# NOTES ON SEGAL'S MACHINE, 5

## Contents

## 1. The equivariant Segal infinite loop space machine; definitions

There are three variants of Segal's infinite loop space machine as originally developed by Segal [9] and Woolfson [12]. Later sources include Bousfield and Friedlander [2], working simplicially, Shimada and Shimakawa [10, 11], and Mandell, May, Schwede, and Shipley [5]. As far as we know, the only source in the literature that proves the crucial group completion property is Segal's original paper [9]. His proof is based on use of his original variant. As we shall see, that variant does not directly generalize to give genuine $G$-spectra. There is no proof in the literature that starts with the second or third variants, which are the ones treated in [2, 5, 10, 12] and, equivariantly, [11]. In particular, there is no published version of the equivariant Segal machine that considers the group completion property. To give the group completion property equivariantly and to prepare for our comparison with the May machine, we give a detailed exposition. This may also be helpful to the modern reader since the original sources can make for hard reading and are quite incomplete in some respects.

1.1. **The first definition of the Segal machine.** Let $\mathscr{F}$ be the opposite of Segal's category $\Gamma$. It is the category of finite based sets $\underline{n} = \{0, 1, \cdots, n\}$ with 0 as basepoint. The morphisms are the based maps. Let $\mathscr{T}$ be the category of nondegenerately based spaces and let $\mathscr{F}\mathscr{T}$ be the category of $\mathscr{F}$-spaces, namely functors $Y \colon \mathscr{F} \longrightarrow \mathscr{T}$, written $\underline{n} \mapsto Y_n$. The Segal maps $\delta_i \colon \underline{n} \longrightarrow \underline{1}$ in $\mathscr{F}$ send $i$ to 1 and $j$ to 0 for $j \neq i$. The Segal map $\delta \colon Y_n \longrightarrow Y_1^n$ has coordinates $\delta_i$. If $n = 0$, we interpret $\delta$ as the terminal map $Y_0 \longrightarrow *$. The "product map" $\phi_n \colon \underline{n} \longrightarrow \underline{1}$ sends $j$ to 1 for $1 \leq j \leq n$ and induces an "$n$-fold multiplication" $Y_n \longrightarrow Y_1$.

Let $G$ be any topological group. We are interested primarily in finite groups, but we want to understand where hypotheses on $G$ come into play. Subgroups of $G$ are understood to be closed.

We define an $\mathscr{F}$-$G$-space to be a functor $Y\colon \mathscr{F} \longrightarrow G\mathscr{T}$, where $G\mathscr{T}$ is the category of nondegenerately based $G$-spaces and based $G$-maps, with left action by $G$. Since the symmetric group $\Sigma_n$ is contained in $\mathscr{F}(\underline{n},\underline{n})$, $Y_n$ and $Y_1^n$ are $(G\times\Sigma_n)$-spaces. Let $\mathbb{F}_n$ be the family of subgroups $\Lambda$ of $G\times\Sigma_n$ such that $\Lambda\cap\Sigma_n=\{e\}$.

**Definition 1.1.** For a topological group $G$ and a family $\mathbb{F}$ of (closed) subgroups of $G$, a map $f\colon X\longrightarrow Y$ of $G$-spaces is a weak $\mathbb{F}$-equivalence if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all $H\in\mathbb{F}$. A map $f$ is a weak $G$-equivalence if $f^H$ is a weak equivalence for all $H\subset G$.

**Definition 1.2.** An $\mathscr{F}$-$G$-space $Y$ is *special* if $\delta\colon Y_n\longrightarrow Y_1^n$ is a weak $\mathbb{F}_n$-equivalence for all $n\geq 0$. It is *very special* if, in addition, $\pi_0(Y_1^H)$ is a group (necessarily abelian) under the induced product for each $H\subset G$; $Y$ is *reduced* if $Y_0$ is a point. A map $f\colon X\longrightarrow Y$ of $\mathscr{F}$-$G$-spaces is a *weak equivalence* if each $f_n$ is a weak $\mathbb{F}_n$-equivalence.

**Remark 1.3.** The unique map $\underline{0}\longrightarrow\underline{n}$ in $\mathscr{F}$ induces a $G$-map $Y_0\longrightarrow Y_n$. If $Y_0$ is $G$-contractible and these maps are $G$-cofibrations, then the quotient maps $q\colon Y_n\longrightarrow Y_n/Y_0\equiv\overline{Y}_n$ give a weak equivalence from $Y$ to the quotient $\mathscr{F}$-$G$-space $\overline{Y}$, which is reduced and is special or very special if $Y$ is special or very special. Thus there is little loss of generality if we restrict attention to reduced $\mathscr{F}$-$G$-spaces.

Let $\Delta$ be the usual simplicial category. A simplicial object is a contravariant functor defined on $\Delta$, and a cosimplicial object is a covariant functor. As we recall in §2.1, there is a canonical functor $F\colon\Delta^{op}\longrightarrow\mathscr{F}$. By pullback along $F$, a $G\mathscr{F}$-space $Y$ can be viewed as a simplicial $G$-space, and it has a geometric realization $|Y|$; we use the standard realization, taking degeneracies into account. If $Y$ is reduced, the evident $G$-map $Y_1\times I\longrightarrow|Y|$ factors through a natural $G$-map $\Sigma Y_1\longrightarrow|Y|$ with adjoint $\eta\colon Y_1\longrightarrow\Omega|Y_1|$. We shall give the essentially standard proof of the following result in §2.2.

