INVERTIBILITY IN BICATEGORIES

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ABSTRACT. We aim to develop a calculational foothold on Picard and Brauer groups in generalized contexts by investigating invertibility. Aside from introduction of basic definitions and foundations, the essential point is the characterization of generalized Azumaya objects. We give tilting theory as an application of this characterization.
Note: this is a preprint draft

1. INTRODUCTION

We study invertibility in autonomous monoidal categories. Our interest is in generalized Picard and Brauer groups, and the main theorem generalizes the classical characterization of Azumaya algebras over a commutative ring.

Definition 1.1 (Picard Group)**.** Let A be a 0-cell of a bicategory $\mathscr{B}$. The *Picard group* of A, denoted $Pic(A)$, is the group of isomorphism classes of invertible 1-cells $A \nrightarrow A$.

Definition 1.2 (Brauer Group)**.** Let $\mathscr{B}$ be a symmetric monoidal bicategory with unit 0-cell R. The *Brauer group* of R, denoted $Br(R)$, is the group of 1-cell-equivalence-classes (Morita equivalence classes) of 0-cells A for which there exists a 0-cell B such that $A \otimes_R B$ is equivalent to R.

Let A be a 0-cell of an autonomous monoidal bicategory $\mathscr{D}$ with unit R. Let A^e denote the 0-cell $A \otimes A^{op}$, and let A_r be the 1-cell $A^e \nrightarrow R$ induced by the unit 1-cell for A. Throughout, we use the notation $\mathrm{Hom}_{\underline{s}}$ for Hom over the source 0-cell, and $\mathrm{Hom}_{\underline{t}}$ for Hom over the target 0-cell. For example, $\mathrm{Hom}_{\underline{s}}(A_r, A_r) = \mathrm{Hom}_{A^e}(A_r, A_r)$ and $\mathrm{Hom}_{\underline{t}}(A_r, A_r) = \mathrm{Hom}_R(A_r, A_r)$.

Our results are summarized below.

Theorem. *The following statements are equivalent:*

- *i.* *A_r is an invertible 1-cell.*
- *ii.* *a)* *A_r is right-dualizable and*
  - *b)* *the evaluation $\mathrm{Hom}_{\underline{s}}(A_r, A^e) \otimes_R A_r \to A^e$ is an isomorphism and*
  - *c)* *the action of R induces $R \cong \mathrm{Hom}_{\underline{s}}(A_r, A_r)$.*
- *iii.* *a)* *A_r is left-dualizable and*
  - *b)* *the evaluation $A_r \otimes_{A^e} \mathrm{Hom}_{\underline{t}}(A_r, R) \to R$ is an isomorphism and*
  - *c)* *the action of A^e induces $A^e \cong \mathrm{Hom}_{\underline{t}}(A_r, A_r)$.*
- *iv.* *There exists a 0-cell B such that $A \otimes_R B$ is Morita equivalent to R.*

If $\mathscr{D}$ is a triangulated bicategory, then these are equivalent to the following:

- *v.* *a)* *A_r is right-dualizable and*
  - *b)* *the action of R induces $R \cong \mathrm{Hom}_{\underline{s}}(A_r, A_r)$ and*
  - *c)* *A^e is left-A_r-local.*
- *vi.* *a)* *A_r is left-dualizable and*
  - *b)* *the action of A^e induces $A^e \cong \mathrm{Hom}_{\underline{t}}(A_r, A_r)$ and*
  - *c)* *R is right-A_r-local.*

Definition 1.3. If A satisfies any of the equivalent conditions above, A is an *Azumaya object* of $\mathscr{D}$.

For motivation, we describe the classical case, where R is a commutative ring and $\mathscr{D}$ is the bicategory of R-algebras and their bimodules. There, we restrict attention to the first four parts of the theorem, and

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Date: March 11, 2009.

the definition of Azumaya objects becomes the usual definition of Azumaya algebras over R. We have the following further translations in this case:

- A 1-cell $X : C \nrightarrow D$ is right-dualizable if and only if it is finitely-generated and projective over its source, C. It is left-dualizable if and only if it is finitely-generated and projective over its target, D.
- A is *separable* over R if and only if *ii.a* holds.
- The center of A is equal to R if and only if *ii.c* holds.
- A is *faithfully projective* over R if and only if *iii.b* and *iii.a* hold.

Note, moreover, that in the classical case *ii.a* and *ii.c* together imply *ii.b*. We do not expect such a result to hold in general, but we do not have a counterexample.

Question 1.4. Is there a symmetric monoidal bicategory with 0-cell, A, for which *ii.a* and *ii.c* hold, but *ii.b* does not? Note that, in such a case, A_r would be right-dualizable, but not left-dualizable; In algebraic settings, this would mean that A_r is finitely-generated and projective over A^e, but not so over R.

