NOTES ON SEGAL'S MACHINE, 3

There is no treatment of the equivariant Segal machine in the literature that considers the group completion property, as far as I know. To test understanding, I'm going to bang out notes about how to think about the nonequivariant Segal machine, in hopes of clarification.

In its most conceptual form, I think of it in the context of MMSS, whose notation I will follow (Mandell, May, Schwede, Shipley). Consider $\mathscr{F} \subset \mathscr{W}$, where $\mathscr{W}$ is the category of based spaces homeomorphic to finite CW complexes. The inclusion induces a forgetful functor

$$\mathbb{U} \colon \mathscr{W}\mathscr{T} \longrightarrow \mathscr{F}\mathscr{T}$$

from the category of $\mathscr{F}$-spaces to the category of $\mathscr{W}$-spaces. That functor has a left adjoint prolongation functor

$$\mathbb{P} \colon \mathscr{F}\mathscr{T} \longrightarrow \mathscr{W}\mathscr{T}$$

There are also forgetful functors

$$\mathbb{U}_? \colon \mathscr{W}\mathscr{T} \longrightarrow ?$$

where ? can be the category of prespectra, symmetric spectra, or orthogonal spectra ($\mathscr{P}$, $\Sigma\mathscr{S}$, $\mathscr{I}\mathscr{S}$). Of course, Segal took ? to be prespectra. We choose orthogonal spectra. Note that for $\mathscr{F}$-spaces and $\mathscr{W}$-spaces, we automatically have structure maps [**?**, 2.13 and 4.9], so we don't need to worry about the distinction between diagram spaces and diagram spectra in these cases. The Segal machine on $\mathscr{F}$-spaces is the composite $\mathbb{U}_? \circ \mathbb{P}$.

Let me explain the functor $\mathbb{P}$ in terms closer to those we have been thinking about. For $X \in \mathscr{W}$, we have the contravariant functor $X^\bullet \colon \mathscr{F} \longrightarrow \mathscr{T}$. Conceptually, it is the represented functor that sends $\mathbf{n}^+$ to the function space $\mathscr{W}(\mathbf{n}^+, X)$. For an $\mathscr{F}$-space Y, I claim that

$$\mathbb{P}(Y)(X) = X^\bullet \otimes_{\mathscr{F}} Y.$$

Taking $X = \mathbf{n}^+$, the unit $\eta \colon Y \longrightarrow \mathbb{U}\mathbb{P}Y$ of the adjunction sends $Y(\mathbf{n}^+)$ to $\mathrm{id} \times Y(\mathbf{n}^+)$ and, by Yoneda, η is a natural isomorphism. For a $\mathscr{W}$-space Z, the counit $\varepsilon \colon \mathbb{P}\mathbb{U}Z \longrightarrow Z$ is given on the space X by the composites

$$\mathscr{W}(\mathbf{n}^+, X) \wedge Z(\mathbf{n}^+) \xrightarrow{Z \wedge \mathrm{id}} \mathscr{T}(Z(\mathbf{n}^+), Z(X)) \wedge Z(\mathbf{n}^+) \xrightarrow{eval} Z(X).$$

Henceforward, we write Y_n for $Y(\mathbf{n}^+)$.

For an $\mathscr{F}$-space Y with prolonged $\mathscr{W}$-space $\mathbb{P}Y$ and for spaces X and X' in $\mathscr{W}$, the identity map on $X \wedge X'$ has adjoint $X' \longrightarrow \mathscr{W}(X, X \wedge X')$, which can be composed with Y to obtain $X' \longrightarrow \mathscr{T}(Y(X), Y(X \wedge X'))$. Its adjoint is a map

$$Y(X) \wedge X' \longrightarrow Y(X \wedge X')$$

Taking $X = S^V$ and $X' = S^W$, this gives the structure maps

$$\Sigma^W Y(S^V) \longrightarrow Y(S^{V \oplus W})$$

of the prespectrum $\mathbb{U}_{\mathscr{P}}\mathbb{P}(Y)$.

