RESEARCH STATEMENT

NICK GURSKI

1. Introduction

Higher dimensional category theory is a field still in its infancy. Bicategories, a weakened version of 2-categories, were defined by Benabou in [8], but no attempt at a weak theory of n-categories for $n > 2$ was made until Street's paper "The algebra of oriented simplexes" in 1987. Since that paper, no less than ten systematic definitions of weak n- or ω-category have been proposed (see [31] for an overview), and the motivations have been as varied as the definitions, coming from sources such as logic, algebra, and homotopy theory.

A central problem in higher category theory is that of categorifying a structure: given a particular kind of structure S with data consisting of sets, functions between them, and equational axioms, how can we replace sets with categories, functions with functors, and equations with natural isomorphisms while adding in new axioms to give a new structure S' that behaves like S "up to homotopy"?

$$\begin{array}{rcl} \text{Underlying sets} & \mapsto & \text{Underlying categories} \\ \text{Functions between them} & \mapsto & \text{Functors between them} \\ \text{Axioms} & \mapsto & \text{Natural isomorphisms} \\ ??? & \mapsto & \text{New axioms} \end{array}$$

For instance, if S denotes the structure of a monoid, the categorification S' is the structure of a monoidal category, now a widely used tool in mathematics. If S is the structure of a category, then the structure S' is that of a bicategory; categorifying once more, we get the structure of a tricategory.

It is here that we see the difference between the categorification described above and the process of strict categorification in which sets are replaced with categories and functions with functors, but the axioms remain the same.

$$\begin{array}{rcl} \text{Underlying sets} & \mapsto & \text{Underlying categories} \\ \text{Functions between them} & \mapsto & \text{Functors between them} \\ \text{Axioms} & \mapsto & \text{Same axioms} \end{array}$$

The strict categorification of the structure of a category is the structure of a 2-category, and the strict categorification of the structure of a 2-category is a 3-category. Every bicategory is appropriately equivalent to a 2-category, but the same does not hold for tricategories and 3-categories. This observation is the starting point for coherence theory in higher dimensions.

The coherence theorem for tricategories has a close connection with the problem of algebraically modeling homotopy 3-types, i.e., spaces X in which $\pi_i X = 0$ if $i > 3$. In [23], Grothendieck posited that weak n-groupoids should model homotopy n-types. Specializing to $n = 3$, it is easy to show that strict 3-groupoids only model those homotopy 3-types in which the Whitehead product $\pi_2 X \times \pi_2 X \to \pi_3 X$ is the zero map. This deficiency can be corrected by weakening the strict 3-groupoids to trigroupoids (see [26] for a good discussion of this modeling problem and a variety of possible solutions), showing how *purely topological* considerations give insight into *categorical* constructions. My research addresses these issues by looking at the deep connections between higher dimensional category theory and other fields, and then by proving important categorical results using the insights gained in the process.

One concept whose categorification is receiving attention these days is that of a vector space. It turns out that categorifying vector spaces in a way that is both computationally useful and categorically approachable is a difficult problem. To demonstrate the difficulties involved in categorifying concepts in algebra, consider the process of categorifying commutative monoids. One might begin by replacing the underlying set with an underlying category, the multiplication function with a multiplication functor, the identity element with an identity object, and the associativity, unit, and commutativity axioms with associaity, unit, and commutativity isomorphisms. When deciding on the axioms for a categorified commutative monoid, one might adopt the following philosophy: whenever it is possible, in a commutative monoid, to derive a new axiom by using a sequence of associativity, unit, and commutativity axioms, then a categorified commutative monoid should have an axiom between the composite of the corresponding isomorphisms.

While this philosophy seems reasonable, it destroys any hope of finding naturally occurring examples of this structure. The equation $a+a=a+a$ can either be seen as the commutativity axiom, or just the identity. Thus the above philosophy would give rise to the equation $\gamma_{a,a}=1$ where γ denotes the commutativity isomorphism. Unfortunately, almost no naturally occurring symmetric monoidal category – the concept which should be viewed as the correct categorification of commutative monoid – satisfies this axiom, and in particular crucially important examples like the category of sets, the category of spaces, and categories of modules do not satisfy this axiom. The example of commutative monoids demonstrates the care that must be taken when categorifying concepts in algebra.