The last two parts of the theorem become relevant if we have a triangulated structure, such as when R is a ring spectrum $\mathscr{D}$ is the homotopy category of bimodules over R-algebras. Note that in topological contexts $\mathrm{Hom}_{\underline{s}}(A_r, A_r)$ is the topological Hochschild cohomology spectrum, $THH_R(A, A)$. In [BL04], Baker and Lazarev prove the equivalence of the last two conditions in the theorem–clearly understanding them as a good definition of Azumaya spectra. The formal connection to classical definitions, however, is new.

As corollaries, we have an application to tilting theory. A bicategorical perspective on the following two results is developed in [Joh08]. Our work with invertibility here allows us to give streamlined proofs.

Corollary 1.5 (Rickard)**.** *Let S be a DG R-algebra, and let T be a DG S-module. If T has the following two properties, then $\mathscr{D}_R(S)$ and $\mathscr{D}_R(End_S(T))$ are equivalent as triangulated categories.*

i. T is a right-dualizable S-module.
ii. T generates the triangulated category $\mathscr{D}_R(S)$.

Note. The object T above (or below) is called a *tilting complex* (or *spectrum*).

Corollary 1.6. *Let R be a commutative ring spectrum, and let $\mathscr{D}_R$ denote the bicategory of R-algebras and homotopy categories of bimodules. Suppose A is an R-algebra, and let T be a fibrant and cofibrant A-module, with endomorphism R-algebra $E = F_A(T, T)$. If T has the following two properties, then $\mathscr{D}_R(A)$ and $\mathscr{D}_R(E)$ are equivalent categories.*

i. T is (right-)dualizable as an A-module.
ii. T generates the triangulated category $\mathscr{D}_R(A)$.

2. Bicategories

We assume the reader is familiar with basic definitions for bicategories, however we do not assume the reader is an expert. We begin by reviewing some important examples and relevant additional structures. A precise and concise introduction can be found in [Lei98], while [Lac07] provides a more expanded guide. A thorough introduction developed for similar purposes can be found in [Joh08, §3 and §5]

We use arrows such as $f : M \to M'$ to denote that f is a 2-cell with source M and target M', and slashed arrows such as $M : A \nrightarrow B$ to denote that M is a 1-cell with source A and target B. We use $\circ$ or juxtaposition to denote vertical composition of 2-cells, and $\odot$ to denote horizontal composition of 1-cells and of 2-cells.

2.1. **Examples.** Let R be a commutative ring. The collection of R-algebras is the collection of 0-cells for the bicategory $\mathscr{M}_R$. The 1-cells of $\mathscr{M}_R(A, B)$ are the B-A-bimodules, and the 2-cells are bimodule homomorphisms. The horizontal composition, $\odot$, is given by the tensor product: If $M \in \mathscr{M}_R(B, C)$ and $N \in \mathscr{M}_R(A, B)$, then

$$M \odot N = M \otimes_B N \in \mathscr{M}_R(A, C).$$

If, more generally, R is a commutative DG algebra, we have the bicategory DG_R, defined similarly: the 0-cells are DG R-algebras, 1-cells are DG bimodules, and 2-cells are maps of bimodules. We also have the derived bicategory $\mathscr{D}_R$, with the same 0-cells as DG_R, but the category of 1- and 2-cells between two 0-cells is the derived category of bimodules.

Still more generally, if R denotes a commutative ring spectrum we likewise have a bicategory of R-algebra spectra, bimodule spectra, and bimodule morphisms. This bicategory is denoted $\mathscr{S}_R$. Like DG_R, $\mathscr{S}_R$ has a derived bicategory, and we extend the notation $\mathscr{D}_R$ to include the case that R is a spectrum also. Note that if R is a commutative DG algebra and HR its Eilenberg-Mac Lane spectrum, then we have $\mathscr{D}_R(A, B) \simeq \mathscr{D}_{HR}(HA, HB)$ for all DG algebras A and B.

Any cocomplete symmetric monoidal category with unit R gives rise to a bicategory whose 0-cells are monoids and 1-cells are bimodules. Colimits are required to construct the $\odot$, just as they are in the cases of tensor and smash above.

2.2. Monoidal bicategories.

A monoidal bicategory can be defined as a tricategory with one object. In practical terms, this means that the bicategory is equipped with an additional monoidal product on 0-, 1, and 2-cells, satisfying reasonable associativity and unit constraints. In $\mathscr{M}_R$, the monoidal product is $\otimes_R$; in $\mathscr{S}_R$, it is $\wedge_R$. In these examples, the monoidal product is symmetric, and hence these are symmetric monoidal bicategories. More generally, if $\mathscr{C}_R$ is a cocomplete monoidal category with unit R, and $\mathscr{B}_R$ the bicategory of monoids and bimodules in $\mathscr{C}_R$, then $\mathscr{B}_R$ is a symmetric monoidal bicategory with monoidal product induced by that of $\mathscr{C}_R$. The following lemma gives one consequence of the symmetry for composition.