That was the conceptual version. Note that it makes no use whatsoever of the forgetful functor from $\mathscr{F}$-spaces to simplicial spaces. In the simplicial world, this is exactly the Bousfield-Friedlander construction. However, the functor $\mathbb{P}$ is not so easy to analyze homotopically (at least before model theoretic cofibrant approximation), so we fatten it up by considering the bar construction and the evident natural map

$$B(X^{\bullet}, \otimes_{\mathscr{F}}, Y) \longrightarrow X^{\bullet} \otimes_{\mathscr{F}} Y = \mathbb{P}(Y)(X).$$

When $X = \mathbf{n}^+$, this map is an equivalence by standard properties of the bar construction. According to [Woolfson, Thm 1.5], this bar construction is homotopy equivalent (he doesn't say homeomorphic, even though he writes an equal sign) to what Segal calls $X \otimes Y$, which is our $\mathbb{P}(\tau Y)(X)$, where τ is Segal's fattening functor on $\mathscr{F}$-spaces. That is, the bar construction is a conceptual variant of one version of Segal's machine.

It seems to me that the comparison map in the previous paragraph should be an equivalence whenever Y is cofibrant in the right model structure, presumably Rekha's, and that should be a way to understand and get around anything like the Woolfson comparison proof.

There is also an equivalent, more calculationally accessible, version of Segal's machine, which is his first construction [Segal, p. 295]. For that, consider the smash product $\wedge\colon \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}$. This is strictly associative and unital using lexicographic ordering. The unit is $\mathbf{1}^+$. For an $\mathscr{F}$-space Y, we have the functor $Y \circ \wedge\colon \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{T}$. For each $\mathbf{n}^+$, we have the $\mathscr{F}$-space $Y[n]$ that sends $\mathbf{m}^+$ to $Y(\mathbf{m}^+ \wedge \mathbf{n}^+)$. Segal defines the classifying space BY to be the $\mathscr{F}$-space whose nth space is the geometric realization $|Y[n]|$, where the realization is defined using the canonical functor $\Delta^{op} \longrightarrow \mathscr{F}$. He then defines a prespectrum $\mathbf{B}Y$ with nth space the iterated classifying space B^nY. According to Segal's [**?**, 3.2], the prespectrum $\mathbf{B}Y$ is equivalent to the prespectrum with nth space his $S^n \otimes Y$, which is the prespectrum $\mathbf{U}_{\mathscr{P}} \circ \mathbf{P}$. By [Woolfson, Thm 1.5], this in turn is equivalent to the prespectrum with nth space $B((S^n)^{\bullet}, \mathscr{F}, Y)$.

It would be useful to have a direct proof that $B((S^n)^{\bullet}, \mathscr{F}, Y)$ is weakly equivalent $\mathbf{B}Y$. Perhaps the evident all at once construction of B^nY as the (diagonal) realization of the evident n-simplicial space would help make this clear. However, equivariantly, we have no interest in the iteration and will focus on the first space.

So think about the first space $Y[1] = |Y|$ of BY. If G is a topological group and we look at $Y = G^{\bullet}$ as a Δ^{op}-space using its product, then $|Y|$ is just the classical classifying space BG. When Y is an $\mathscr{F}$-space regarded as a simplicial space, its first space Y_1 plays the role of G, and if Y is special then Y_n is equivalent to $(Y_1)^n$. With the evident product, Y_1 is an H-space (it is convenient to require Y_0 to be a point to see the unit), and spaces of the form Y_1 for an $\mathscr{F}$-space Y give the Segalic version of an E_∞-space. Ignoring $\mathscr{F}$ and just looking at Y as a simplicial space, its first space is a homotopy commutative Segalic A_∞-space. These are considered in [**?**, §15], where it is proven but not stated that the canonical map $Y_1 \longrightarrow \Omega BY$ is a group completion and thus a weak equivalence if Y_1 is connected. This gives the group completion property for the Segal machine, and, since BY is connected, it also shows that the Segal machine delivers an almost Ω-prespectrum. Probably the last can be seen directly from the conceptual construction, but I don't see the group completion property that way.

In my first note in this series, I wrote: Question: Ignoring group completion, do you know a good proof from the conceptual definition that we do get an almost Ω-prespectrum? In my second note, I wrote about that, so let's recall how that goes.