The concept of vector space is much more complex than that of commutative monoid – it involves a group structure, the notion of a field, and the definition of a bilinear map. Categorifying all of these ideas and the *process* by which one defines vector spaces is a project Weiwei Pan and I are currently involved in, but it need not be the only valid categorification of vector space as "valid categorification" depends as much on the application in mind as the results obtained. Understanding both the categorical subtleties and the motivating applications for a particular categorified concept is crucial if research on topics such as categorified vector spaces is to be successful and remain relevant. An awareness of these issues is reflected in my work on 2-vector spaces as I strive to clarify the categorical and computational aspects of a variety of proposed definitions in order to produce good working definitions of categorified concepts in linear algebra.

My research in higher category theory is both technical and mindful of future applications. I have focused on low-dimensional problems, such as coherence for tricategories, that address central questions certain to be important in higher dimensions, and I am particularly interested in categorical problems that contribute to or are informed by other areas of mathematics such as topology, algebraic geometry, and representation theory.

2. Demystifying tricategories

Tricategories are a particular model for weak 3-dimensional categories introduced by Gordon, Power, and Street in [22]. In this sort of structure, composition of 3-cells is associative and unital, composition of 2-cells is associative and unital up to isomorphism, and composition of 1-cells is associative and unital up to equivalence, all subject to appropriate coherence axioms.

While this definition captures the essential features one would expect from a weak 3-dimensional category, it suffers from certain technical difficulties that make it difficult to use in practice. My 2006 University of Chicago thesis [24] studied a fully algebraic version of this definition, and greatly improved upon the coherence results in [22] using this new definition. My research has contributed to the coherence theory for tricategories in four very important ways. First, my algebraic version of the coherence theorem makes diagram-checking in an arbitrary tricategory simple in much the same spirit as the coherence theory for bicategories. This result should be compared with the original coherence theorem for tricategories in [22] which merely gives a strictification result, bringing us to the second contribution of my thesis: an explicit strictification construction. Third, I proved a coherence theorem for *functors* between tricategories, something that was missing from the original work of Gordon, Power, and Street. Finally, this coherence theorem for functors allows for definitions of weak 4-category such as given by Trimble in [40] – the very definition of a tricategory depends upon the coherence theorem for functors between bicategories, as does Trimble's definition of tetracategory. I have continued my research on this algebraic version of tricategories and studied two distinct problems: degenerate tricategories, or tricategories with a single object or single object and single 1-cell; and the low dimensional structures that tricategories form.

2.1. **Degenerate tricategories.** For many mathematicians, the most familiar object from higher dimensional category theory is that of the monoidal category – a category with a tensor product that is associative and unital up to coherent isomorphism. Categories such as the category of sets and functions, the category of spaces and continuous maps, and various categories of modules and module homomorphisms all have natural monoidal structures. But monoidal categories are 2-dimensional structures in disguise as they are actually bicategories with a single object. The objects of the monoidal category are now treated as 1-cells of this new bicategory, and the morphisms become the 2-cells. The old tensor product is now composition of 1-cells, with the rest of the monoidal data filling in the data for a bicategory.

Just as naturally occurring categories are often monoidal, so we expect naturally occurring n-categories also to often be monoidal. Inspired by the case $n = 2$, it is reasonable to conjecture that monoidal n-categories are the same as $(n + 1)$-categories with a single object. Lack actually proves more for the case $n = 2$ in [30] where he constructs a 2-category of bicategories in which the 2-category of monoidal categories fully embeds. This embedding has the property that every bicategory with a single object is in the essential image.

This naturally raises the question of what sort of structure on an n-category is the same as an $(n + 2)$-category with a single object and single 1-cell. Baez and Dolan formulated the Stabilization Hypothesis in [3] to suggest an answer to the general question: what structure on an n-category is equivalent to that of being an

$(n+k)$-category with a single 0-cell, single 1-cell, etc., up through single $(k-1)$-cell? Their answer is that when $k = 1$, the structure is that of a monoidal n-category, and as k increases the monoidal structure becomes more and more commutative until it stabilizes when $k = n+2$, at which point we should call the resulting object a symmetric monoidal n-category.

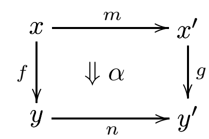
Eugenia Cheng and I studied this closely when $n + k = 2$ in [12], and continued in the paper [14] by studying when $n + k = 3$. We prove two key results in [14] that help explain the structure of a tricategory. First, we compare tricategories with a single object and single 1-cell to braided monoidal categories. While there are natural functors between "doubly degenerate tricategories" and braided monoidal categories, they do not give an equivalence as transformations between doubly degenerate tricategories contain more information than monoidal transformations. Constructing both of these functors does give a rigorous proof of the folklore result that braided monoidal categories give rise to tricategories in which not every diagram of constraint 3-cells commutes. The second key result of [14] is an examination of the stable case, where the tricategories involved have a single object, single 1-cell, and single 2-cell. There we prove an equivalence between these structures and the expected value of commutative monoids, but only when we examine a certain *tricategory* of these structures.

