Notes towards explicit comparison of May and Segal, heading towards equivariant comparison.

1. Segal, Shimakawa, Woolfson

Nonequivariantly, think about MMSS. The Segal machine is the composite of $\mathbb{P}\colon \mathscr{F}\mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$ and $\mathbb{U}_?$, where the target of $\mathbb{U}_?$ can be symmetric spectra, orthogonal spectra, or prespectra. Segal of course chose prespectra. We will choose orthogonal spectra. The functor $\mathbb{U}\colon \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T}$ sends a $\mathscr{W}$-space X to its restriction to $\mathscr{F}$. $\mathbb{P}$ was only spelled out simplicially, by reference, in MMSS. But it seems easy and relevant. The bar construction $B(X^\bullet, \mathscr{F}, Y)$ is giving a homotopically well behaved approximation of the prolongation functor, and similarly on forgetting down to orthogonal G-spectra. This explains the Segal-Shimakawa-Woolfson machine conceptually. Etc. [More detailed notes on Segal elsewhere]

Segal: $B(\underline{S}^\bullet, \mathscr{F}, X)$; Vth space $B((S^V)^\bullet, \mathscr{F}, X)$.

For a based space X, $X^\bullet$ is the contravariant functor $\mathscr{F} \longrightarrow \mathscr{T}$ that takes $\underline{n} = \{0, 1, \cdots, n\}$ to X^n. [$\underline{n} = \mathbf{n}^+$ elsewhere] Crucially, note that this uses the coproduct given by the diagonal map of X (as contrasted with the product that covariant functors on $\mathscr{F}$ impose). $\underline{S}$ is the orthogonal sphere spectrum, with Vth space S^V. Apply $(-)^\bullet$ spacewise.

Claim: Almost Ω-spectrum with the group completion property.

Query: What is the zeroth space of this G-prespectrum, the case $V = 0$?

Can replace $\mathscr{F}$ by $\hat{\mathscr{C}}$ for any operad $\mathscr{C}$, but we assume that $\mathscr{C}$ is an E_∞ operad. Then

$$B((S^V)^\bullet, \hat{\mathscr{C}}, X) \longrightarrow B((S^V)^\bullet, \mathscr{F}, X)$$

is a levelwise equivalence of orthogonal spectra. Note technically: no passage to orbits under symmetric groups, so local contractibility suffices for the equivalence. Write $\mathscr{C}_V = \mathscr{C} \times \mathscr{K}_V$, as in [1]. The projections $\mathscr{C}_V \longrightarrow \mathscr{C}$ induce

$$B((S^V)^\bullet, \hat{\mathscr{C}}_V, X) \longrightarrow B((S^V)^\bullet, \mathscr{C}, X).$$

This is not a level equivalence, but it is a weak equivalence of orthogonal spectra. In the colimits used to define the homotopy groups, V is approaching ∞, where we are seeing the E_∞ operad $\mathscr{C} \times \mathscr{K}_\infty$. In all of these we are just using the contravariant $\mathscr{F}$-spaces $(S^V)^\bullet$.

We only see a right action of a monad on the suspension functor by using the adjoint of the map of the approximation theorem, which is invisible to the Segal machine. In a sense, the idea below is to make it visible via a crazy chain of comparisons.

May-Thomason: $B(\Sigma^\infty L, \hat{C}, X)$, where in modern terms $\mathscr{C}$ is any E_∞ operad over the infinite Steiner operad $\mathscr{K}_\infty$, such as $\mathscr{C}_\infty = \mathscr{C} \times \mathscr{K}_\infty$.

May-Thomason orthogonal variant: Vth space $B(\Sigma^V L, \hat{C}_V, X)$; with # standing for the variable V, we use the notation $B(\Sigma^\bullet L, \hat{\mathbb{C}}_\bullet, X)$ for this variant. Here again $\mathscr{C}_V$ is the product of any chosen E_∞ operad, $\mathscr{C}$ say, and the Steiner operad $\mathscr{K}_V$. Note that the bullet in the previous machines stood for taking cartesian powers, whereas it refers the variable indexing space V here. We will use it both ways in what follows.

