^q \rightarrow D^{q+1}$ and $D^q \rightarrow D^q \otimes I$ to generate the model structure for DG modules over A . This is formalized by any of several standard results for model structures, and we quote one such result here.

Theorem 7.6 ([MS06, 4.5.6]). *Let \mathcal{C} be a bicomplete category with a subcategory of weak equivalences (that is, a subcategory containing all isomorphisms in \mathcal{C} and closed under retracts and the two out of three property). Let \mathcal{S} and \mathcal{J} be compact sets of maps in \mathcal{C} . If the following two conditions hold, then \mathcal{C} is a compactly generated model category, with generating cofibrations \mathcal{S} and generating acyclic fibrations \mathcal{J} :*

- (i) (Acyclicity) *Every relative \mathcal{J} -cell complex is a weak equivalence.*
- (ii) (Compatibility) *A map has the RLP with respect to \mathcal{S} if and only if it is a weak equivalence and has the RLP with respect to \mathcal{J} .*

Note. The term *compactly generated* is a specialization of the notion of cofibrantly generated for the case that \mathcal{S} and \mathcal{J} are compact sets of maps (recall Definition 7.2). It means that the fibrations are defined by RLP with respect to \mathcal{J} , the acyclic fibrations are defined by RLP with respect to \mathcal{S} , the cofibrations are the retracts of relative \mathcal{S} -cell complexes, and the acyclic cofibrations are the retracts of relative \mathcal{J} -cell complexes.

The main advantages of compact generation over cofibrant generation are that it does not require one to use the full version of the small-object argument, but only a small-object argument over countable colimits, and that it is sufficiently general for many topological and algebraic applications, including the one which concerns us here.

7.7 Application. In our application, \mathcal{C} will be the category of (DG) A -modules, and the weak equivalences will be the quasi-isomorphisms. The set $\mathcal{I} = \{S^{n-1} \rightarrow D^n \mid n \in \mathbb{Z}\}$, and the set $\mathcal{J} = \{D^n \xrightarrow{i_0} D^n \otimes I \mid n \in \mathbb{Z}\}$. By definition, the relative \mathcal{I} -cell modules are the relative cell A -modules, and since D^n and $D^n \otimes I$ are cell A -modules, any relative \mathcal{J} -cell module is also a cell A -module. Since S^{n-1} and D^n are finite cell A -modules, the next lemma shows that \mathcal{I} and \mathcal{J} are compact.

Lemma 7.8 (compactness). *If $Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \dots \rightarrow Z$ is a relative cell complex, and if C is a finite cell A -module, then the natural map below is an isomorphism.*

$$\operatorname{colim} \operatorname{Hom}_A(C, Z_r) \xrightarrow{\cong} \operatorname{Hom}_A(C, \operatorname{colim} Z_r)$$

Proof. Assume first that C is a bounded complex of finitely-generated free A -modules, with generators x_1, \dots, x_n . Then an A -module map $f : C \rightarrow \operatorname{colim} Z_r$ is uniquely determined by the elements $f(x_i) \in \operatorname{colim} Z_r$. Since n is finite, there is some s such that $f(x_i) \in \operatorname{im}(Z_s \rightarrow \operatorname{colim}_r Z_r)$ for all i . Hence the lemma holds when C is free; in particular, the lemma holds when $C = S^n$ or $C = D^n$. Now if M is a cell complex for which the lemma holds, and $S^n \rightarrow M$ is any map of A -modules, then the pushout of this map along $S^n \rightarrow D^{n+1}$ is an A -module for which the lemma holds. Since the lemma holds for the A -module 0, it holds for every finite cell A -module. \square

7.9 Acyclicity. We note first that the inclusion $i_0 : A \rightarrow I$ of free A - A -bimodules given by $1 \mapsto \langle 0 \rangle$ has a deformation retraction, r , given by $r \langle I \rangle = 0$ and $r(a \langle 0 \rangle + b \langle 1 \rangle) = a + b$. An explicit homotopy $h : i_0 r \simeq \operatorname{id}$ is easily constructed. For any A -module, M , the inclusion $i_0 : M \rightarrow M \otimes I$ has a deformation retraction induced by r , and therefore also any map $M \rightarrow N$ given by pushout along $D^n \rightarrow D^n \otimes I$ has a deformation retraction.

Since S^n and $S^n \otimes I$ are finite cell A -modules, we apply Lemma 7.8 to see that any relative \mathcal{J} -cell module is a weak equivalence.