These results suggest that the Stabilization Hypothesis presents a very subtle problem. Degenerate tricategories should be similar to existing structures like braided monoidal categories, and my work with Eugenia Cheng indicates to what extent that is true. Our research raises new coherence questions concerning k-tuply degenerate n-categories, and solidifies the understanding of a key foundational hypothesis in higher dimensional category theory.

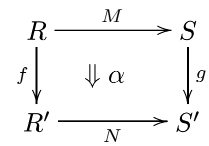
2.2. **Structures of tricategories.** While my work with Eugenia Cheng on degenerate tricategories was aimed at understanding the structure internal to tricategories, my work with Richard Garner was aimed at understanding the total structures that have tricategories as their objects. The problem we considered was quite simple: what is the "smallest" structure having tricategories as objects and trihomomorphisms as 1-cells? Trihomomorphisms do not compose associatively, so the answer cannot be that we have a category of tricategories and trihomomorphisms. Garner and I show that it is possible to construct both a bicategory and a tricategory whose objects are tricategories and whose 1-cells are trihomomorphisms.

Constructing a bicategory of tricategories was simple, with proofs largely consisting of using the coherence theorems in my thesis [24]. It would be possible to construct a tricategory of tricategories in the same fashion, but the number of diagrams to be checked grows quickly when passing from dimension two to dimension three and the proofs would amount to pages of unenlightening coherence calculations. Instead of pursuing the direct approach, Garner and I found a simple and elegant way of encoding the necessary coherence data, a structure we called a *locally cubical bicategory.*

Cubical structures have long been a part of category theory in one form or another. The easiest example of a cubical structure is a double category (see [17]) which has objects, horizontal 1-cells, vertical 1-cells, and 2-cells that exist in squares as below.



A simple example of a (weak) double category is **Ring**. Objects are rings R, horizontal 1-cells $M : R \to S$ are left $S-$, right $R-$bimodules, vertical 1-cells $f : R \to R'$ are ring homomorphisms, and 2-cells



are group homomorphisms $\alpha : M \to N$ such that $\alpha(s \cdot m \cdot r) = g(s) \cdot \alpha(m) \cdot f(r)$.

The strategy that Garner and I used to construct a tricategory of tricategories was to define a new structure which replaced the hom-categories of a bicategory with (weak) double categories. This produced a structure which behaves as a bicategory would with respect to coherence, i.e., the 1-cells $(fg)h$ and $f(gh)$ are *isomorphic* instead of merely *equivalent* as in a tricategory, but the total structure is genuinely 3-dimensional, having objects, 1-cells, two different kinds of 2-cells (vertical and horizontal), and 3-cells living in squares as above. Using our construction of the bicategory of tricategories, it turned out to be quite easy to organize tricategories into this new sort of hybrid structure we called a locally cubical bicategory. Finally, we showed that given certainly easily-checked conditions, satisfied by our locally cubical bicategory of tricategories, a locally cubical bicategory gives rise to a tricategory.

Perhaps the most important consequence of my work with Garner was to show how it is possible to build tricategories from far simpler structures. A major obstacle to the wider use of higher categories in other areas of mathematics is the perception that higher dimensional categories are very difficult objects to construct and manipulate. Even within category theory itself, tricategories are regarded as notoriously difficult objects. Our construction of a tricategory of tricategories – an object whose complexity is compounded, being a complicated structure composed itself of cells that are themselves quite complicated – is elegant and an example of how it is possible to build interesting and natural tricategories without employing any constructions more complicated than that found in a monoidal category. Just like my work with Eugenia Cheng on degenerate tricategories, this project with Richard Garner aims at understanding tricategories with the express goal of making 3-dimensional category theory *computable* and *easily accessible* to the non-expert.

3. Higher category theory and topology

Higher dimensional category theory has long drawn many of its inspirations from topology. For instance, Street's simplicial definition presented in [38] is essentially a study in taking seriously the idea of simplicial horn-filling as a kind of composition. Many definitions of weak n-category use approaches that resemble the techniques employed by algebraic topologists studying iterated loop spaces, see for example work of Batanin [6], [7] and the Tamsamani-Simpson school of definitions [39], [35].