Idea: for each $t \in I$ and $V \in \mathscr{I}$, there is a contravariant enriched functor $(S^V)^\bullet_t \colon \hat{\mathscr{K}}_V \longrightarrow \mathscr{T}$ that takes $\underline{n}$ to $(S^V)^n$. The evaluation maps fit together to give

$$(S^V)^n \times I \times \hat{\mathscr{K}}_V(\underline{m}, \underline{n}) \longrightarrow (S^V)^m. \tag{1.1}$$

This goes roughly as follows. Steiner paths are paths. (I'll be hazy for the moment). At time t, each element $(k_1, \cdots, k_j)$ of any $\mathscr{K}(j)$ gives a map $S^V \longrightarrow (S^V)^j$, which at time $t = 0$ is the diagonal and at time $t = 1$ factors through the j-fold wedge ${}^j(S^V)$ If we were looking at little n-cubes this last would be the obvious thing where we collapse points of I^n not in the interior of the image of one of the k_i to a point and blow up the little cubes to full size. Each k_i is canonically homotopic to the identity map, so we get a homotopy. Where to put it so it makes sense? In the left hand variable, where nothing stops us from considering the functor $\underline{S}^\bullet \wedge I_+$ from $\mathscr{F}$ to orthogonal spectra.

The following tiny construction is supposed to be the heart of the matter. A Steiner path in V is a map $h \colon I \longrightarrow R_V$ such that $h(0) = \mathrm{id}$, where R_V is the space of distance reducing embeddings $V \longrightarrow V$. Then $\mathscr{K}_V(q)$ is the space of q-tuples of Steiner paths h_i such that the $h_i(1)$ have disjoint images. We define a homotopy

$$\gamma \colon \mathscr{K}_V(q) \times S^V \times I \longrightarrow (S^V)^q \tag{1.2}$$

with coordinates γ_i by letting

$$\gamma_i(\langle h_1, \cdots, h_q\rangle, v, t) = \begin{cases} w & \text{if } h_i(t)(w) = v \\ * & \text{if } v \notin \mathrm{im}(h_i(t)). \end{cases} \tag{1.3}$$

If $t = 0$, this is just the diagonal map $S^V \longrightarrow (S^V)^q$, which makes this map relevant to the Segal machine. If $t = 1$, this map lands in the q-fold wedge ${}^q(S^V)$ of copies of V since the conditions $h_i(1)(w) = v$ as i varies are mutually exclusive; that is, there is at most one i such that $h_i(1)(w) = v$. This is relevant since the map

$$\tilde{\alpha} \colon \Sigma^V K_V X \longrightarrow \Sigma^V X$$

that figures in the May machine is given by

$$\tilde{\alpha}(\langle h_1, \cdots, h_q\rangle, v, x_1, \cdots, x_q) = \begin{cases} (x_i, w_i) & \text{if } h_i(1)(w_i) = v \\ * & \text{if } v \notin \mathrm{im}(h_i(t)) \text{ for } 1 \le i \le q. \end{cases} \tag{1.4}$$

This feels uncannily prescient, as if the Steiner operads were designed to make an explicit comparison between the May and Segal machines possible.