**Proposition 1.4.** *If $Y$ is a reduced special $\mathscr{F}$-$G$-space, then $\eta\colon Y_1\longrightarrow\Omega|Y|$ is a group completion of Hopf $G$-spaces.*

From here, the Segal machine in its first avatar is constructed as follows [9, p. 295]. We work equivariantly and use a slight reformulation given in [8].[1] We have the smash product $\wedge\colon\mathscr{F}\times\mathscr{F}\longrightarrow\mathscr{F}$. It sends $(\underline{m},\underline{n})$ to $\underline{mn}$ and is strictly associative and unital using lexicographic ordering. The unit is $\underline{1}$. We shall later also use the wedge sum $\vee\colon\mathscr{F}\times\mathscr{F}\longrightarrow\mathscr{F}$ which sends $(\underline{m},\underline{n})$ to $\underline{m+n}$. In fact, $\mathscr{F}$ is bipermutative under this sum and product.

For a reduced special $\mathscr{F}$-$G$-space $Y$, we have the functor

$$Y\circ\wedge\colon\mathscr{F}\times\mathscr{F}\longrightarrow G\mathscr{T}.$$

For each $\underline{n}$, we have the $\mathscr{F}$-$G$-space $Y[n]$ that sends $\underline{m}$ to $Y(\underline{m}\wedge\underline{n})$; thus $Y[1]=Y$. Following Segal, define the classifying $\mathscr{F}$-$G$-space $\mathbb{B}Y$ to be the $\mathscr{F}$-$G$-space whose $n$th $G$-space is the geometric realization $|Y[n]|$. Iterating, define $\mathbb{B}^qY=\mathbb{B}(\mathbb{B}^{q-1}Y)$ for $q\geq 1$. We obtain a *naive* $G$-prespectrum $\mathbb{S}Y$ with $q$th space $\mathbb{S}_qY=(\mathbb{B}^qY)_1$ for $q\geq 1$. The $\mathscr{F}$-$G$-spaces $\mathbb{B}^qY$ for $q\geq 1$ are reduced and special; since $(\mathbb{B}^qY)_1$

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[1] Perversely, [8] takes $\Delta$ to be the opposite of the category everybody else calls $\Delta$.

is $G$-connected, they are very special. Proposition 1.4 implies that the structure maps $\mathbb{S}_qY \longrightarrow \Omega\mathbb{S}_{q+1}Y$, $q \geq 1$, are weak $G$-equivalences.

There are two alternative choices for the zeroth $G$-space $\mathbb{S}_0Y$. We can take it to be $\Omega|Y| = \Omega\mathbb{S}_1Y$, and then $\mathbb{S}Y$ is a naive $\Omega$-$G$-spectrum. With that choice, the group completion $\eta\colon Y_1 \longrightarrow \Omega BY$ is additional structure for the $\Omega$-$G$-spectrum $\mathbb{S}Y$. This choice was preferred in [8]. Alternatively, we can take $\mathbb{S}_0Y$ to be $Y_1$, in which case $\mathbb{S}Y$ is a naive positive $\Omega$-$G$-spectrum with the group completion $\eta\colon Y_1 \longrightarrow \Omega|Y|$ as its first structure map. We prefer the second choice as our guide in this paper.

With the first choice, the nonequivariant Segal machine plays a special role. As is proven in [8], any other infinite loop space machine that takes $\mathscr{F}$-spaces, or appropriate more general input, to $\Omega$-spectra and has a natural group completion map from $Y_1$ to its zeroth space is equivalent to the Segal machine. The proof makes essential use of the fact that Segal's machine produces $\mathscr{F}\mathscr{F}$-spaces, namely functors $\mathscr{F} \longrightarrow \mathscr{F}\mathscr{T}$.

This construction actually works for any topological group $G$, not necessarily finite. However, it does *not* work to construct *genuine* $G$-spectra from $\mathscr{F}$-$G$-spaces when $G$ is finite: there is no evident way to build in deloopings by non-trivial representations of $G$.

1.2. **The conceptual variant of Segal's machine.** Returning to $\mathscr{F}$-$G$-spaces, the more conceptual variants of Segal's machine do give genuine $G$-spectra. In principle, these variants make no use of the functor $F\colon \Delta \longrightarrow \mathscr{F}$. We follow the nonequivariant exposition of [5], but [1] has relevant equivariant details.

The category $G\mathscr{T}$ is enriched over $\mathscr{T}$ via the morphism spaces $G\mathscr{T}(X,Y)$ of based $G$-maps $X \longrightarrow Y$. It can be viewed as the $G$-fixed category of the $G$-category $\mathscr{T}_G$ with the same objects as $G\mathscr{T}$. The morphism $G$-spaces $\mathscr{T}_G(X,Y)$ consist of all maps $X \longrightarrow Y$, with $G$ acting by conjugation. We obtain an essentially small full topological subcategory $G\mathscr{W}$ of $G\mathscr{T}$ and an essentially small full topological $G$-subcategory $\mathscr{W}_G$ of $\mathscr{T}_G$ by restricting to those based $G$-spaces that are homeomorphic to finite $G$-CW complexes. We introduce a convenient general notation for functor categories.

**Notation 1.5.** For an essentially small topological $G$-category $\mathscr{D}_G$, let $\mathrm{Fun}(\mathscr{D}_G, \mathscr{T}_G)$ denote the category of continuous $G$-functors $Z\colon \mathscr{D}_G \longrightarrow \mathscr{T}_G$ and continuous natural transformations of $G$-functors between them. Thus $Z$ assigns a $G$-space $Z(X)$ to each object $X \in \mathscr{D}_G$ and it assigns based $G$-maps

$$Z\colon \mathscr{D}_G(X,Y) \longrightarrow \mathscr{T}_G(Z(X), Z(Y))$$

to each pair of objects $X, Y$. Together with the unit maps $S^0 \longrightarrow \mathscr{T}_G(Z(X), Z(X)$ that send 1 to the identity map, these maps must satisfy the evident unit and associativity axioms. If $G$ acts trivially on $\mathscr{D}_G$, we denote it by $\mathscr{D}$. Then $\mathrm{Fun}\,\mathscr{D}, \mathscr{T}_G$ is isomorphic to the category $\mathrm{Fun}\,\mathscr{D}, G\mathscr{T}$ of continuous functors $\mathscr{D} \longrightarrow G\mathscr{T}$ and continuous natural transformations.