In particular, two proto-examples of weak higher categories from topology continue to stimulate current research: the n-category of n-dimensional cobordisms with corners, and the ω-category of spaces, continuous maps, and all higher homotopies. My research has touched on both of these examples, but with very different purposes. My work on higher cobordism categories was primarily aimed at constructing this geometrically-motivated structure using machinery in higher category theory, while my research involving higher categories of spaces used techniques in algebraic topology to prove a coherence result for braided monoidal bicategories.

3.1. **Higher categories of cobordisms.** The idea for constructing an n-category of n-dimensional cobordisms is motivated by a process common in category theory: taking a traditional definition with a long list of data and axioms (in this case, that of a 2-dimensional topological quantum field theory) and rewriting it succinctly using categorical terminology (here, expressing such a TQFT not as a vector space with a variety of operations and required axioms but merely as a symmetric monoidal functor $\mathbf{2Cob} \to \mathbf{Vect}$). This process often highlights similarities with other interesting problems and reveals available categorical techniques.

The compact definition of a 2-dimensional TQFT as a symmetric monoidal functor $\mathbf{2Cob} \to \mathbf{Vect}$ requires two essential ingredients: the definition of a symmetric monoidal functor, and the construction of the symmetric monoidal category $\mathbf{2Cob}$. Going up to dimension three, we have two choices, either to study 3-dimensional cobordisms between closed 2-manifolds, or to study 3-dimensional cobordisms between 2-manifolds which are themselves cobordisms between their boundaries. Using this latter notion for a field theory gives what is called an *extended* 3-dimensional topological quantum field theory, and it is this structure that we wish to formulate using higher dimensional categories. In analogy with the two-dimensional case, we might then make the definition that an extended $(n+1)$-dimensional TQFT is a symmetric monoidal n-functor $(\mathbf{n+1})\mathbf{Cob} \to \mathbf{nVect}$; even though none of these terms is defined in general, this approach provides a clear strategy for defining and studying higher dimensional TQFT's.

Constructing $\mathbf{2Cob}$ by hand is relatively straightforward, and it would be possible to construct a bicategory $\mathbf{3Cob}$ in a similar fashion, but this approach makes little headway into the problem of defining $\mathbf{nCob}$ in general as hands-on definitions of weak n-categories do not exist above $n = 4$ (see work of Trimble [41] for a definition of tetracategory). Thus Eugenia Cheng and I set out to use one of the general definitions of weak n-category to give the first construction of $\mathbf{nCob}$ as an n-category in [13].

Instead of using abstract manifolds, we settled on the modest goal of constructing higher dimensional categories of manifolds embedded in cubes. Our approach was similar to that of Baez and Langford in [**?**] in which they constructed a 2-category of 2-tangles embedded in 4-space. Stacking cubes along any of their sides provides different ways to "compose" manifolds, but the result is no longer a manifold embedded in a cube and is instead a manifold embedded in a rectangular prism. In order to rectify this situation, we modified the definition of Trimble [40] and used an operad to parametrize composition.

Our construction of higher categories of cobordisms is the first such construction to use general higher categorical tools to produce n-categories of cobordisms. The techniques used not only show how one can construct higher dimensional categories of cobordisms by changing the operad, they also open up the possibility of applying a Trimble-like definition to other types of field theories (like conformal field theory) by using operads appropriate to that field theory.

3.2. **Operads and coherence theorems.** A second research project of mine that is topologically motivated is a proof of coherence for braided monoidal bicategories using techniques from geometry and homotopy theory. Coherence results are some of the most powerful results in category theory, and the generic coherence theorem states that some large class of diagrams always commute. Coherence for monoidal bicategories is essentially a corollary of coherence for tricategories, but the coherence story when braidings or symmetries are involved is much more complicated.

Joyal and Street give an excellent treatment of coherence for both monoidal and braided monoidal categories in their paper [27]. The coherence result for monoidal categories states that, in a free monoidal category, every diagram created from the associativity and unit isomorphisms commutes. This allows one, for instance, to ignore the difference between the sets $(X \times Y) \times Z$ and $X \times (Y \times Z)$ – of course these sets are not equal, but they are isomorphic in a strong enough sense that one can do mathematics as if they were the same.

The analagous result does not hold for braided monoidal categories in that there are diagrams that do not commute in free braided monoidal categories. In particular, the isomorphism obtained from braiding twice is not in general equal to the identity: $\gamma_{b,a} \circ \gamma_{a,b} \neq 1_{ab}$. The work of Joyal and Street shows that the free braided monoidal category on a single object is in fact the disjoint union of the braid groups viewed as a category, and my work on coherence for braided monoidal bicategories in [25] extends this theory up one categorical dimension.