Lemma 2.3. *Let $M : C \nrightarrow R$ and $N : R \nrightarrow D$ be 1-cells of a symmetric monoidal bicategory $\mathscr{B}$ with unit R. The symmetry gives*

$$N \odot M \cong (M \otimes_R D) \odot (C \otimes_R N)$$

in $\mathscr{B}(C, D)$.

2.4. Autonomous structure.

[[describe the autonomous structure necessary for Proposition 3.10]]

2.5. Closed structure.

A *closed structure* for a bicategory, $\mathscr{B}$, defines right adjoints for $\odot$. For a 1-cell M, the right adjoint to $- \odot M$ is called "right-hom", or "source-hom", and denoted $\operatorname{Hom}_{\underline{s}}(M, -)$. The adjoint to $M \odot -$ is called "left-hom", or "target-hom", and denoted $\operatorname{Hom}_{\underline{t}}(M, -)$. The adjunctions are written as

$$\begin{aligned} \mathscr{B}(V \odot M, W) &\cong \mathscr{B}(V, \operatorname{Hom}_{\underline{s}}(M, W)) \\ \mathscr{B}(M \odot T, U) &\cong \mathscr{B}(T, \operatorname{Hom}_{\underline{t}}(M, U)) \end{aligned}$$

The existence of left and right hom functors defines a *closed bicategory.* Formal definitions and a complete description of closed structures can be found in [MS06].

Note that our examples above have internal hom functors, and this defines a closed structure. In general, if $\mathscr{C}_R$ is a bicomplete closed symmetric monoidal category, then the bicategory $\mathscr{B}_R$ built from $\mathscr{C}_R$ is a closed symmetric monoidal bicategory.

3. DUALITY AND INVERTIBILITY IN BICATEGORIES

For general discussion about duality, we consider fixed 1-cells $X : B \nrightarrow A$ and $Y : A \nrightarrow B$ in a closed bicategory $\mathscr{B}$.

Definition 3.1 (Dual pair)**.** We say (X, Y) is a dual pair, or 'X is left-dual to Y' ('Y is right-dual to X'), or 'X is right-dualizable' ('Y is left-dualizable') to mean that we have 2-cells

$$\eta : A \to X \odot Y \quad \text{and} \quad \varepsilon : Y \odot X \to B$$

such that the following composites are the respective identity 2-cells.

$$X \cong A \odot X \xrightarrow{\eta \odot \mathrm{id}} X \odot Y \odot X \xrightarrow{\mathrm{id} \odot \varepsilon} X \odot B \cong X$$

$$Y \cong Y \odot A \xrightarrow{\mathrm{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \mathrm{id}} B \odot Y \cong Y$$

Definition 3.2 (Base and cobase for a dual pair)**.** When (X, Y) is a dual pair in a bicategory $\mathscr{B}$, we term the source of X (the target of Y) the *base* of the dual pair, and we term the source of Y (the target of X) the *cobase* of the dual pair. Thus, the evaluation map of the dual pair is a two-cell from $Y \odot X$ to the base 1-cell, and the coevaluation (unit) is a two-cell from the cobase 1-cell to $X \odot Y$.

Definition 3.3 (Invertible pair)**.** A dual pair (X, Y) is called invertible if the maps η and ε are isomorphisms. Equivalently, the adjoint pairs described above are adjoint equivalences.

Duality for monoidal categories has been studied at length, and duality in a bicategorical context has been introduced in [MS06, §16.4]. The definition of duality does not require $\mathscr{B}$ to be closed, but we will make use of the following basic facts about duality, some of which do require a closed structure on $\mathscr{B}$.

Proposition 3.4. *A 1-cell $X \in \mathscr{B}(A, B)$ is right-dualizable if and only if the coevaluation*

$$\nu : X \odot (\operatorname{Hom}_{\underline{s}}(X, A)) \to \operatorname{Hom}_{\underline{s}}(X, X)$$

is an isomorphism. Moreover, this is the case if and only if the map

$$\nu_Z : X \odot (\operatorname{Hom}_{\underline{s}}(X, Z)) \to \operatorname{Hom}_{\underline{s}}(X, X \odot Z)$$

is an isomorphism for all 1-cells Z with target A.