To complete the definition of (1.1), we use these maps to parametrize, using

$$\hat{\mathscr{K}}_V(\underline{m}, \underline{n}) = \bigvee_{\phi \colon \underline{m} \to \underline{n}} \prod_{1 \le j \le n} \mathscr{K}_V(s_j), \tag{1.5}$$

where $s_j = |\phi^{-1}(j)|$ for $1 \le j \le n$. The $\phi^{-1}(j)$ are ordered subsets of $\underline{m}$. We must define the m coordinates of $\hat{\delta}$, starting with points of $(S^V)^n$ and we use the map γ with q replaced by s_i and v taken in the jth coordinate of $(S^V)^n$ to obtain the coordinates indexed on the set $\phi^{-1}(i)$. The coordinates indexed on those k not in any of the $\phi^{-1}(j)$ are $*$ (since $\phi(k) = 0$ for those k). Write γ_t for γ at time $t \in I$. We may view the target of $\hat{\gamma}_1$ as the wedge ${}^m(S^V)$, and it then makes sense to restrict to wedges in the source as well, thus obtaining the variant

$$\hat{\gamma}_1 \colon {}^n(S^V) \times I \times \hat{\mathscr{K}}_V(\underline{m}, \underline{n}) \longrightarrow {}^m(S^V). \tag{1.6}$$

Using γ_1, we obtain a contravariant functor ${}^{\bullet}\underline{S})$ from $\hat{\mathscr{K}}_V$ to orthogonal spectra that uses wedges rather than products of the sphere spectrum $\underline{S}$.

There are going to be finicky details, especially with regard to basepoints, but from here the idea is to display a sequence of weak equivalences of machines as follows, where we start with an $\mathscr{F}$-space X. In fact, the construction will give directly that all except the last of the sequence of vertical maps is a weak equivalence of orthogonal G-spectra. The Segal machine will give group completions for the top bar construction, and the May machine will give group completions for the bottom construction, and both will be almost Ω-spectra. Therefore, getting compatible maps at the bottom level, the group completion property, proven totally differently for the two machines, will give that the bottom vertical arrow is also an equivalence: we won't have to prove that directly at all.

$$\begin{tikzcd}
B(\underline{S}^{\bullet}, \mathscr{F}, X) \\
B(\underline{S}^{\bullet}, \hat{\mathscr{C}}, X) \arrow[u] \\
B(\underline{S}^{\bullet}, \hat{\mathscr{C}}_{\bullet}, X) \arrow[u] \arrow[d, "i_0"] \\
B(\underline{S}^{\bullet} \times I, \hat{\mathscr{C}}_{\bullet}, X) \\
B(\underline{S}^{\bullet}, \hat{\mathscr{C}}_{\bullet}, X) \arrow[u, "i_1"] \\
B({}^{\bullet}\underline{S}, \hat{\mathscr{C}}_{\bullet}, X) \arrow[u, "\subset"] \arrow[d] \\
B(\Sigma^{\bullet} L, \hat{\mathbb{C}}_{\bullet}, X).
\end{tikzcd}$$

The bullets in superscripts in the first six bar constructions denote cartesian or wedge powers. The remaining bullets denote functors on $V \in \mathscr{I}$. The top two arrows are induced by the projections $\hat{\mathscr{C}} \longrightarrow \mathscr{F}$ and $\hat{\mathscr{C}}_V \longrightarrow \hat{\mathscr{C}}$, respectively. By pullback along these projections, $(S^V)^{\bullet}$ and X are viewed as functors defined on $\hat{\mathscr{C}}$ and $\hat{\mathscr{C}}_V$. In the middle, $(S^V)^{\bullet} \times I$ is viewed as a functor defined on $\hat{\mathscr{C}}_V$ via its projection to $\hat{\mathscr{K}}_V$. At time $t = 0$, this agrees with the action defined via the projection to $\mathscr{F}$. The maps i_0 and i_1 denote the restriction of the homotopy displayed in the middle to times $t = 0$ and $t = 1$, and the map denoted $\subset$ is induced by the inclusions ${}^{\bullet}(S^V) \subset (S^V)^{\bullet}$. These five arrows are all weak equivalences of orthogonal spectra, the last because the inclusion of a finite wedge in a finite product of orthogonal spectra is a weak equivalence.