The first definition of braided monoidal 2-category was given by Kapranov and Voevodsky [29] and was motivated by the study of higher dimensional knots. Baez and Neuchl [5] improved this definition and Crans [15] later added additional unit conditions, giving what is now the final version of the definition of braided monoidal 2-category. This definition had no fully weak cousin to which it was related, much as 2-categories were defined before bicategories. Giving a fully weak version of the definition was straightforward, but proving that the weak definition and the strict one given by Crans were equivalent required using operadic tools from the study of double loop spaces.

Operads were originally defined by algebraic topologists to study the problem of detecting up-to-homotopy algebraic structures on a space (see [9] and [33]). A topological operad P consists of spaces $P(n)$ for each natural number n, a unit $1 \in P(1)$, and composition maps

$$P(n) \times P(k_1) \times \cdots \times P(k_n) \to P(k_1 + \cdots + k_n)$$

satisfying simple associativity and unit conditions. The space $P(n)$ is thought of as the space of n-ary operations, and often we require symmetric group actions which record what happens when the input variables are permuted. An operad P should be thought of as codifying a particular kind of structure, and then algebras over P are objects equipped with an algebraic structure of type P.

May defined certain operads called the little n-cubes operads C_n in [33] that were relevant to iterated loop space theory and also feature in my work. May proved that if X is a path-connected space with an algebra structure over the operad C_n, then X has the weak homotopy type of $\Omega^n Y$ for some space Y. My work begins by showing that the fundamental 2-groupoid $\Pi_2 X$ of a space X is sensitive to the presence of

C_1- and C_2-algebra structures . Recall that the fundamental 2-groupoid of X is the bicategory with objects the points of X, 1-cells the paths between them, and 2-cells homotopy classes of homotopies between paths. This construction is actually the assignment on objects of a functor $\Pi_2 : \mathbf{Top}_3 \to \mathbf{Bicat}$, where $\mathbf{Top}_3$ is a tricategory of spaces, maps, and homotopies. I extend the fundamental 2-groupoid functor to one which equips $\Pi_2 X$ with the structure of a monoidal bicategory when X is a C_1-algebra and the structure of a braided monoidal bicategory when X is a C_2-algebra.

With this construction in hand, we are ready to prove a coherence theorem for braided monoidal bicategories. I show that the free braided monoidal bicategory on one object is equivalent to the braided monoidal bicategory $\Pi_2(C_2*)$ (here C_2* is the free C_2-algebra on the terminal space). The proof of this equivalence relies heavily on work of Carter and Saito [11] classifying surface braids in $\mathbb{R}^4$ via Morse theory. An immediate corollary of this theorem is that every diagram of constraint 2-cells in the free braided monoidal bicategory on one object commutes: this is just the categorical interpretation of the simple fact that C_2* is homotopy equivalent to a disjoint union of Eilenberg-Mac Lane space of type $K(B_k, 1)$ (where B_k is the Artin braid group on k strands).

Coherence for braided monoidal bicategories is an important theorem for a variety of reasons. First, coherence makes the structure usable for both theoretical and practical applications as it removes the need to check a large number of diagrams by hand. A second important consequence of my coherence theorem is that it shows how a braided structure is essentially the same thing as an action of the braid groups on a monoidal bicategory – no higher dimensional action is required. Third, my proof suggests a uniform strategy for proving coherence for monoidal kinds of weak n-categories in higher dimensions by using well-known operads in algebraic topology and Morse-theoretic classifications of certain kinds of disks embedded in Euclidean space, uniting important concepts in geometry and category theory. My proof represents a significant step forward in the understanding of coherence for monoidal structures on higher categories by employing techniques from geometry and topology.

Both of these projects – constructing higher categories of cobordisms and proving coherence for braided monoidal bicategories using topological techniques – build important links between higher category theory and topology. Exploring examples of higher categorical structures in other branches of mathematics helps keep higher dimensional category theory relevant and useful, one of the top priorities of my research.

4. Ongoing research

My current research projects, much like those listed above, reflect a wide interest in higher dimensional category theory and other subjects to which I believe higher dimensional category theory can make important contributions. I would like to highlight here two different projects that I am currently involved in actively: joint work with Richard Garner on using tricategorical techniques to study fibred 2-categories, and joint work with Jeremy Copeland and Weiwei Pan on the foundations of categorified linear algebra and representation theory.