Proposition 3.5. *Let (X, Y) be a dual pair in $\mathscr{B}$, with $X : B \nrightarrow A$ and $Y : A \nrightarrow B$.*

i. For any 0-cell C, we have two adjoint pairs of functors, with left adjoints written on top:

$$\begin{tikzcd} \mathscr{B}(A, C) \arrow[r, shift left, "- \odot X"] & \mathscr{B}(B, C) \arrow[l, shift left, "- \odot Y"] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{B}(C, A) \arrow[r, shift left, "Y \odot -"] & \mathscr{B}(C, B) \arrow[l, shift left, "X \odot -"] \end{tikzcd}$$

The structure maps for the dual pair give the triangle identities necessary to show that the displayed functors are adjoint pairs.

ii. If $\mathscr{B}$ is closed, then Y is canonically isomorphic to $\operatorname{Hom}_{\underline{s}}(X, B)$, and for any 1-cell $Z : B \nrightarrow D$, the natural map $Z \odot \operatorname{Hom}_{\underline{s}}(X, B) \to \operatorname{Hom}_{\underline{s}}(X, Z)$ is an isomorphism.

iii. Moreover, in the case that $\mathscr{B}$ is closed, X is canonically isomorphic to $\operatorname{Hom}_{\underline{t}}(Y, A)$, and for any 1-cell $W : D' \nrightarrow A$, the natural map $\operatorname{Hom}_{\underline{t}}(Y, A) \odot W \to \operatorname{Hom}_{\underline{t}}(Y, W)$ is an isomorphism.

Lemma 3.6. *Let $X : A \nrightarrow B$ be a 1-cell in $\mathscr{B}(A, B)$. If X is right-dualizable and the unit $B \to X \triangleright X$ is an isomorphism, then the evaluation $X \odot (\operatorname{Hom}_{\underline{t}}(X, B)) \to B$ is an isomorphism. Likewise, if X is left-dualizable and the unit $A \to \operatorname{Hom}_{\underline{t}}(X, X)$ is an isomorphism, then the evaluation $(\operatorname{Hom}_{\underline{s}}(X, A)) \odot X \to A$ is an isomorphism.*

Proof. We prove the first statement, leaving the second as an exercise in opposites. Let Y denote the canonical right dual of X. Since X is right-dualizable, Y is left-dualizable and X is isomorphic to the canonical left dual of Y: $X \cong \operatorname{Hom}_{\underline{t}}(Y, A)$. The isomorphism $B \xrightarrow{\cong} \operatorname{Hom}_{\underline{s}}(X, X)$ implies that the unit for the duality is an isomorphism: $B \xrightarrow{\cong} X \odot Y$. Now we have the following commutative square:

$$\begin{tikzcd} X \odot (\operatorname{Hom}_{\underline{t}}(B, X)) \arrow[rr, "\text{evaluation}"] \arrow[d, "\cong"'] & & B \arrow[d, "\cong"] \\ (\operatorname{Hom}_{\underline{t}}(Y, A)) \odot (\operatorname{Hom}_{\underline{t}}(B, X)) \arrow[r, "\cong"] & \operatorname{Hom}_{\underline{t}}(Y, (\operatorname{Hom}_{\underline{t}}(X, B))) \arrow[r, "\cong"] & \operatorname{Hom}_{\underline{t}}((X \odot Y), B) \end{tikzcd}$$

where the two vertical isomorphisms are described above, the left-hand isomorphism is a consequence of dualizability for Y, and the right-hand isomorphism is an exercise in adjunction. ☐

3.7. **Examples.** The right-dualizable 1-cells in the bicategory $\mathscr{M}_R$ are the finitely-generated projective bimodules. More precisely, they are finitely-generated projective as right-modules over their source (the base of the duality). In $\mathscr{D}_R$, the dualizable objects are the retracts of finite-cell bimodules.

3.8. **Azumaya objects and the Brauer group.** For the following, we let A be a fixed 0-cell of $\mathscr{D}$. We denote the enveloping 0-cell, $A \otimes_R A^{op}$ by A^e. Let $A_r \in \mathscr{D}(A^e, R)$ denote A regarded as a 1-cell $A^e \nrightarrow R$.

restate this? **Proposition 3.9.** *For A_r as above, the following are equivalent.*

- *i.* *A_r is an invertible 1-cell.*
- *ii.*
  - *a) The coevaluation $A_r \otimes_{A^e} (\mathrm{Hom}_{\underline{s}}(A_r, A^e)) \to \mathrm{Hom}_{\underline{s}}(A_r, A_r)$is an isomorphism.*
  - *b) The evaluation $(\mathrm{Hom}_{\underline{s}}(A_r, A^e)) \otimes_R A_r \to A^e$ is an isomorphism.*
  - *c) The unit map $R \to \mathrm{Hom}_{\underline{s}}(A_r, A_r)$ is an isomorphism.*
- *iii.*
  - *a) The coevaluation $(\mathrm{Hom}_{\underline{t}}(A_r, R)) \otimes_R A_r \to \mathrm{Hom}_{\underline{t}}(A_r, A_r)$is an isomorphism.*
  - *b) The evaluation $A_r \otimes_{A^e} \mathrm{Hom}_{\underline{t}}(A_r, R) \to R$ is an isomorphism.*
  - *c) The unit map $A^e \to \mathrm{Hom}_{\underline{t}}(A_r, A_r)$ is an isomorphism.*