7.10 Compatibility. Let $p : X \rightarrow Y$ be a map of A -modules. Assume first that p has the RLP with respect to \mathcal{I} . Then for maps

$$\begin{array}{ccccc} \bigoplus S^{q_i-1} & \longrightarrow & C_r & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ \bigoplus D^{q_i} & \longrightarrow & C_{r+1} & \longrightarrow & Y \end{array}$$

where C_{r+1} is a pushout, we have a lift $\bigoplus D^{q_i} \rightarrow X$ and hence a lift $C_{r+1} \rightarrow X$. Therefore p lifts with respect to any relative (\mathcal{I} -)cell module, and in particular has the RLP with respect to \mathcal{J} . Moreover, p has the RLP with respect to all maps $0 \rightarrow S^n$ and $S^n \otimes I \rightarrow S^n$, and hence p is a weak equivalence.

Now suppose only that p is a weak equivalence and has the RLP with respect to \mathcal{J} . Being a weak equivalence, p satisfies the homotopy extension and lifting property (Theorem 7.4). To see that this implies the result, note first that there is an isomorphism of free A - A -bimodules $I \cong D^1 \oplus S^0$ given by changing basis in degree 0, and hence a projection $I \rightarrow D^1$ which equalizes the two inclusions i_0 and $i_1 : S^0 \rightarrow I$. Moreover, the composite of either inclusion with this projection is the standard inclusion $S^0 \rightarrow D^1$. We tensor with S^n and use the isomorphism $D^{n+1} \cong S^n \otimes D^1$ to define a map $S^n \otimes I \rightarrow S^n \otimes D^1 \cong D^{n+1}$ equalizing the inclusions i_0 and $i_1 : S^n \rightarrow S^n \otimes I$ and such that either composite $S^n \rightarrow S^n \otimes I \rightarrow D^{n+1}$ is the standard inclusion. In other words, the diagram below commutes.

$$\begin{array}{ccccc} S^n & \xrightarrow{i_0} & S^n \otimes I & \xleftarrow{i_1} & S^n \\ & & \swarrow & & \parallel \\ & & D^{n+1} & \longleftarrow & S^n \\ \downarrow & & \parallel & & \\ D^{n+1} & & & & \end{array}$$

Notation 8.1. In contrast with Section 7, here we let S^n , D^m , and I denote the corresponding chain complexes over k . For any chain complex, M , over k , we let ${}_C M_A$ denote the DG (C, A) -bimodule $C \otimes_k M \otimes_k A$. So ${}_C M_A \in DG_k(A, C)$. We denote the generating cofibrations and acyclic cofibrations of $DG_k(A, B)$ by $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$, respectively. We let \otimes denote \otimes_k .

Proposition 8.2 (Push-out Products). *The local model structure on each 1-cell category $DG_k(A, B)$ is compatible with \odot -composition of 1-cells in the following sense: If i and j are generating cofibrations, then their pushout-product is a cofibration, and if one of i or j is a generating acyclic cofibration and the other is a generating cofibration, then their pushout-product is an acyclic cofibration.*

Proof. Suppose first that $i : S^q \rightarrow D^{q+1}$ and $j : S^r \rightarrow D^{r+1}$ are generating cofibrations in $Ch(k)$. If we denote by \mathbb{P} the pushout below, then the pushout product of i and j is the induced map $\mathbb{P} \rightarrow D^{q+1} \otimes D^{r+1}$.

$$\begin{array}{ccc} S^q \otimes S^r & \longrightarrow & S^q \otimes D^{r+1} \\ \downarrow & & \downarrow \\ D^{q+1} \otimes S^r & \longrightarrow & \mathbb{P} \\ & & \searrow \text{---} \\ & & D^{q+1} \otimes D^{r+1} \end{array}$$

This map can be obtained explicitly by attaching a cell of dimension $q + r + 2$ to \mathbb{P} , and hence is a cofibration. Likewise, if j is taken to be a generating acyclic cofibration $D^{r+1} \rightarrow D^{r+1} \otimes I$, the pushout product can be seen explicitly to be a cofibration. Since extension of scalars to an arbitrary DG k -algebra preserves cofibrations, the pushout products over \odot are still cofibrations. When i or j is taken to be a generating acyclic cofibration, we see that the pushout product is still acyclic by recalling the deformation retraction $D^{r+1} \xrightarrow{i_0} D^{r+1} \otimes I \xrightarrow{\cong} D^{r+1}$ of Subsection 7.9. Extending to another DG k -algebra, we still have these deformation retractions, so the pushout products over \odot remain weak equivalences. \square

Proposition 8.3. *If $f : A \rightarrow B$ is a map of DG k -algebras, then the adjoint pair $(f_!, f^*)$ is a Quillen adjoint pair for each C .*

$$DG_k(A, C) \begin{array}{c} \xleftarrow{f_!} \\ \xrightarrow{f^*} \end{array} DG_k(B, C)$$