4.1. **Fibred objects in geometry and category theory.** Fibred objects play an important role in geometry. The collection of bundles (of a specified type) over a fixed base X is an invariant of X, and studying these collections often reveals the geometric structure of the base. This process is made simpler by the construction of a universal bundle. This universal bundle $E \to B$ should have the property that maps $X \to B$ correspond to bundles over X, or in other words the universal bundle is a *representing object* for the functor

$$\text{base } X \mapsto \text{bundles over } X.$$

Thus the study of bundles can begin with the identification of an appropriate representing object for the functor above.

In algebraic geometry, the situation is complicated by the fact that representing object needs to record nontrivial automorphisms in the fibers. Thus instead of just having a universal bundle, we have a universal object whose fibers are groupoids that remember not only what the fibers of a bundle $V \to X$ are but also their automorphisms. This was the motivation for the introduction of *stacks* in [16]. Stacks can be thought of as sheaves of groupoids in which the gluing conditions now keep track of both object- and the morphism-level information. But now higher dimensional analogues of stacks are being considered (see [35], [10]), so it is time to ask the basic categorical questions regarding fibred objects in higher category theory.

I am currently studying fibred 2-categories and bicategories in joint work with Richard Garner. Our work begins by studying the following moduli-type problem: given a 2-category K, can we find a representing object for the construction $K \mapsto \mathbf{2Cat}/K$? This work arose naturally from thinking about fibrations of 2-categories, but curiously has a solution using the locally cubical bicategories we studied in constructing our tricategory of tricategories.

The analogous problem for categories is much simpler. Starting with a base category C, we are interested in finding an object P such that maps $F : C^{\mathrm{op}} \to P$ are categories over C; that representing object P is given by the bicategory of profunctors. A profunctor from A to B is a functor $B^{op} \times A \to \mathbf{Sets}$. The bicategory **Prof** has objects small categories, 1-cells profunctors, and 2-cells natural transformations. Treating C^{op} as a discrete 2-category, we get a correspondence between normal lax functors $C^{op} \to \mathbf{Prof}$ and categories over C. It is this correspondence that Garner and I are studying for 2-categories and bicategories.

We study two problems. The first is, given a 2-category K, characterizing strict 2-functors $A \to K$ with A a strict 2-category. The second is, given a bicategory B, characterizing weak functors $A \to B$ with A a bicategory. These problems are related, but the 2-categorical one can be approached solely within the context of locally cubical bicategories, leading to a much simpler representing object.

Our next goal is to understand fibrations of 2-categories and bicategories from this point of view. In the case of categories, fibrations can be expressed not as normal lax functors into **Prof** but as weak functors into **Cat**. Garner and I intend to explore fibrations in two dimensions using these representations, with the eventual goal being an understanding of the categorical underpinnings of the constructions used in the study of 2-gerbes and 2-stacks [10].

4.2. **Higher categories and representation theory.** The last research problem I would like to discuss is my joint work with Jeremy Copeland and Weiwei Pan on the foundations of categorified linear algebra and representation theory. This subject is receiving increasing attention as it becomes relevant to important problems in mainstream mathematics, such as in the study of elliptic cohomology [1], [36]. Our work focuses on understanding the basic categorical properties of various notions of categorified abelian group and categorified vector space.

There are a variety of definitions of categorified vector spaces, or 2-vector spaces, in the literature today. The first was given by Kapranov and Voevodsky in [28]; their definition is that a 2-vector space is an abelian category V that is a module over **Vect** which is equivalent to a module of the form $\mathbf{Vect}^n$. A second definition was given by Baez and Crans in [2]; their definition is that a 2-vector space is a category internal to **Vect**, or a category in which the objects and morphisms each form a vector space such that source, target, identities, and composition are all linear maps.

While the Kapranov-Voevodsky definition has been explored by a number of authors (see [19], [1]), the Baez-Crans definition has not been thoroughly studied for the purpose of categorified representation theory (the thesis of Forrester-Barker [20] treats the strict case). Jeremy Copeland and I have begun an extensive investigation into the structure of 2-representations using the Baez-Crans definition. Since their 2-category **2Vect** is 2-equivalent to the 2-category of 2-term chain complexes $A_1 \to A_0$, chain maps, and chain homotopies, these 2-representations are very similar to group representations but with a homological flavor.