Proof. In general, a pair of 1-cells (X, Y) is invertible if and only if X is right-dualizable, Y is isomorphic to the canonical right dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *ii*. Likewise, (Y, X) is invertible if and only if X is left-dualizable, Y is isomorphic to the canonical left dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *iii*. Since (X, Y) is an invertible pair if and only if (Y, X) is such, the proof is complete. □

Proposition 3.10. *Let A be a 0-cell of a symmetric monoidal bicategory $\mathscr{D}$. Then A is Azumaya in $\mathscr{D}$ if and only if there is a 0-cell, B such that $A \otimes_R B$ is Morita equivalent to R.*

Proof. If A is Azumaya, then taking $B = A^{(op)}$ gives one direction. For the other direction, suppose that $P : R \nrightarrow A \otimes_R B$ is an invertible 1-cell between R and $A \otimes_R B$. Let $P^* = \mathrm{Hom}_{\underline{s}}(P, R)$, and let $P^e = P \otimes_R (P^*)^{op}$, $P^{*e} = P^* \otimes P^{op}$. Then (P^e, P^{*e}) is an invertible pair of 1-cells between the 0-cells R^e and $(A \otimes_R B)^e \cong A^e \otimes_R B^e$. Moreover, since (P, P^*) is an invertible pair, $P^e = P \otimes_R (P^*)^{op} \cong (A \otimes_R B)_r \cong A_r \otimes_R B_r$.

Let $Q = P^{*e}$ and let $T = (A^e \otimes_R B_r) \odot Q$. We have shown that $(A_r \otimes_R B_r, Q)$ is an invertible pair of 1-cells, and we will now show that (A_r, T) is an invertible pair between A_e and R. A straightforward use of associativity shows that $A_r \odot T \cong R$. We use the symmetry lemma now for the reverse composite:

$$\begin{aligned} T \odot A_r = (A^e \otimes_R B_r) \odot Q \odot A_r &\cong (A^e \otimes_R B_r) \odot (A_r \otimes_R A^e \otimes_R B^e) \odot (A^e \otimes_R Q) \\ &\cong (A^e \otimes_R B_r) \odot (A^e \otimes_R A_r \otimes_R B^e) \odot (A^e \otimes_R Q) \\ &\cong (A^e \otimes_R A_r \otimes_R B_r) \odot (A^e \otimes_R Q) \\ &\cong (A^e \otimes_R R) \cong A^e \end{aligned}$$

□

4. TRIANGULATED BICATEGORIES

We recall first the definitions of localizing subcategory and generator for a triangulated category, and then give a definition (4.4) of triangulated bicategory suitable for our purposes. In particular, under this definition $\mathscr{D}_R$ is a triangulated bicategory.

Definition 4.1 (Localizing subcategory)**.**
If $\mathscr{T}$ is a triangulated category with infinite coproducts, a *localizing* subcategory, $\mathscr{S}$, is a full triangulated subcategory of $\mathscr{T}$ which is closed under coproducts from $\mathscr{T}$.

Remark 4.2. This is equivalent to the definition for arbitrary triangulated categories of [Hov99], (which requires that a localizing subcategory be thick) because a triangulated subcategory automatically satisfies the 2-out-of-3 property and because in any triangulated category with countable coproducts, idempotents have splittings. See [Nee01, 1.5.2, 1.6.8, and 3.2.7] for details.

Definition 4.3 (Triangulated generator)**.**
A set, $\mathscr{P}$, of objects in $\mathscr{T}$ (triangulated category with infinite coproducts, as above) is a set of *triangulated generators* (or simply *generators*) if the only localizing subcategory containing $\mathscr{P}$ is $\mathscr{T}$ itself.

Definition 4.4 (Triangulated bicategory [MS06, §16.7])**.**
A closed bicategory $\mathscr{B}$ will be called a *triangulated bicategory* if for each pair of 0-cells, A and B, $\mathscr{B}(A,B)$ is a triangulated category with infinite coproducts, and if the suspension, Σ, is a pseudofunctor on $\mathscr{B}$, and furthermore the local triangulations on $\mathscr{B}$ are compatible as described in the following two axioms.

(TC1) For a 1-cell $X : A \nrightarrow B$, there is a natural isomorphism

$$\alpha : X \odot \Sigma A \to \Sigma X$$

such that the composite below is multiplication by -1.

$$\Sigma^2 A = \Sigma(\Sigma A) \xrightarrow{\alpha^{-1}} \Sigma A \odot \Sigma A \xrightarrow{\gamma} \Sigma A \odot \Sigma A \xrightarrow{\alpha} \Sigma(\Sigma A) = \Sigma^2 A$$

(TC2) For any 1-cell, W, the functors $W \odot -$, $- \odot W$, $W \rhd -$, and $- \rhd W$ are exact.