**Definition 1.6.** A *$\mathscr{W}_G$-space* $Z$ is a continuous $G$-functor $\mathscr{W}_G \longrightarrow \mathscr{T}_G$. Regard $\mathscr{F}$ as a $G$-trivial $G$-category; it is both a subcategory of $G\mathscr{W}$ and a $G$-subcategory of $\mathscr{W}_G$, We have the functor categories $\mathrm{Fun}(\mathscr{F}, G\mathscr{T}) = \mathrm{Fun}(\mathscr{F}, \mathscr{T}_G)$ of $\mathscr{F}$-$G$-spaces and $\mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G)$ of $\mathscr{W}_G$-spaces. The inclusion $\mathscr{F} \subset \mathscr{W}_G$ induces a forgetful functor

$$\mathbb{U}\colon \mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G) \longrightarrow \mathrm{Fun}(\mathscr{F}, \mathscr{T}_G)$$

The functor $\mathbb{U}$ has a left adjoint prolongation functor

$$\mathbb{P}\colon \operatorname{Fun}(\mathscr{F}, \mathscr{T}_G) \longrightarrow \operatorname{Fun}(\mathscr{W}_G, \mathscr{T}_G).$$

There are also forgetful functors

$$\mathbb{U}_{\mathscr{C}}\colon \operatorname{Fun}(\mathscr{W}_G, \mathscr{T}_G) \longrightarrow \mathscr{C},$$

where $\mathscr{C}$ can be the category of $G$-prespectra, symmetric $G$-spectra, or orthogonal $G$-spectra, denoted $G\mathscr{P}$, $G\Sigma\mathscr{S}$, and $G\mathscr{I}\mathscr{S}$. Of course, nonequivariantly, Segal took $\mathscr{C}$ to be prespectra. We choose orthogonal $G$-spectra and we abbreviate $G\mathscr{I}\mathscr{S}$ to $G\mathscr{S}$. As recalled below, for $\mathscr{W}_G$-spaces, we automatically have structure maps [5, 2.13 and 4.9], so that there is no distinction between $G\mathscr{W}$-spaces and $G\mathscr{W}$-spectra. This is what leads to the functors $\mathbb{U}_{\mathscr{C}}$. The conceptual Segal machine on $G\mathscr{F}$-spaces is the composite $\mathbb{U}_{G\mathscr{S}} \circ \mathbb{P}$.

The functor $\mathbb{P}$ is a left Kan extension that is best viewed as a tensor product of functors. For $X \in G\mathscr{W}$, we have the contravariant functor $X^{\bullet}\colon \mathscr{F} \longrightarrow G\mathscr{T}$. Conceptually, it is the represented functor that sends $\underline{n}$ to the function space $\mathscr{W}(\underline{n}, X) \cong X^n$ with its induced action by $G$. By definition,

$$\mathbb{P}(Y)(X) = X^{\bullet} \otimes_{\mathscr{F}} Y.$$

Taking $X = \underline{n}$, the unit $\eta\colon Y \longrightarrow \mathbb{U}\mathbb{P}Y$ of the adjunction sends $Y_n$ to $\mathrm{id}_{\underline{n}} \times Y_n$; by Yoneda, $\eta$ is a natural isomorphism. For a $\mathscr{W}_G$-space $Z\colon \mathscr{W}_G \longrightarrow \mathscr{T}_G$, the counit $\varepsilon\colon \mathbb{P}\mathbb{U}Z \longrightarrow Z$ is given on $X \in G\mathscr{W}$ by the composites

$$\mathscr{W}_G(\underline{n}, X) \wedge Z(\underline{n}) \xrightarrow{Z \wedge \mathrm{id}} \mathscr{T}_G(Z(\underline{n}), Z(X)) \wedge Z(\underline{n}) \xrightarrow{eval} Z(X).$$

For an $\mathscr{F}$-$G$-space $Y\colon \mathscr{F} \longrightarrow G\mathscr{T}$, we write $Y_n$ for $Y(\underline{n})$ as before, but we follow the usual convention of abbreviating notation by writing $\mathbb{P}Y(X) = Y(X)$ for $X \in G\mathscr{W}$. For spaces $X, X' \in G\mathscr{W}$, the adjoint $X' \longrightarrow \mathscr{W}_G(X, X \wedge X')$ of the identity map on $X \wedge X'$ can be composed with $\mathbb{P}Y$ to obtain a map

$$X' \longrightarrow \mathscr{T}_G(Y(X), Y(X \wedge X').$$

Its adjoint is a map

$$Y(X) \wedge X' \longrightarrow Y(X \wedge X').$$

Applying 1-point compactification to real $G$-inner product spaces $V$, we obtain based $G$-spheres $S^V$. Taking $X' = S^W$ and $X = S^V$, the map just displayed gives the structure maps

$$\Sigma^W Y(S^V) \longrightarrow Y(S^{V \oplus W})$$

of the orthogonal $G$-spectrum $\mathbb{U}_{G\mathscr{S}}\mathbb{P}(Y)$.

**Remark 1.7.** Nonequivariantly, Segal [9, 3.2] states that, in our notation, the prespectrum $\mathbb{S}Y$ is isomorphic to the prespectrum with $q$th space $\mathbb{P}(Y)(S^q)$, which is our prespectrum $\mathbf{U}_{\mathscr{P}} \circ \mathbf{P}Y$. In his notation, he says this holds because "$X \otimes (Y \otimes A) = (X \wedge Y) \otimes A$", where $A$ is an $\mathscr{F}$-space and $X$ and $Y$ are based spaces. I can't make sense of that statement, since $Y \otimes A$ in his paper is just a space, not an $\mathscr{F}$-space, but the proof of [9, 3.7] is saying something. Maybe $Y \otimes \mathbb{P}(A)$ could mean tensoring over based spaces, and then applying $\mathbb{U}$ to give $Y \otimes A = \mathbb{U}(Y \otimes \mathbb{P}(A)$. Then the statement could mean

gap

$$(\mathbb{P}Y)(W \wedge X) \cong \mathbb{P}(\mathbb{U}((\mathbb{P}Y) \otimes W)(X) \quad \text{or just} \quad ((\mathbb{P}Y) \otimes W)(X)$$

But that doesn't look right to me. I would like to understand this.