Copeland and I have proven that this 2-category **2Vect** shares many of the categorical features that **Vect** has, but it also lacks some key properties. For instance, we proved that **2Vect** has a forgetful functor to the 2-category **Cat** which has a left 2-adjoint. We have also shown that **2Vect** has a tensor product, and the left 2-adjoint $\mathbf{Cat} \to \mathbf{2Vect}$ is monoidal with respect to this tensor. The existence of a monoidal left adjoint is precisely analogous to that of the relationship between **Vect** and **Sets**. The fact that the free vector space functor $\mathbf{Sets} \to \mathbf{Vect}$ is monoidal gives rise to the construction of group algebras, and we produce analogous constructions in the context of 2-vector spaces.

It is well known that the category of representations of a finite group G over a field k of characteristic zero is equivalent to the category of $k[G]$-modules, and Copeland and I have shown that this remains true up one categorical dimension. Using Morita theory, it is possible to show that if the category of $k[G]$-modules is equivalent to the category of $k[G']$-modules, then the groups G and G' are in fact isomorphic, and it is here that this categorified version of representation theory gives a decidedly less satisfactory result. Before explaining our result, a definition is necessary. A 2-group is a monoidal groupoid in which every object has an inverse, up to isomorphism, under the tensor product. Examples of 2-groups are automorphism 2-groups of categories, crossed modules, and Picard groupoids. The result that Copeland and I prove is that, if k has characteristic zero, then 2-representations of a finite 2-group $\mathfrak{G}$ are equivalent to 2-representations of the group of components of $\mathfrak{G}$, $\pi_0(\mathfrak{G})$.

Our proof of this result highlights the importance of the coefficients used – the fact that k is characteristic zero is crucial. The automorphisms of the unit object in a 2-group $\mathfrak{G}$ forms an abelian group, and we have shown how it is possible to recover this abelian group by studying different 2-categories of 2-representations using torsion coefficients. Thus while no single field can distuingish two arbitrary 2-groups, we are showing how the *family* of 2-categories $\mathbf{2Rep}_k(\mathfrak{G})$ is collectively faithful using a combination of categorical and representation-theoretic techniques.

Another line of research prompted by these results is joint work with Weiwei Pan on a much more elementary definition of 2-vector spaces that, instead of categorifying vector spaces directly, attempts to categorify the *process* in passing from groups to abelian groups to rings to fields, and then to define 2-vector spaces as modules over these categorified fields. There is already a good candidate for categorified abelian groups, namely symmetric categorical groups [34], also known as commutative Picard groupoids. Pan and I are in the process of understanding various coherence problems associated with constructing a good tensor product on the 2-category of symmetric categorical groups and in understanding how free such objects relate to important calculations in stable homotopy theory.

5. Concluding remarks

Higher dimensional category theory is a field that, while technically challenging to the outsider, has much to offer other branches of mathematics, from giving a coherent approach to structures with many operations to clarifying the algebra of strings and higher dimensional submanifolds propogating through spacetime. My research has had and will continue to have two primary goals: first, to advance the understanding of low-dimensional structures that hold the key to further research, and second to build further connections between higher dimensional category theory and other disciplines, both by studying examples outside category theory and by incorporating techniques from traditional areas of mathematics.

References

[1] Nils Baas, Bjorn Dundas, and John Rognes, *Two-vector bundles and forms of elliptic cohomology*, in Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal, editor U. Tillmann. Cambridge University Press, 2004.

[2] John Baez and Alissa Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory and Appl. of Categories **12** (2004), 492–528.

[3] John Baez and James Dolan, *Higher-dimensional algebra and topological quantum field theory*, Jour. Math. Phys. **36** (1995), 6073–6105.

[4] John Baez and Laurel Langford, *Higher dimensional algebra IV: 2-tangles*,Adv. Math. **180** (2003), 705–764.

[5] John Baez and Martin Neuchl, *Higher-dimensional algebra I: braided monoidal 2-categories*, Adv. Math. **121** (1996), 196–244.

[6] Michael Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Adv. Math. **136** (1998), 39-103.

[7] Michael Batanin, *The Eckman-Hilton argument and higher operads*, e-print available at arXiv:0207281.

[8] Jean Bénabou, *Introduction to bicategories*, Lecture notes in mathematics **47** (1967).

[9] J. M. Boardman and Rainer Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics **347**. Springer-Verlag, Berlin, 1973.

[10] Lawrence Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque **225**. Soc. Math. France, Paris, 1994.

[11] Scott Carter and Masahico Saito, *Knotted surfaces and their diagrams*, Math. Surveys and Monographs, **55**. Amer. Math. Soc., Providence, RI, 1998.

