H_∞ STRUCTURE ON BP

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1. Introduction

(1) introduce BP
(2) "review" H_∞ structure
(3) review the Quillen idempotent $\varepsilon : MU \xrightarrow{r} BP \xrightarrow{s} MU$
(4) s does not induce an H_∞ structure on BP
(5) introduce McClure's formula and related definitions
   (a) For a multi-index $\beta = (\beta_0, \beta_1, \cdots)$,
      The *weight* of β is $wt(\beta) = \sum i\beta_i$.
      The *degree* of β is $|\beta| = \sum \beta_i$.
      The *length* of β is the smallest n such that $\beta_i = 0$ for $i > n$.
   (b) $a_i(x)$ is the coefficient of y^i in the series

$$P(x, y) = \prod_{i=1}^{p-1} ([i]x +_{BP} y)$$

      $a(x)^\beta = a_0(x)^{\beta_0} a_1(x)^{\beta_1} a_2(x)^{\beta_2} \cdots$
   (c) $b = b_0 + b_1 + b_2 + \cdots$; $b_0 = 1$
      $(b^k)_\beta \in \mathbb{Z}$ denotes the coefficient of $b_0^{\beta_0} b_1^{\beta_1} b_2^{\beta_2} \cdots$ in the series b^k.
   (d)

$$MC_n(x) = \sum_{|\beta|=n} (b^{-(n+1)})_\beta \; r_*[\mathbb{C}\mathrm{P}^{n-wt(\beta)}] \; a(x)^\beta$$

(6) $BP^*(B\mathbb{Z}/p) = BP^*(pt.)[[x]]/[p](x)$; $x = r_*(u)$

Proposition 1.1 (McClure 7.8)**.** *The map $r : MU \to BP$ induces an H_∞ structure on BP if and only if $MC_n(x)$ is zero in $BP^*(B\mathbb{Z}/p)$ for each n not of the form $p^k - 1$.*

We calculate the degree of $MC_n(x)$ as follows. Let $q = 2(p-1)$, and write

$$P(x, y) = \prod_{i=1}^{p-1} ([i]x +_{BP} y) = a_0(x) + a_1(x)y + a_2(x)y^2 + \cdots$$

Now x and y each have degree 2, so the whole product has degree q and hence each $a_i(x)$ has degree $q - 2i$. Therefore the degree of $a(x)^\beta$ is $\sum \beta_i(q - 2i) = q|\beta| - 2 \cdot wt(\beta) = qn - 2 \cdot wt(\beta)$. The degree of $[\mathbb{C}\mathrm{P}^{n-wt(\beta)}]$ is $-2(n - wt(\beta))$. Since the coefficients $(b^k)_\beta$ have degree zero, the total degree of $MC_n(x)$ is

$$qn - 2 \cdot wt(\beta) - 2(n - wt(\beta)) = 2n(p-2)$$

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2. BP at the prime 2

Theorem 2.1. *If $p = 2$, then r does not induce an H_∞ structure on BP.*

Proof. At $p = 2$, the degree of $MC_n(x)$ is zero for all n. Since the constant term of the 2-series is zero, we show that $MC_n(x)$ is non-zero in $BP^*(B\mathbb{Z}/p)$ for all n by showing that the constant term of $MC_n(x)$ is non-zero.

Since $p = 2$, we have $P(x, y) = x +_{BP} y$ and therfore the constant term of $a_0(x)$ is zero. Hence to determine the constant term of $MC_n(x)$ we consider only multi-indices, β, for which $\beta_0 = 0$. Now when $\beta_0 = 0$, we have $wt(\beta) \geq |\beta| = n$. However, the term $r_*[\mathbb{C}\mathrm{P}^{n-wt(\beta)}]$ is non-zero only if $wt(\beta) \leq n$, and hence we consider only β for which $|\beta| = wt(\beta) = n$. The only multi-index with this property is $\beta = (0, n, 0, 0 \cdots)$.

The constant term of $a_1(x)$ is 1, so the constant term of $MC_n(x)$ is the coefficient of b_1^n in b^{-n-1}.

Writing $\bar{b} = b_1 + b_2 + \cdots$, we have

$$b^k = \sum_{i \geq 0} \binom{k}{i} \bar{b}^i.$$

Recall that the formula $\binom{-N}{K} = (-1)^k \binom{N+K-1}{K}$ extends the binomial coefficients, and the formula above, to the case $k < 0$, and so we find the coefficient of b_1^n in b^{-n-1} is $\binom{-n-1}{n} = (-1)^n \binom{2n}{n}$. □

Remark 2.1. The n^{th} Catalan number is $\frac{1}{n+1}\binom{2n}{n}$.

3. Other primes