This isomorphism would not make sense equivariantly [BUT IT DOES: it is just a statement about a naive $G$-spectrum which, if everything is correct, should be the underlying naive $G$-spectrum of a genuine $G$-spectrum]. However, Segal's key observation does make sense and reads as follows. We give the proof in §2.1.

**Proposition 1.8.** *For $\mathscr{F}$-$G$-spaces $Y$, there is a natural $G$-homeomorphism*

$$|Y| \longrightarrow (S^1)^\bullet \otimes_{\mathscr{F}} Y = Y(S^1).$$

1.3. **A factorization of Segal's conceptual machine.** Digression. Finite $G$-spaces $\mathscr{F}_G$ and $\mathscr{F} \subset \mathscr{F}_G \subset \mathscr{W}_G$. To be written. (No unresolved mathematics here.)

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G) \arrow[r, "\mathbb{U}"] & \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \arrow[r, "\mathbb{U}"] & \mathrm{Fun}(\mathscr{F}, \mathscr{T}_G)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{F}, \mathscr{T}_G) \arrow[r, "\mathbb{P}"] & \mathrm{Fun}(\mathscr{F}_G, \mathscr{T}_G) \arrow[r, "\mathbb{P}"] & \mathrm{Fun}(\mathscr{W}_G, \mathscr{T}_G)
\end{tikzcd}$$

1.4. **A homotopical variant of Segal's conceptual machine.** While $\mathbb{U}_{G\mathscr{S}}\mathbb{P}(Y)$ is the conceptually natural equivariant version of Segal's machine, the functor $\mathbb{P}$ is not so easy to analyze homotopically, at least before model theoretic cofibrant approximation. Therefore we fatten it up by considering the two-sided bar construction $B(X^\bullet, \mathscr{F}, Y)$ determined by the contravariant functor $X^\bullet$ and the covariant functor $Y$ defined on $\mathscr{F}$, following Woolfson [12].[2] We have an evident natural map

$$\varepsilon\colon B(X^\bullet, \mathscr{F}, Y) \longrightarrow X^\bullet \otimes_{\mathscr{F}} Y = Y(X).$$

It is the geometric realization of a map of simplicial spaces to the constant simplicial space at $Y(X)$. By standard properties of the bar construction, when $X = \underline{n}$ the domain is the $n$th space of the $\mathscr{F}$-space $B(\mathscr{F}, \mathscr{F}, Y)$, the target is isomorphic to $Y_n$, and $\varepsilon$ is an equivalence with a natural homotopy inverse $\eta\colon Y_n \longrightarrow B(\mathscr{F}(-,\underline{n}), \mathscr{F}, Y)$ given by the evident inclusion of zero simplices. We shall discuss model structures and prove the following result in §2.4.

**Lemma 1.9.** *If $Y$ is Reedy(?) cofibrant, then $\varepsilon$ is a levelwise weak equivalence of $\mathscr{W}_G$-spaces.*

*Proof.* This is our substitute for use of $\tau$ and fat geometric realization. □ gap

Letting $G$ be a compact Lie group and letting $X$ run through the $G$-spheres $S^V$, the bar construction gives an orthogonal $G$-spectrum, which we denote by $\mathbb{S}_G Y$. This gives a homotopically well-behaved variant of the conceptual version of Segal's machine, and it is our preferred version of the equivariant Segal machine. Since, equivariantly, we can no longer use the group completion property to see that $\mathbb{U}_{\mathscr{S}}\mathbb{P}(Y)$ is a positive $\Omega$-spectrum, we instead use the following result. It is the equivariant version of results in [2, 12] and will be proven in §2.3. It holds for any topological group $G$.

**Theorem 1.10.** *Let $Y$ be a special $\mathscr{F}$-$G$-space. If $f\colon A \longrightarrow X$ is a based map with cofiber $i\colon X \longrightarrow Cf$, where $A$ is $G$-connected (or, more generally, grouplike), then*

$$\begin{tikzcd}
B(A^\bullet, \mathscr{F}, Y) \arrow[r, "f_*"] & B(X^\bullet, \mathscr{F}, Y) \arrow[r, "i_*"] & B(Cf^\bullet, \mathscr{F}, Y)
\end{tikzcd}$$

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[2] Segal and Woolfson also use a different kind of fattening, denoted $\tau$ in [9, 12] where it is specified as a functor from simplicial spaces to simplicial spaces but is applied as if it had been specified as a functor from $\mathscr{F}$-spaces to $\mathscr{F}$-spaces. We subsume it in cofibrant approximation.

*is a fibration sequence of $G$-spaces; that is, the canical induced map from $B(A^\bullet, \mathscr{F}, Y)$ to the homotopy fiber of $i_*$ is a weak $G$-equivalence.*

**Remark 1.11.** This result is stated without proof by Shimakawa [11, p. 248], who says the proof is quite similar to the nonequivariant proof of [12, 1.7]. I believe that he is right about this, at least I see no problem on a first reading of [12], even if the group $G$ is arbitrary!

gap?

When $G$ is a compact Lie group, an orthogonal $G$-spectrum $E$ is said to be a positive $\Omega$-$G$-spectrum if the structure map $E(V) \longrightarrow \Omega^{W-V}EW$ is a weak equivalence of $G$-spaces whenever $V^G \neq 0$. These are the fibrant objects in the positive stable model structure on $G\mathscr{S}$ [4, §5].