Proof. This follows from the observation that, for a chain complex $M \in Ch(k)$,

$$f_!({}_C M_A) = {}_C M_A \odot_A B_B \cong {}_C M_B.$$

Hence $f_!$ induces an isomorphism of sets $\mathcal{I}(A, C) \cong \mathcal{I}(B, C)$ and $\mathcal{J}(A, C) \cong \mathcal{J}(B, C)$, where \mathcal{I} and \mathcal{J} are the generating cofibrations and acyclic cofibrations for $DG_k(A, C)$ and $DG_k(B, C)$. \square

Remark 8.4. It seems that a similar statement for (f^*, f_*) is not true unless ${}_A B_A$ is cofibrant as an A -module, since otherwise f^* does not preserve cofibrations in general.

Lemma 8.5 (f^* creates weak equivalences). *If $e : X \rightarrow Y$ is a map of 1-cells in $DG_k(B, C)$ for which $f^*e : f^*X \rightarrow f^*Y$ is a weak equivalence, then the original map $e : X \rightarrow Y$ is a weak equivalence.*

Proof. If $L \rightarrow M$ is a cofibration in $Ch(k)$, we have noted already that the induced map ${}_C L_B \rightarrow {}_C M_B$ is isomorphic to $f_!({}_C L_A \rightarrow {}_C M_A)$. To show that $e : X \rightarrow Y$ is a weak equivalence, it suffices to show that e has HELP with respect to maps of this form (Lemma 7.5). Consider the adjoint lifting diagrams below.

$$\begin{array}{ccc} f_!{}_C L_A & \longrightarrow & f_!{}_C(L \otimes I)_A \longleftarrow f_!{}_C L_A \\ \downarrow & \nearrow & \downarrow \\ & Y & \longleftarrow X \\ & \downarrow & \downarrow \\ f_!{}_C M_A & \longrightarrow & f_!{}_C(M \otimes I)_A \longleftarrow f_!{}_C M_A \end{array} \quad \begin{array}{ccc} {}_C L_A & \longrightarrow & {}_C(L \otimes I)_A \longleftarrow {}_C L_A \\ \downarrow & \nearrow & \downarrow \\ & f^*Y & \longleftarrow f^*X \\ & \downarrow & \downarrow \\ {}_C M_A & \longrightarrow & {}_C(M \otimes I)_A \longleftarrow {}_C M_A \end{array}$$

Lifts exist on the right by hypothesis, and hence on the left by adjunction. \square

Proposition 8.6. *If f above is a weak equivalence, then the Quillen pair $(f_!, f^*)$ is a Quillen equivalence for all C .*

Proof. By [Hov99, Prop. 1.3.13] it suffices to prove that, for cofibrant 1-cells $M \in DG_k(A, C)$ and $N \in DG_k(B, C)$, the composites

$$M \rightarrow f^* f_! M = M \odot_A B_B \odot_B B_A \cong M \odot_A B_A$$

and

$$f_! Q(f^* N) = Q(N \odot_B B_A) \odot_A B_B \rightarrow N \odot_B B_A \odot_A B_B \rightarrow N$$

are weak equivalences. The functor Q is cofibrant replacement; no fibrant replacement is necessary since every object is fibrant.

That the top composite is a weak equivalence follows because M is a cofibrant (C, A) -bimodule and $A \rightarrow {}_A B_A$ is a weak equivalence of (A, A) -bimodules (Proposition 7.14). To see that the bottom composite is a weak equivalence, we consider the composite

$$Q(N \odot_B B_A) \rightarrow Q(N \odot_B B_A) \odot_A B_B \odot_B B_A \rightarrow N \odot_B B_A$$

where the first map is the unit of the adjunction, and the second is f^* applied to the composite above. This total composite is the cofibrant replacement map $Q(N \odot_B B_A) \rightarrow N \odot_B B_A$, and hence is a weak equivalence by definition. The first map in this composite is a weak equivalence because $Q(N \odot_B B_A)$ is cofibrant (as above), and hence by the two-out-of-three property, $f^*(Q(N \odot_B B_A) \odot_A B_B \rightarrow N)$ is a weak equivalence. By Lemma 8.5 then, the map $Q(N \odot_B B_A) \odot_A B_B \rightarrow N$ is a weak equivalence. \square

Corollary 8.7. *For $f : A \xrightarrow{\cong} B$ as above, the dual pair $({}_A B_B, {}_B B_A)$ is invertible when considered as a pair of 1-cells in the derived categories $\mathcal{D}_k(B, A)$ and $\mathcal{D}_k(A, B)$, respectively.*