Note the conflict here between the categories used in the top five bar constructions and the monads $\hat{\mathbb{C}}_V$ used at the bottom. The third through sixth bar constructions all use the categories $\hat{\mathscr{C}}_V$ and not their associated monads. Everything so far works equally well equivariantly. There is flab in these bar constructions which is related to basepoints and visible to the eyes of the category Π. We may want to reduce the categorical bar constructions to eliminate flab and make them more resemble the monadic construction at the bottom.

We must still construct the bottom arrow, and here we must recall how to relate the categories $\hat{\mathscr{C}}_V$ to the monads $\hat{\mathbb{C}}_V$. Either way, the action on X is induced in the evident way from the projection $s\hat{C}_V \longrightarrow \mathscr{F}$. Similarly, the contravariant action of $\hat{\mathscr{C}}_V$ on ${}^{\bullet}(S^V)$ and the right action of the monad $\hat{\mathbb{C}}$ on the functor $\Sigma^V L$ are induced from the projection $\mathscr{C}_V \longrightarrow \mathscr{K}_V$.

We certainly have maps

$$\bigvee_{\phi\colon \underline{m}\longrightarrow \underline{n}} \hat{\mathscr{C}}_V(\underline{m},\underline{n}) \times X_m \longrightarrow X_n$$

for all m and n coming from the projections of the $\hat{\mathscr{C}}_V$ to $\hat{\mathscr{C}}$ and the given action of $\hat{\mathscr{C}}$ on X. We can replace X by $\hat{\mathscr{C}}_V P$ for any Π-space P and still have such maps, since the category $\hat{\mathscr{C}}_V$ acts on the Π-space $\hat{\mathscr{C}}_V P$. Iterating and ignoring the left-hand variable, we almost have a map of simplicial spaces that gives the bottom arrow. However, we must reconcile the contravariant functor ${}^{\bullet}\underline{S}$ that we see at $t = 1$ with the $\hat{\mathbb{C}}_{\bullet}$-functor $\Sigma^{\bullet} L$. We construct the following monstrous diagram.

$$\begin{tikzcd}
\bigvee_{m,n} {}^{n}(S^V) \times \hat{\mathscr{C}}_V(\underline{m},\underline{n}) \times P_m \arrow[rr, "\hat{\gamma}_1 \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \delta \times \mathrm{id}"'] & & \bigvee_m {}^{m}(S^V) \times P_m \arrow[d, "\mathrm{id} \times \delta"] \\
\bigvee_{m,n} {}^{n}(S^V) \times \hat{\mathscr{C}}_V(\underline{m},\underline{1})^n \times P_m \arrow[d, "\nu \times \mathrm{id}"'] & & \bigvee_m {}^{m}(S^V) \times P_1^m \arrow[d, "\nu"] \\
\bigvee_{m,n} {}^{n}(S^V \times \hat{\mathscr{C}}_V(\underline{m},\underline{1})) \times P_m \arrow[d, "\nabla \times \mathrm{id}"'] & & \bigvee_m {}^{m}(S^V \times P_1) \arrow[d, "\nabla"] \\
\bigvee_m S^V \times \mathscr{C}_V(m) \times P_m \arrow[r, "\mathrm{id} \times \delta"] \arrow[uuurr, "\hat{\gamma}_1 \times \mathrm{id}"] \arrow[d] & \bigvee_m S^V \times \mathscr{C}_V(m) \times P_1^m \arrow[uur, "\hat{\gamma}_1 \times \mathrm{id}"] \arrow[d] & S^V \times P_1 \arrow[d] \\
\Sigma^V L \hat{\mathbb{C}}_V P \arrow[r, "\Sigma^V L \hat{\mathbb{C}}_V \delta"'] & \Sigma^V C_V L P \arrow[r, "\tilde{\alpha}"'] & \Sigma^V L P = \Sigma^V P_1
\end{tikzcd}$$