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**Corollary 1.12.** *Let $Y$ be a reduced special Reedy(?) cofibrant $\mathscr{F}$-$G$-space. Then $\mathbb{S}_G Y$ is a positive $\Omega$-$G$-spectrum and, if $V^G \neq 0$, the structure $G$-map*

$$(\mathbb{S}_G Y)(S^0) \longrightarrow \Omega^V(\mathbb{S}_G Y)(S^V)$$

*is a group completion.*

gap

**Remark 1.13.** For finite $G$, Shimakawa [11, p. 248–250] gives an impenetrable proof of the first statement based on a result whose manifold theoretic proof he may not have understand; he took it from an unpublished preprint of Segal. We must find a better argument. If the previous remark is correct, then this is the first place that the assumption that $G$ is finite enters, and it seems possible that $G$ could be a compact Lie group and it would all still work.

*Proof of the second statement.* We may write $V = \mathbb{R} \oplus W$ and thus $S^V = S^1 \wedge S^W$. Consider the following diagram, in which we have implicitly identified the functor $\Omega^V$ with $\Omega\Omega^W$.

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[dd, "\eta"', "\simeq"] & \Omega|Y| \arrow[d, "\cong"] & \\
& \Omega Y(S^1) & \\
B((S^0)^\bullet, \mathscr{F}, Y) \arrow[r] & \Omega B((S^1)^\bullet, \mathscr{F}, Y) \arrow[u, "\simeq", "\varepsilon"'] \arrow[r, "\simeq"] & \Omega^V B((S^V)^\bullet, \mathscr{F}, Y)
\end{tikzcd}$$

check

The top horizontal arrow is a group completion by Proposition 1.4. The left map $\eta$ is a weak equivalence since $S^0 = \underline{1}$. The right vertical homeomorphism is given by Proposition 1.8 and the equivalence $\varepsilon$ is given by Lemma 1.9. The square commutes. The bottom composite is the structure $G$-map of interest, and it is the composite of a structure map that is a group completion and the loops on a structure map that is a weak $G$-equivalence by the first statement. □

## 2. The equivariant Segal infinite loop space machine; proofs

2.1. **The proof that $|Y|$ is $G$-homeomorphic to $\mathbb{P}Y(S^1)$.** Recall that the objects of $\Delta$ are the unbased finite sets $\mathbf{n} = \{0, 1, \cdots, n\}$ and its morphisms are the nondecreasing functions.

**Definition 2.1.** We define a functor $F\colon \Delta^{op} \longrightarrow \mathscr{F}$. On objects, $F\mathbf{n} = \underline{n}$. For a map $\phi\colon \mathbf{n} \longrightarrow \mathbf{m}$ in $\Delta$, define $F\phi\colon F\underline{m} \longrightarrow F\underline{n}$ by sending $i$ to $j$ whenever $\phi(j-1) < i \leq \phi(j)$, where $1 \leq j \leq n$, and sending $i$ to 0 if there is no such $j$. Thus

$$(F\phi)^{-1}(j) = \{i|\phi(j-1) < i \leq \phi(j)\} \ \text{ for } \ 1 \leq j \leq n.$$

**Remark 2.2.** Observe that if $\delta_i\colon \mathbf{n-1} \longrightarrow \mathbf{n}$ and $\sigma_i\colon \mathbf{n+1} \longrightarrow \mathbf{n}$, $0 \leq i \leq n$, are the standard "face and degeneracy maps" that skip or repeat $i$ in the target, then $F\delta_i\colon \underline{n} \longrightarrow \underline{n-1}$ is the ordered surjection that repeats $i$ and $F\sigma_i\colon \underline{n} \longrightarrow \underline{n+1}$ is the ordered injection that skips $i$. Note in particular that $F\delta_1 = \phi_2\colon \underline{2} \longrightarrow \underline{1}$.

To prove Proposition 1.8, we compare definitions. We have

$$|Y| = Y \otimes_\Delta \Delta \ \text{ and } \ Y(X) \equiv \mathbb{P}(Y)(X) = X^\bullet \otimes_{\mathscr{F}} Y.$$

We are interested in the case $X = S^1$, but we shall analyze the target in general. To aid in the comparison, we rewrite $|Y|$ as $\Delta \otimes_{\Delta^{op}} Y$. Here $\Delta$ on the left is the covariant functor $\Delta \longrightarrow \mathscr{U}$ that sends $\mathbf{n}$ to the standard topological simplex

$$\Delta_n = \{(t_1, \cdots, t_n)|0 \leq t_1 \leq \cdots \leq t_n \leq 1\}.$$

Nowadays it is more usual to use tuples $(s_0, s_1, \cdots, s_n)$ such that $0 \leq s_i \leq 1$ and $\sum_i s_i = 1$, but the formulae $s_i = t_{i+1} - t_i$ and $t_i = s_0 + \cdots s_{i-1}$ translate between the two descriptions. For $0 \leq i \leq n$, the face map $\delta_i\colon \Delta_n \longrightarrow \Delta_{n+1}$ and the degeneracy map $\sigma_i\colon \Delta_{n+1} \longrightarrow \Delta_n$ are given by

$$\delta_i(t_1, \cdots, t_n) = \begin{cases} (0, t_1, \cdots, t_n) & \text{if } i = 0 \\ (t_1, \cdots t_{i-1}, t_i, t_i, t_{i+1}, \cdots t_n) & \text{if } 0 < i < n \\ (t_1, \cdots, t_n, 1) & \text{if } i = n \end{cases}$$

$$\sigma_i(t_1, \cdots, t_n) = (t_1, \cdots, t_i, t_{i+2}, \cdots, t_n)$$

We think of the circle as $I/\{0, 1\}$. Map $\Delta_n$ to $(S^1)^n$ by sending $(t_1, \cdots, t_n) \in \Delta_n$ to $(t_1, \cdots, t_n) \in (S^1)^n$. Looking at the definition of the functor $F$, we see that this defines a map $\xi\colon \Delta \longrightarrow (S^1)^\bullet$ of cosimplicial spaces,[3] where $(S^1)^\bullet$ is a cosimplicial check space by pullback along $F$. Therefore $\xi$ induces a natural map

$$\xi_*\colon |Y| = \Delta \otimes_{\Delta^{op}} Y \longrightarrow (S^1)^\bullet \otimes_{\mathscr{F}} Y = Y(S^1).$$