[12] Eugenia Cheng and Nick Gurski, *The periodic table of n-categories for low dimensions I: degenerate categories and degenerate bicategories* in Categories in Algebra, Geometry, and Mathematical Physics, proceedings of Streetfest, editors Batanin, et al, Contemporary Math. AMS **431** (2007), 143–164.

[13] Eugenia Cheng and Nick Gurski, *Towards an n-category of cobordisms*, Theory and Appl. of Categories **18** (2007), 274–302.

[14] Eugenia Cheng and Nick Gurski, *The periodic table of n-categories for low dimensions II: degenerate tricategories*, Submitted for publication, e-print available at arXiv:0706.2307.

[15] Sjoerd Crans, *Generalized centers of braided and sylleptic monoidal 2-categories*, Adv. Math., **136** (1998), 183–223.

[16] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Pub. Math. l'IHÉS, **36** (1960), 75–109.

[17] Charles Ehresmann, *Catégories structurées*, Ann. Sci. École Norm. Sup. **80** (1963), 349–426.

[18] Josep Elgueta, *Representation theory of 2-groups on finite dimensional 2-vector spaces*, e-print available at arXiv:0408120.

[19] Josep Elgueta, *Generalized 2-vector spaces and general linear 2-groups*, e-print available at arXiv:0606472.

[20] Magnus Forrester-Barker, *Representation Theory of Crossed Modules and cat^1-groups*, 2004 University of Wales, Bangor Ph.D. thesis.

[21] Richard Garner and Nick Gurski, *The low-dimensional structures formed by tricategories*, To appear in Math. Proc. Cam. Phil. Soc.

[22] Robert Gordon, John Power, and Ross Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. **117** (1995), no. 558.

[23] Alexander Grothendieck, *Pursuing stacks*, letter to Daniel Quillen, 1983.

[24] Nick Gurski, *An algebraic theory of tricategories*, 2006 University of Chicago Ph.D. thesis.

[25] Nick Gurski, *Loop spaces, and coherence for monoidal and braided monoidal bicategories*, Submitted for publication.

[26] André Joyal and Joachim Kock, *Weak units and homotopy 3-types*, in Categories in Algebra, Geometry, and Mathematical Physics, proceedings of Streetfest, editors Batanin, et al, Contemporary Math. AMS **431** (2007), 257–276.

[27] André Joyal and Ross Street, *Braided tensor categories*, Advances in Mathematics **102** (1983), 20–78.

[28] Mikhail Kapranov and Vladimir Voevodsky, *2-categories and Zamolodchikov tetrahedron equations*, Proc. Symp. Pure Math. **56** Part 2 (1994), AMS, Providence, 177–260.

[29] Mikhail Kapranov and Vladimir Voevodsky, *Braided monoidal 2-categories and Manin-Schechtman higher braid groups*, Jour. Pure Appl. Algebra **92** (1994), 241–267.

[30] Stephen Lack, *Icons*, e-print available at arXiv:0711.4657.

[31] Tom Leinster, *A survey of definitions of n-category*, Theory and Appl. of Categories **10** (2002), 1–70.

[32] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, no. 298, Cambridge University Press, 2004.

[33] J. Peter May, *The geometry of iterated loop spaces*, Lec. Notes in Math., **271**. Springer-Verlag, Berlin-New York, 1972.

[34] J. Martínez Moreno, A. del Río, and E. Vitale, *Chain complexes of symmetric categorical groups*, Journal of Pure and Appl. Algebra **196** (2005), 279–312.

[35] Carlos Simpson, *A closed model structure for n-categories, internal $\underline{Hom}$, n-stacks, and generalized Seifert-Van Kampen*, e-print available at arXiv:9704006.

[36] Stephan Stolz and Peter Teichner, *What is an elliptic object?*, in Proceedings of the 2002 Oxford Symposium in Honour of the 60th Birthday of Graeme Segal, editor U. Tillmann. Cambridge University Press, 2004.

[37] Ross Street, *The algebra of oriented simplexes*, Journal of Pure and Applied Algebra **49** (1987), no. 3, 283–335.

[38] R. Street, *Weak omega-categories*. Diagrammatic morphisms and applications (San Francisco, CA, 2000), 207–213, Contemp. Math., 318, Amer. Math. Soc., Providence, RI, 2003.

[39] Zouhair Tamsamani, *Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux*, K-theory **16** (1999), no.1, 51–99.

[40] Todd Trimble, *What are 'fundamental n-groupoids'?*, Seminar at DPMMS, Cambridge, August 24, 1999.

[41] Todd Trimble, *Notes on tetracategories*, e-print available at http://math.ucr.edu/home/baez/trimble/tetracategories.html.