If $\mathscr{B}$ is a triangulated bicategory and P, Q are 1-cells in $\mathscr{B}(A,B)$, we emphasize that $\mathscr{B}$ is triangulated by writing the abelian group of 2-cells $P \to Q$ as $\mathscr{B}[P,Q]$ and by writing the graded abelian group obtained by taking shifts of Q as $\mathscr{B}[P,Q]_*$. To emphasize the source and target of P and Q, we may also write $\mathscr{B}(A,B)[P,Q]_*$.

Definition 4.5 ($\odot$-faithful 1-cells)**.**
In any locally additive bicategory, $\mathscr{B}$, a 1-cell $W : A \nrightarrow B$ is called *left-faithful* if triviality for any 1-cell $Z : C \nrightarrow A$ is detected by triviality of the composite $W \odot Z$. That is, $Z : C \nrightarrow A$ is zero if and only if $W \odot Z = 0$. A collection of 1-cells, $\mathscr{E}$, in $\mathscr{B}(A,B)$ is called *jointly left-faithful* if the objects have this property jointly; that is, $Z = 0$ if and only if $W \odot Z = 0$ for all $W \in \mathscr{E}$. The term *left-faithful* is defined similarly, considering $- \odot W$ instead of $W \odot -$.

Remark 4.6. If $\mathscr{B}$ is a monoidal additive category with monoidal product $\odot$, the unit object is both left- and right-faithful. In arbitrary locally additive bicategories, if $A \neq B$ then $\mathscr{B}(A,B)$ may not have a single object with this property, but in relevant examples the collection of all 1-cells, $\mathrm{ob}\mathscr{B}(A,B)$, does have this property jointly. As a counter-point to this remark, we have the following lemma.

Lemma 4.7. *Let $\mathscr{B}$ be a triangulated bicategory, and let $P : A \nrightarrow B$ be a generator for $\mathscr{B}(A,B)$. If the collection of all 1-cells, $\mathscr{B}(A,B)$, is jointly left-faithful (resp. right-faithful), then P is left-faithful (resp. right-faithful).*

Proof. Consider the left-faithful case; the right-faithful case is similar. Given any 1-cell $Z : C \nrightarrow A$ with $P \odot Z = 0$, let $\mathscr{S}$ be the full subcategory of 1-cells, $W : A \nrightarrow B$ for which $W \odot Z = 0$. This is a localizing subcategory of $\mathscr{B}(A,B)$, and by assumption $P \in \mathscr{S}$, so $\mathscr{S} = \mathscr{B}(A,B)$, and hence $Z = 0$. ☐

Remark 4.8. Since the functors $P \odot -$ are exact, the property of $P \odot -$ detecting trivial objects is equivalent to $P \odot -$ detecting isomorphisms (meaning that a 2-cell f is an isomorphism if and only if $P \odot f$ is so).

4.9. **Tilting theory.** For this subsection, we let $\mathscr{D}$ denote a closed, symmetric monoidal bicategory with unit 0-cell R. We do not require that $\mathscr{D}$ be triangulated, but simply that $\mathscr{D}$ have a 0-object in each 1-cell category.

Definition 4.10. Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}(A,B)$.

A 1-cell $M : C \nrightarrow A$ is *left-T-acyclic* if $T \odot M = 0$. A 1-cell $N : C \nrightarrow A$ is *left-T-local* if $\mathscr{D}(C,A)[M,N]_* = 0$ for all T-acyclic 1-cells $M \in \mathscr{D}(C,A)$. The full subcategory of left-T-local 1-cells in $\mathscr{D}(C,A)$ is denoted $\mathscr{D}(C,A)_{\langle T_{\odot} \rangle}$. (The notation $T_{\odot}$ is intended to remind the reader of push-forward via $\odot$-composition.)

A 1-cell $M' : B \nrightarrow C$ is *right-T-acyclic* if $M \odot T = 0$. A 1-cell $N' : B \nrightarrow C$ is *right-T-local* if $\mathscr{D}(B,C)[M',N']_* = 0$ for all right-T-acyclic 1-cells $M' \in \mathscr{D}(B,C)$. The full subcategory of right-T-local 1-cells in $\mathscr{D}(B,C)$ is denoted $\mathscr{D}(B,C)_{\langle T^{\odot} \rangle}$.(The notation $T^{\odot}$ is intended to remind the reader of pull-back via $\odot$-composition.)