Recall that every point of $|Y|$ is represented by a unique point $(u, y)$ such that $u \in \Delta_p$ is an interior point and $y \in Y_p$ is a nondegenerate point [6, 14.2]. Said another way, $|Y|$ is filtered with strata

$$F_p|Y| - F_{p-1}|Y| = (\Delta_p - \partial DE_p) \times Y_p - sY_{p-1},$$

where $sY_{p-1}$ denotes the union of the subspaces $s_i(Y_{p-1})$. We shall describe $Y(X)$ similarly for all $X \in G\mathscr{W}$, and we shall specialize to $X = S^1$ to see that $\xi_*$ is a natural homeomorphism.

To this end, we describe the structure of $\mathscr{F}$, partially following [8, §5].

**Definition 2.3.** Define the subcategory $\Pi$ of $\mathscr{F}$ by letting $\Pi$ have the same objects as $\mathscr{F}$ and those maps $\phi\colon \underline{m} \longrightarrow \underline{n}$ such that $|\phi^{-1}(j)| \leq 1$ for $1 \leq j \leq n$. A map $\pi \in \Pi$ is a projection if $|\pi^{-1}(j)| = 1$ for $1 \leq j \leq n$. A map $\iota \in \Pi$ is an injection if $|\iota^{-1}(0)| = \{0\}$. The permutations are the maps in $\Pi$ that are both injections and projections. A map $\varepsilon \in \mathscr{F}$ is *effective* if $\varepsilon^{-1}(0) = 0$ and an effective map $\varepsilon$ is *ordered* if $\varepsilon(i) < \varepsilon(j)$ implies $i < j$. An effective map is *essential* if it is surjective, that is if $|\varepsilon^{-1}(j)| \geq 1$ for $1 \leq j \leq n$.

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[3] Warning: we are thinking of both source and target as cosimplicial *unbased* spaces.

Observe that $\mathscr{F}$ is generated under the wedge sum and composition by $\Pi$ and the single product morphism $\phi_2 \colon \underline{2} \longrightarrow \underline{1}$.

**Lemma 2.4.** *A map* $\phi \colon \underline{m} \longrightarrow \underline{n}$ *in* $\mathscr{F}$ *factors as the composite* $\iota\varepsilon\pi$ *of a projection* $\pi$*, an essential map* $\varepsilon$ *and an injection* $\iota$*, uniquely up to permutation. That is, given two such decompositions of* $\phi$*, there are permutations* $\sigma$ *and* $\tau$ *making the following diagram commute.*

$$\begin{tikzcd}
& \underline{q} \arrow[r, "\varepsilon"] \arrow[dd, "\sigma"] & \underline{r} \arrow[dr, "\iota"] \arrow[dd, "\tau"] & \\
\underline{m} \arrow[ur, "\pi"] \arrow[dr, "\pi'"'] & & & \underline{n} \\
& \underline{q} \arrow[r, "\varepsilon'"'] & \underline{r} \arrow[ur, "\iota'"'] &
\end{tikzcd}$$

*Moreover, if* $\varepsilon \colon \underline{m} \longrightarrow \underline{n}$ *is effective, there is a permutation* $\nu \in \Sigma_m$ *such that* $\varepsilon \circ \nu$ *is ordered. If* $\varepsilon$ *is itself ordered, then* $\varepsilon\nu$ *is ordered if and only* $\nu$ *is in the subgroup* $\Sigma_{e_1} \times \cdots \times \Sigma_{e_n}$ *of* $\Sigma_m$*, where* $e_j = |\varepsilon^{-1}(j)|$ *for* $1 \leq j \leq n$*.*

*Proof.* The projection $\pi$ is determined up to order by which $i \geq 1$ in $\underline{m}$ are mapped to 0 in $\underline{n}$. The injection $\iota$ is determined up to order by which $j \geq 1$ in $\underline{n}$ are not in the image of $\phi$. Up to order, $\varepsilon$ is the wedge sum in $\mathscr{F}$ of the product maps $\phi_j : \underline{e_j} \longrightarrow \underline{1}$, where $e_j = |\phi^{-1}(j)|$ for those $j$ such that $1 \leq j \leq n$ and $\phi^{-1}(j)$ is nonempty. Up to permutation, these $e_j$ run through the numbers $|\varepsilon^{-1}(j)|$, $1 \leq j \leq r$. $\square$

The $G$-space $Y(X)$ is the quotient of $\coprod_{n \geq 0} X^n \times Y_n$ obtained by identifying $(\phi^*(v), y)$ with $(v, \phi_*(y)$ for all $\phi \colon \underline{m} \longrightarrow \underline{n}$, $v \in X^n$, and $y \in Y_m$. Here $\phi^*(v_1, \cdots, v_n) = (u_1, \cdots, u_m)$ where $u_i = v_{\phi(i)}$, with $u_i = *$ if $\phi(i) = 0$, and $\phi_*(y)$ is given by the covariant functoriality of $Y$. The image of $\coprod_{n \leq p} X^n \times Y_n$ is topologized as a quotient and denoted $F_pY(X)$, and $Y(X)$ is given the topology of the union of the $F_pY(X)$.