8.8 Duality in DG_k and \mathcal{D}_k .

Lemma 8.9. *If M is right-dualizable in $DG_k(A, B)$, then M is a retract of a finite free (right-)DG-module over $A \otimes_k B^{op}$.*

Proof. Since M is right-dualizable, the coevaluation map $\nu : M \odot (M \triangleright A) \rightarrow M \triangleright M$ is an isomorphism, and hence there is a map $B \rightarrow M \odot (M \triangleright A)$ lifting the unit $B \rightarrow M \triangleright M$. We let $\Sigma_i(m_i \otimes \varphi_i)$ denote the image of the multiplicative identity, 1_B , under this map, where $m_i \in M$, $\varphi_i \in M \triangleright A = \text{Hom}_A(M, A)$, and the sum has only finitely many terms.

We now show that there is an $A \otimes_k B^{op}$ -module map p , with a section $\tilde{\varphi}$, making M a retract of a finitely-generated free DG-module, where each e_i is a generator of degree $|e_i| = |m_i|$:

$$\bigoplus_i (A \otimes_k B^{op} \cdot e_i) \begin{array}{c} \xleftarrow{\tilde{\varphi}} \\ \xrightarrow{p} \end{array} M.$$

The map p is defined by $p(a \otimes b \cdot e_i) = b \cdot m_i \cdot a$, and the section $\tilde{\varphi}$ is defined by $\tilde{\varphi}(m) = \Sigma_i \varphi_i(m) \otimes 1_B$. It is easy to see that $\tilde{\varphi}$ is a section for p by making use of the fact that $id_M = \nu(\Sigma_i m_i \otimes \varphi_i) = \Sigma_i m_i \cdot \varphi_i$, \square

Lemma 8.10. *Let $M \in \mathcal{D}_k(A, B)$ and suppose M is a retract of a finite cell (B, A) -bimodule. Then $M : A \rightarrow B$ is (right-)dualizable in \mathcal{D}_k and therefore the coevaluation $M \odot (M \triangleright A) \rightarrow M \triangleright M$ is an isomorphism in \mathcal{D}_k .*

Note. Since we are working in the derived bicategory, \mathcal{D}_k , the source-homs, \triangleright , above are understood to be the derived homs. Since M is cofibrant, the derived and underived homs are equal on M .

Proof. One characterization of duality is that the map induced by evaluation $\mathcal{D}_k[W, Z \odot (M \triangleright A)] \rightarrow \mathcal{D}_k[W \odot M, Z]$ be an isomorphism for all 1-cells W, Z with appropriate source and target. From this point of view, the five lemma shows that the full subcategory of dualizable objects in $\mathcal{D}_k(A, B)$ is a thick subcategory (see, for example, [MS06, 16.8.1]). Since the pushouts which build cell modules are examples of exact triangles in \mathcal{D}_K , the result follows by noting that the spheres and disks, S^q and D^q , are dualizable.

The coevaluation map, ν , is defined as the adjoint to $M \odot (M \triangleright A) \odot M \rightarrow M \odot A \cong M$, induced by the evaluation map, so if M is dualizable in the sense described above, then taking $W = M \triangleright M$ and $Z = M$ produces the inverse to coevaluation. \square

Lemma 8.11. *Let $M : A \rightarrow B$ be a 1-cell in \mathcal{D}_k , and suppose the coevaluation $M \odot (M \triangleright A) \rightarrow M \triangleright M$ is an isomorphism. Then M is (quasi-)isomorphic to a retract of a finite cell (B, A) -module.*

Proof. Following the usual argument, we implicitly take a cofibrant replacement for M as a (B, A) -bimodule, $\text{colim } M_r \xrightarrow{\cong} M$. The inverse to coevaluation, composed with the monoid map $B \rightarrow M \triangleright M$ gives $\eta : B \rightarrow M \odot (M \triangleright A)$. Since $-\odot (M \triangleright A)$ preserves colimits, and since B is compact in $\mathcal{D}_k(B, B)$, this map factors through some finite stage of $\text{colim}(M_r \odot (M \triangleright A))$, and we have a lift in the diagram below for some r .

$$\begin{array}{ccc} & M_r \odot (M \triangleright A) \odot M & \longrightarrow M_r \\ & \downarrow & \downarrow \\ M \cong B \odot M & \xrightarrow{\eta \odot \text{id}} M \odot (M \triangleright A) \odot M & \xrightarrow{\text{id} \odot \text{eval}} M \end{array}$$

The bottom composite is the identity, and we see that M is a retract of M_r . \square

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