Baker and Lazarev describe the following in the context of spectra, but their methods generalize to our setting. The key observation is that for any 1-cell P whose source is A, $\mathrm{Hom}_{\underline{s}}(T,P)$ is right-T-local. Likewise, if P' is any 1-cell whose target is B, $\mathrm{Hom}_{\underline{t}}(T,P')$ is left-T-local.

Proposition 4.11 (Baker-Lazarev factorization [BL04])**.** *Let $T : A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ be a 1-cell in $\mathscr{D}(A,B)$. The adjunctions induced by T factor through the T-local pseudofunctors; we have the following diagrams of adjoint transformations:*

$$\begin{tikzcd}
\mathscr{D}(B,-) \arrow[rr, shift left, "-\odot T"] \arrow[dr, shift left] & & \mathscr{D}(A,-) \arrow[ll, shift left, "\mathrm{Hom}_{\underline{s}}(T,-)"] \arrow[dl, shift left, "\mathrm{Hom}_{\underline{s}}(T,-)"] \\
& \mathscr{D}(B,-)_{\langle T^{\odot}\rangle} \arrow[ul, shift left] \arrow[ur, shift left, "-\odot T"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{D}(-,A) \arrow[rr, shift left, "T\odot -"] \arrow[dr, shift left] & & \mathscr{D}(-,B) \arrow[ll, shift left, "\mathrm{Hom}_{\underline{t}}(T,-)"] \arrow[dl, shift left, "\mathrm{Hom}_{\underline{t}}(T,-)"] \\
& \mathscr{D}(-,A)_{\langle T_{\odot}\rangle} \arrow[ul, shift left] \arrow[ur, shift left, "T\odot -"] &
\end{tikzcd}$$

Proposition 4.12 ([BL04])**.** *If a 1-cell $T \in \mathscr{D}(A,E)$ is right-dualizable and the unit map induces an isomorphism $E \cong \mathrm{Hom}_{\underline{s}}(T,T) = {}_E[\mathrm{Hom}_A({}_ET_A, {}_ET_A)]_E$, then the induced adjoint pair is an equivalence $\mathscr{D}(-,A)_{\langle T_{\odot}\rangle} \simeq \mathscr{D}(-,E)$. We have a corresponding statement for the case of left-dualizability.*

Proof. Let T^* denote the right-dual to T. Since T is right-dualizable, T^* is left-dualizable and the evaluation map $T \odot (\mathrm{Hom}_{\underline{t}}(T,E)) \to E$ is an isomorphism (Lemma 3.6). Moreover, $\mathrm{Hom}_{\underline{t}}(T,-)$ takes values in the T-local category and hence the fact that the unit of the adjunction is an isomorphism follows from the fact that the evaluation is so. ☐

Corollary 4.13. *If T satisfies the hypotheses of 4.12 and if in addition T is left-faithful (Definition 4.5), then all three of the adjoint pairs above are equivalences.*

Proof. If T is left-faithful, then all left-T-acyclics are trivial, and hence $\mathscr{D}(-,A) \simeq \mathscr{D}(-,A)_{\langle T_{\odot}\rangle}$. The result then follows from 4.12. ☐

Proposition 4.14. *Let $T : A \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} B$ be a 1-cell in $\mathscr{D}$. The following are equivalent:*

i. *T is invertible.*

ii. *a) T is right-dualizable.*
  b) The unit induces $B \cong \mathrm{Hom}_{\underline{s}}(T,T)$.
  c) A is left-T-local.

iii. *a) T is left-dualizable.*
  b) The unit induces $A \cong \mathrm{Hom}_{\underline{t}}(T,T)$.
  c) B is right-T-local.

Corollary 4.15 ([BL04, 2.1,2.3])**.** *The following are equivalent:*

i. *A is Azumaya in $\mathscr{D}$.*

ii. *a) A_r is right-dualizable.*
  b) The unit induces $R \cong \mathrm{Hom}_{\underline{s}}(A_r, A_r)$.
  c) A^e is left-A_r-local.

iii. *a) A_r is left-dualizable.*
  b) The unit induces $A^e \cong \mathrm{Hom}_{\underline{t}}(A_r, A_r)$.
  c) R is right-A_r-local.

Proof of 1.5. Let $\widetilde{T}$ denote T regarded as a bimodule over $E = {}_R[\mathrm{Hom}_S({}_RT_S, {}_RT_S)]_R$. Since T is (right-) dualizable, $\widetilde{T}$ is (right-) dualizable in $\mathscr{D}_R(S,E)$.