**Definition 2.5.** For an unbased $G$-space $U$, the configuration space $F(U, p) \subset U^p$ is the $G$-subspace of points $(v_1, \cdots, v_p)$ such that $v_i \neq v_j$ for $i \neq j$. For an $\mathscr{F}$-$G$-space $Y$, a point $y \in Y_p$, $p \geq 1$, is *degenerate* if it is in $\iota_*(Y_{p-1})$ for some injection $\iota \colon \underline{p-1} \longrightarrow \underline{p}$. Define $sY_{p-1}$ to be the subspace of degenerate points in $Y_p$.

**Lemma 2.6.** $F_0Y(X) = * \times Y_0$. *For* $p \geq 1$*, the stratum* $F_pY(X) - F_{p-1}Y(X)$ *is*

$$F(X - \{*\}, p) \times_{\Sigma_p} (Y - sY_{p-1}).$$

*Proof.* Using projections, every point of $\coprod_{n \geq 1} X^n \times Y_n$ is equivalent to a point $(v, y)$ such that either $n = 0$ or no coordinate of $v$ is the basepoint of $X$. Using injections, every point is equivalent to a point $(v, y)$ such that $y$ is nondegenerate. Using permutations and canonical maps $\phi_i \colon \underline{i} \longrightarrow \underline{1}$ when $i$ coordinates of $v$ are equal, every point is equivalent to a point $(v, y)$ such that $v$ has no repeated coordinates. We must take orbits under the action of $\Sigma_p$ as stated (or find a canonical way to choose orbit representatives) to avoid double counting of elements. Taking care of the order in which the cited operations are taken, the conclusion follows. $\square$

check

It is now easy to see that $\xi \colon |Y| \longrightarrow Y(S^1)$ is a homeomorphism.

*Proof of Proposition 1.8.* The points $(t_1, \cdots, t_p)$ of $F(S^1 - *, p) = F(I - \{0,1\}, p)$ can be ordered so that $0 < t_1 < \cdots < t_p < 1$, and $\xi$ maps $\Delta_p - \partial\Delta_p$ homeomorphically onto this space. The degeneracy subspaces $s(Y_{p-1}) \subset Y_p$ for $Y$ as an $\mathscr{F}$-$G$-space and $Y$ as a $\Delta^{op}$-$G$-space coincide. The gluing instructions for attaching the filtrations agree. □

2.2. **The proof of the group completion property.** Say that a Hopf space $X$ is *admissible* if it is homotopy associative and if left translation by any element is homotopic to right translation by the same element; of course, this holds if $X$ is homotopy commutative. A Hopf $G$-space $X$ is admissible if each $X^H$ is admissible.

**Definition 2.7.** A Hopf map $f\colon X \longrightarrow Y$ between admissible Hopf spaces is a *group completion* if $\pi_0(Y)$ is a group (necessarily abelian), $f_*\colon \pi_0(X) \longrightarrow \pi_0(Y)$ is the group completion of the commutative monoid $\pi_0(X)$, and, for every commutative ring $R$ or equivalently every field $R$, the map $f_*\colon H_*(X;R) \longrightarrow H_*(Y;R)$ is algebraic localization at the multiplicative submonoid $\pi_0(R)$. Thus $H_*(Y;R)$ is the localization $H_*(X;R)[\pi_0(X)^{-1}]$. A map $f$ of Hopf $G$-spaces is a group completion if its fixed point maps $f^H$ are all group completions.[4]

Let $Y$ be a reduced special $\mathscr{F}$-$G$-space. Observe that all subgroups $H$ of $G$ are in the family $\mathbb{F}_n$ of subgroups of $G \times \Sigma_n$ since $H \cap \Sigma_n = e$. Therefore the Segal maps

$$\delta^H\colon Y_n^H \longrightarrow (Y_1^n)^H = (Y_1^H)^n$$

are weak equivalences and $Y^H$ is a nonequivariant reduced special $\mathscr{F}$-space. The full strength of the assumption that the Segal maps are weak $\mathbb{F}_n$-equivalences plays no role in the work of this section.

It is convenient but not essential to modify the definition of a special $\mathscr{F}$-$G$-space by requiring the Segal maps $\delta$ to be $G$-homotopy equivalences rather than just weak $G$-equivalences, and then their fixed point maps $\delta^H$ are also homotopy equivalences. We give $Y_1$ a Hopf $G$-space structure by choosing a homotopy inverse when $n = 2$ and using $\phi_2$. Then $Y_1$ and each $Y_1^H$ is admissible. We could instead work with weak Hopf $G$-spaces, but doing so explicitly only obscures the exposition.

We must prove that the canonical map $\eta\colon Y_1 \longrightarrow \Omega|Y|$ is a group completion. Geometric realization commutes with passage to $H$-fixed points. It also commutes with taking loops since $G$ acts trivially on $S^1$. Thus the equivariant case of Proposition 1.4 follows directly from the nonequivariant case. We therefore take $G = e$ and ignore equivariance in the rest of this section.

If $M$ is a topological monoid, we use its product to define a simplicial space $B_*M$ with $B_nM = M^n$. Then $|B_*M|$ is just the classical classifying space $BM$. When $M$ is commutative, $B_*M$ is the simplicial space obtained by pullback of the evident reduced special $\mathscr{F}$-space with $n$th space $M^n$. When $Y$ is a special $\mathscr{F}$-space, its first space $Y_1$ plays the role of $M$. Assuming that $Y$ is reduced ensures that $Y_1$ has a specified unit element $e$. Spaces of the form $Y_1$ for a reduced special $\mathscr{F}$-space $Y$ give the Segalic version of an $E_\infty$-space.