Woolfson implicitly proves almost Ω in his Theorem 1.7, which says that the prolonged functor $Y \equiv \mathbb{P}Y$ turns cofibration sequences into fibration sequences (provided the domain space is connected and Y is special). Apply that to

$$S^n \longrightarrow D^{n+1} \longrightarrow S^{n+1}$$

for $n > 0$. The fibration sequence

$$Y(S^n) \longrightarrow Y(D^{n+1}) \longrightarrow Y(S^{n+1})$$

implies that $Y(S^n)$ is equivalent to $\Omega Y(S^{n+1})$ and a diagram chase from the definition of the structure maps in your note shows that the equivalence is given by the adjoint structure maps. The argument in Bousfield-Friedlander (Lemma 4.1 and Theorem 4.4) works similarly and goes just a bit further to deal with the very special case. I imagine that Shimakawa works similarly but haven't looked yet.

While the iterates B^nY are not useful equivariantly, throwing away B leaves the equivariant group completion property in limbo for Segal's machine. Probably a comparison that only works at "the bottom levels", with no iteration, suffices. That is, we consider those V with $V^G \neq 0$, so that we can identify V with $\mathbb{R} \oplus W$ for some W. Then we can construct a functor $(BY)(V)$ that depends on the $\mathbb{R}$ coordinate to build the classifying space as a realization and is equivalent to $Y(S^V)$. The equivalence should be the hard work and should amount to the bottom step, without iteration, of what is in Segal and Woolfson (where of course the details need careful parsing). Ideally, we would like the proof not to be the same, but rather a direct comparison with the bar construction version, without introduction of Segal's τ. Under the equivalence, the adjoint structure map should factor as the composite of a group completion

$$Y_1 \longrightarrow \Omega BY(\mathbb{R})$$

and Ω applied to an equivalence

$$BY(\mathbb{R}) \longrightarrow \Omega^W BY(\mathbb{V}).$$

This equivalence should be given by application of the almost Ω-spectrum structure. If that makes sense to you, I leave it to you to start on filling in the gaps here and in the sketch below that I sent you yesterday. This will amount to a reworking of Segal's machine equivariantly that, for the first time, includes the group completion property (without which the machine is useless for equivariant algebraic K-theory!)

Note that BY makes perfect sense, without qualification, and $Y_1 \longrightarrow \Omega BY$ should be a group completion, without qualification. Then BY should be equivalent to $B((S^1)^\bullet, \mathscr{F}, Y)$. Maybe we want to assume that Y is cofibrant and use the actual prolongation in the proof. There is also an evident natural inclusion of Y, regarded as a simplicial space, into the simplicial bar construction: just use base points in the circle coordinates and identity maps in the $\mathscr{F}$ coordinates. I don't see why it should be an equivalence though, and maybe something closer to the Segal-Woolfson route is necessary (shudder): certainly the literature is lousy. Then the almost Ω theorem should kick in to give that

$$B((S^1)^\bullet, \mathscr{F}, Y) \longrightarrow \Omega^W B((S^V)^\bullet, \mathscr{F}, Y)$$

is an equivalence, as I said With this route, it looks to me as if the work remaining here is to clarify the Segal-Woolfson equivalence and the group completion $Y_1 \longrightarrow \Omega BY$. This looks almost like too little work, but maybe I'm missing something (besides details).

Equivariantly, I prefer to use $\mathscr{F}$, reserving $\mathscr{F}_G$ and $G\mathscr{F}$ for emergencies. I want to use $G\mathscr{T}$ as the target category of my $\mathscr{F}$-spaces and $G\mathscr{W}$ spaces, where $G\mathscr{W}$ is the full subcategory of based G-spaces homeomorphic to finite G-CW complexes. All of [**?**] adapts equivariantly, and the conceptual description of Segal's machine works verbatim. The bar construction also works verbatim, and should give a homotopically more manageable version of the equivariant Segal machine. The definition of the $Y[n]$ also works just as well. However, it is no longer clear how valuable the space BY and its iterates B^nY are to us. A priori, these just seem to give a naive G-prespectrum. However, an easy commutation with fixed points argument does show that $Y_1 \longrightarrow \Omega BY$ is an equivariant group completion.

References

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