Since R is the unit of the symmetric monoidal bicategory $\mathscr{D}_R$, the 1-cells of $\mathscr{D}_R(S,R)$ are jointly left-faithful (Definition 4.5). Hence Lemma 4.7 shows that T is left-faithful. This means that $\widetilde{T}$ is also left-faithful, and thus the evaluation $(\widetilde{T} \rhd S) \odot \widetilde{T} \to S$ is an isomorphism in $\mathscr{D}_R(S,S)$: The composite below is the identity and the first map, induced by the unit of the adjunction, is an isomorphism so the second must be also.

$$\widetilde{T} \xrightarrow{\cong} \widetilde{T} \odot (\widetilde{T} \rhd S) \odot \widetilde{T} \xrightarrow{1\odot\mathrm{eval}} \widetilde{T}$$

☐

Notation 4.16. Given a map of R-algebras $\iota : B \to E$, we have two restriction-of-scalars functors: one for restriction of left modules, and another for restriction of right modules. For any R-algebra A, We let $\iota_\ell^* : \mathscr{S}_R(A,E) \to \mathscr{S}_R(A,B)$ denote restriction on the left (target), and $\iota_r^* : \mathscr{S}_R(E,A) \to \mathscr{S}_R(B,A)$ denote restriction on the right (source). Both functors create weak-equivalences and fibrations.

Proof of 1.6. Let $E = \mathrm{Hom}_{\underline{\mathsf{s}}}(T,T) = F_A(T,T)$. The unit map $R \to E$ is obtained as the composite of algebra maps $R \to B \to E$. Let $\widetilde{T}$ be a cofibrant replacement for T in $\mathscr{S}_R(A,E)$. Recall that T is cofibrant in $\mathscr{S}_R(A,B)$, and hence has the LLP with respect to acyclic fibrations. We construct $\widetilde{T}$ by the usual factorization of the map from the initial object, and the forgetful functor ι_ℓ^* creates weak equivalences and fibrations, so the lifting property for T gives a weak equivalence $T \xrightarrow{\simeq} \iota_\ell^* \widetilde{T}$.

The canonical dual of T is $F_A(T,A) = \mathrm{Hom}_{\underline{\mathsf{s}}}(T,A) \in \mathscr{S}_R(B,A)$, and we let D denote a cofibrant replacement for $F_A(T,A)$ in $\mathscr{S}_R(B,A)$, so that we have a weak equivalence $D \xrightarrow{\simeq} F_A(T,A)$. The canonical dual of T has a right-action of the endomorphism k-algebra, E, and we let $\widetilde{D}$ be a cofibrant replacement for $F_A(T,A)$ in $\mathscr{S}_R(E,A)$, constructed again by the usual factorization. Since the forgetful functor ι_r^* creates weak equivalences and fibrations, we have an acyclic fibration $\iota_r^*\widetilde{D} \xrightarrow{\simeq} F_A(T,A)$ in $\mathscr{S}_R(B,A)$. Because D is cofibrant, the weak equivalence $D \xrightarrow{\simeq} F_A(T,A)$ lifts with respect to acyclic fibrations and hence we have a weak equivalence $D \xrightarrow{\simeq} \iota_r^* \widetilde{D}$.

Now we show that $(\widetilde{T}, \widetilde{D})$ is a dual pair in $\mathscr{D}_R$. The weak equivalences $\widetilde{T} \to T$ and $\widetilde{D} \to F_A(T,A)$ in $\mathscr{S}_R(A,E)$ and $\mathscr{S}_R(E,A)$, respectively, give maps

$$\widetilde{T} \odot \widetilde{D} \to T \odot F_A(T,A) \to E \text{ and } \widetilde{D} \odot \widetilde{T} \to F_A(T,A) \odot T \to A$$

in $\mathscr{S}_R(E,E)$ and $\mathscr{S}_R(A,A)$, respectively. Moreover, the first map is an isomorphism in $\mathscr{D}_R(E,E)$ because its image under $\iota_\ell^* \iota_r^*$ is a composite of two isomorphisms in $\mathscr{D}_R(B,B)$:

$$\iota_\ell^* \widetilde{T} \odot \iota_r^* \widetilde{D} \cong T \odot D \cong \iota_\ell^* \iota_r^* E.$$

The inverse to this map gives the unit for the dual pair, and the duality diagrams commute because the corresponding diagrams for T and $F_A(T,A)$ do. Hence the functors $- \odot \widetilde{T}$ and $- \odot \widetilde{D}$ induce an adjunction

$$\begin{tikzcd} \mathscr{D}_R(A,C) \arrow[r, shift left, "- \odot \widetilde{T}"] & \mathscr{D}_R(E,C) \arrow[l, shift left, "- \odot \widetilde{D}"] \end{tikzcd}$$

and the unit of this adjunction is an isomorphism.

As in the algebraic case, the 1-cells of $\mathscr{D}_R(A,R)$ are jointly left-faithful and hence the generator T is left-faithful. Since ι_ℓ^* creates weak equivalences, $\widetilde{T}$ is also left-faithful and the result follows just as in the algebraic case. ☐

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