It makes sense to ask that a simplicial space $Y$ be reduced and special, since we can use iterated face maps to define Segal maps $Y_n \longrightarrow Y_1^n$. The Segal maps

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[4] Segal [9, §4] describes the nonequivariant notion of group completion a bit differently, in a form less amenable to equivariant generalization, and he makes several reasonable restrictive hypotheses in his proof of the group completion property. In particular, he assumes that $Y_1$ is a topological monoid and $\pi_0(Y_1)$ contains a cofinal free abelian monoid.

of $\mathscr{F}$-spaces are the images under $F$ of these more general Segal maps. Then $Y_1$ is a Hopf space with product induced by a homotopy inverse to the second Segal map and $d_1\colon Y_2 \longrightarrow Y_1$. When $Y$ is an $\mathscr{F}$-space, $\phi_2 = Fd_1$. Spaces of the form $Y_1$ for a reduced special simplicial space $Y$ give the Segalic version of an $A_\infty$-space. The group completion theorem for reduced special $\mathscr{F}$-spaces is a special case of the following more general group completion theorem.

**Theorem 2.8.** *If $Y$ is a reduced special simplicial space such that $Y_1$ and $\Omega|Y_1|$ are admissible Hopf spaces, then $\eta\colon Y_1 \longrightarrow \Omega|Y_1|$ is a group completion.*

This result was proven but not stated in [7, §15]. The result actually stated there, [7, 15.1], is the case when $Y_n = M^n$ for a topological monoid $M$. However, the proof is given in the generality of Theorem 2.8. To make that clear, we summarize the argument, referring to [7] for details.

Since $\pi_1(|Y|)$ is an abelian group, it is isomorphic to $H_1(Y;\mathbb{Z})$. Now using the Künneth theorem and inspection, we can check that $\pi_0(\Omega|Y|)$ is the group completion of $\pi_0(Y)$ by the chain level argument used in [7, 15.2].

By [7, 15.3], there is an adjunction $(T,S)$ relating the category of reduced special simplicial spaces, denoted $\mathscr{S}^+\mathscr{T}$ there, to the category $\mathscr{T}$. The functor $T = |-|$ is geometric realization. The functor $S$ is a reduced version of the total singular complex. For a based space $X$, $S_pX$ is the set of $p$-simplices $\Delta_p \longrightarrow X$ that map all vertices to the basepoint. In particular, $S_1X = \Omega X$. Let $\phi\colon TSX \longrightarrow X$ and $\psi\colon Y \longrightarrow STY$ be the unit and counit of the adjunction. Then [7, 15.5] gives the following result.

**Proposition 2.9.** *If $X \in \mathscr{T}$ is path connected, then $\phi\colon TSX \longrightarrow X$ is a weak equivalence. For any $Y \in \mathscr{S}^+\mathscr{T}$, $T\psi\colon TY \longrightarrow TSTY$ is a weak equivalence.*

From here, the main tool is the standard homology spectral sequence of the filtered space $|Y|$. We take coefficients in a field $R$. Then, using the Künneth theorem and the fact that $Y$ is special, we see that $E^1Y$ is the algebraic bar construction on $H_*(Y_1)$, so that $E^2_{p,q}Y = \mathrm{Tor}^{H_*(Y_1)}_{p,q}(R,R)$. Clearly $E^2_{0,0}Y = R$ and $E^2_{0,q}Y = 0$ for $q > 0$. The spectral sequence converges to $H_*(|Y|)$. We have the analogous spectral sequence for $STY$. The idea is to apply an appropriate version of the comparison theorem for spectral sequences, [7, 15.6], to the map of spectral sequences induced by the map of simplicial spaces $\psi\colon Y \longrightarrow STY$. On 1-simplices, $\psi_1 = \eta\colon Y_1 \longrightarrow \Omega|Y|$ and therefore $E^2\psi = \mathrm{Tor}^{\eta_*}(\mathrm{id},\mathrm{id})$. The map $\{E^r\psi\}$ of spectral sequences converges to the weak equivalence $TY \longrightarrow TSTY$. Therefore $E^\infty\psi$ is an isomorphism.

Write $A = H_*(Y_1)$ and let $\iota\colon A \longrightarrow \overline{A}$ be its localization at the monoid $\pi_0(Y_1)$. Write $B = H_*(\Omega|Y|)$ and let $\zeta\colon \overline{A} \longrightarrow B$ be the map of $R$-algebras such that $\zeta\circ\iota = \eta_*$; it is given by the universal property of localization. We must prove that $\zeta$ is an isomorphism. It is a classical algebraic result [3, VI 4.1.1] that

$$Tor^\iota(\mathrm{id},\mathrm{id})\colon\ \mathrm{Tor}^A(R,R) \longrightarrow Tor^{\overline{A}}(R,R)$$

is an isomorphism in our situation. Therefore we can identify $E^2\psi$ with

$$Tor^\zeta(\mathrm{id},\mathrm{id})\colon\ \mathrm{Tor}^{\overline{A}}(R,R) \longrightarrow Tor^B(R,R).$$

The rest of the argument is given in detail in [7, p. 93]. Both $\overline{A}$ and $B$ are the tensor products of their identity components with the group ring $R[\pi_0(\Omega|Y|)]$. The cited version of the comparison theorem shows how to prove that $E^2_{1,*}\psi$ is an isomorphism and $E^2_{2,*}\psi$ is an epimorphism. For connected graded algebras $A$, $\mathrm{Tor}^A_{1,*}$ and $\mathrm{Tor}^A_{2,*}$ compute the generators and relations of $A$. Now the detailed argument of [7, p. 93] proves by induction on degree that $\zeta$ is an isomorphism between the identity components of $A$ and $B$ and is therefore an isomorphism.

2.3. **The conversion of cofibration sequences to fibration sequences.** Follow Woolfson. His description of $B(X^\bullet, \mathscr{F}, Y)$ as the classifying space of a category does seem convenient here, but where it plays a real role, in the proof of [12, 1.8], is where things most need checking.

2.4. **The Reedy? model structure on $G\mathscr{F}$-spaces.** This may be in Rekha's paper and is what you and Emily were or should have been discussing.

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