The bottom three unlabelled arrows are simply quotient maps sending in the building blocks of the three constructions on the bottom. The functor Σ^V is $S^V \wedge (\mathrm{id})$ (except that we generally define it as $(\mathrm{id}) \wedge S^V$; we suppress the transposition here). The functor L send P to P_1. The monad $\mathbf{C}$ on based spaces X is built from the spaces $\mathscr{C}_V(m) \times X^m$. The spaces $\mathscr{C}_V(m)$ are the effective component of $\hat{\mathscr{C}}(\underline{m},\underline{1})$, and the monad $\hat{\mathbf{C}}_V$ evaluated on P has nth space built from the spaces $\hat{\mathscr{C}}(\underline{m},\underline{n}) \times P_m$. Applying L means taking $n = 1$. The term "effective component" has a precise technical meaning. A map $\phi\colon \underline{m} \longrightarrow \underline{n}$ in $\mathscr{F}$ is effective if $\phi^{-1}(\underline{0}) = \{0\}$ [2, 5.3], and the component of $\hat{\mathscr{C}}_V(\underline{m},\underline{1})$ indexed on the unique effective map $\underline{m} \longrightarrow \underline{1}$ is $\mathscr{C}_V(m)$.

The maps labelled δ are the Segal maps, which have ith component the map induced by the projection $\underline{n} \longrightarrow \underline{1}$ that sends i to 1 and j to 0 for $j \neq i$. In the upper left instance of δ we are ignoring the ineffective components of $\hat{\mathscr{C}}_V(\underline{m},\underline{n})$ for the moment [later we use the correct definition, without restriction, and deal with the ineffective components properly: they will give things that see $\mathscr{C}_V(q)$ for $q < m$, but there should be no problem with that. Surely some but not all of the products in sight should be smash products] Thus this map δ is written as if it sees only the effective component $\mathscr{C}_V(m)$ of $\hat{\mathscr{C}}_V(\underline{m},\underline{n})$, but dealing with the ineffective components should still land us where we want to land: just take account ofthe "correct" q, which is still an allowable choice for m.

The maps ∇ are fold maps, the identity on each wedge summand. The map ν on the right sends a point $(v_i, x_1, \cdots x_m)$ with v_i in the ith wedge summand and all x_j in P_1 to the point (v_i, x_i) in the ith wedge summand $\Sigma^V \times P_1$. The map ν on the left is defined in the same way.

I'm pretty sure the bottom right trapezoid does commute, by inspection of the definitions of $\hat{\gamma}_1$ and $\tilde{\alpha}$ above. This is a rewritten version of part of the diagram that I first thought of, but this version eliminates all consideration of anything but spaces: we can replace P_1 by any based space X.

The reason we can expect the top left part of the diagram to commute is the way that the morphism spaces $\hat{\mathscr{C}}_V(\underline{m},\underline{n})$ are built out of the spaces $\mathscr{C}_V(m)$, as we see in (1.5). The map δ has the effect of picking out the factors, its jth component projecting onto $\mathscr{C}_V(s_j)$. This is vague and wishy-washy, since we have to be more precise about the ineffective components — I don't know where to switch from $\hat{\mathscr{C}}_V(\underline{m},\underline{1})$ to $\mathscr{C}(m)$ — but it will work.

Maybe the point with effectiveness is to notice that the wedge summand indexed on $\phi\colon \underline{m} \longrightarrow \underline{n}$ is mapped to the something that wants to be indexed on the n-tuple of maps $\delta_i\phi\colon \underline{m} \longrightarrow \underline{1}$. By [2, 5.3-5.5], $\delta_i\phi$ is a composite $\varepsilon\pi$ where epz is ordered and effective and $\pi\colon \underline{m} \longrightarrow \underline{q}$ is a projection. When we get down to the bottom of the diagram, we should only see the effective part.

References

[1] B. Guillou and J.P. May. Permutative G-categories in equivariant infinite loop space theory. Preprint, 2012.

[2] J.P. May and R. Thomason. The uniqueness of infinite loop space machines. Topology 17(1978), 205-224.