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DOUBLE CATEGORIES AND BASE CHANGE IN HOMOTOPY THEORY

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For my mother

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ABSTRACT

We begin a combination of three existing theories: abstract homotopy theory, monoidal category theory, and indexed category theory. The first studies categories with weak equivalences, such as topological spaces and chain complexes, and their homotopy categories and derived functors. The second studies categories with a product, such as the smash product of based spaces or the tensor product of modules. The third studies families of categories indexed by objects of a base category, such as modules indexed by rings or bundles indexed by their base spaces.

Many situations in mathematics include two, or all three, of these structures. Based spaces or spectra with their smash products, and chain complexes with their tensor product, are examples of monoidal homotopy theories. Modules over rings, sheaves over spaces, and diagrams over enriched categories are examples of indexed monoidal categories. Sectioned spaces, parametrized spectra, chain complexes of sheaves, and homotopical diagrams are examples of indexed homotopy theories—and most of them also have a monoidal structure. Despite this ubiquity, however, 'indexed monoidal abstract homotopy theory' does not seem to have been widely studied.

Interesting phenomena already appear when we combine our three theories in pairs. Monoidal abstract homotopy theory is fairly well studied. But indexed abstract homotopy theory has to deal with derived base change functors, frequently including composites of left and right derived functors, which the standard technology is often insufficient to deal with. And in addition to a naive notion of indexed monoidal category, there is another sort of 'indexed monoidal structure' analogous to the tensor product of bimodules.

We deal with both of these subtleties by using double categories. Firstly, we show that passage to derived functors is a 'double pseudofunctor,' providing a general framework to compare composites of left and right derived functors. This is also useful in monoidal abstract homotopy theory, once we move beyond monoidal homotopical categories to consider monoidal derived functors and derived enriched categories. Secondly, we show that bimodule-like tensor products can be modeled by a special sort of double category which we call a framed bicategory. In good cases, a naive indexed monoidal category gives rise to a framed bicategory, but not all framed bicategories arise in this way; another important construction consists of monoids and bimodules in some other framed bicategory.

Finally, we combine all three theories to obtain indexed monoidal abstract homotopy theory, and in particular a homotopical theory of framed bicategories. This is not hard for framed bicategories arising from naive indexed monoidal categories, but for those composed of monoids and bimodules, we also need a bar construction to compose bimodules homotopically. We give a detailed analysis for distributors (bimodules between enriched categories), with applications to homotopy limits; the general case is left for future work.

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CHAPTER 1
INTRODUCTION

The goal of this thesis is to begin the study of what we call *indexed monoidal abstract homotopy theory*. The meaning of 'abstract homotopy theory' is fairly well-established by now; it can be called the study of 1-categorical presentations of $(\infty, 1)$-categories (aka 'homotopy theories', cf. [Lur07, Rez01, Ber]). By the addition of 'indexed' we mean to indicate the study of *families* of homotopy theories indexed by some base category, such as parametrized spectra [MS06] or homotopy sheaves [Jar87] indexed by their base spaces, or enriched diagrams indexed by their domain categories. And, of course, the word 'monoidal' indicates the presence of a multiplication on the categories in question.

The combination of these three ideas involves a couple of important new ideas, which appear already when we combine them in pairs. Monoidal abstract homotopy theory is fairly well-understood, at least in the context of model categories; see for instance [Hov99, ch. 4]. But *indexed* abstract homotopy theory seems to be less well-studied. This may be due partly to the complexity of the relationship between left and right derived functors, since 'base change' functors in indexed category theory come in left and right versions. Finally, indexed monoidal category theory also contains unexpected subtleties, even before the addition of homotopy theory: in addition to the naive notion of an indexed family of monoidal categories, there are other types of indexed monoidal structure requiring new categorical notions to properly describe them.

Our solutions to both of these issues involve *double categories*. A double category is much like a bicategory (or 2-category), except that it has two different types of arrows (or 1-cells), called 'vertical' and 'horizontal,' and its 2-cells are shaped like squares. While, as we have said, both solutions involve double categories, the particular double categories involved in the two cases are quite different.

The first flavor of double category is analogous to strict 2-categories such as the 2-category $\mathcal{C}at$ of categories, functors, and natural transformations. This sort of double category often arises when we have a collection of categories that admit two 'dual' types of functor between them. A good example is $\mathbb{M}\mathsf{onCat}$, which consists of monoidal categories, lax *and* oplax monoidal functors, and monoidal natural transformations. It turns out that this flavor of double category is perfectly adapted to describe the relationship between left and right derived functors: roughly speaking, 'left derivable' and 'right derivable' functors between presentations of homotopy theories form a double category of this sort.

The second flavor of double category is more analogous to bicategories such as the bicategory $\mathcal{M}od$ of rings, bimodules, and bimodule homomorphisms. In fact, the easiest example of this flavor of double category is $\mathbb{M}\mathsf{od}$, which extends $\mathcal{M}od$ by including ring homomorphisms as the second type of 1-cell. In such double categories, the two types of 1-cells are not 'dual' but fundamentally different. For example, in $\mathbb{M}\mathsf{od}$, the ring homomorphisms are truly 'morphisms' between the objects (rings), but the bimodules are a different sort of 'object' that just happen to be 'indexed' by pairs of rings. The composition of module-like 1-cells in such a double category supplies an appropriate definition of one important type of 'indexed monoidal structure.' Double categories of this sort, which satisfy a 'base change' property saying that the module-like 1-cells can be 'reindexed' along ring-homomorphism-like ones, we call *framed bicategories*.

The three-way combination (indexed monoidal abstract homotopy theory), however, presents new issues that have not been completely resolved. There are few problems adding homotopy theory to the naive notion of 'indexed monoidal category,' but the homotopical theory of framed bicategories presents new challenges, particularly relating to the construction of monoids and modules. We do not yet have a complete solution to this problem. Instead we study in more detail the particular example of enriched categories and distributors, which illustrates many aspects that are sure to feature in the general theory. In particular, we will see the importance of the *two-sided bar construction* in defining derived monoidal structures.

An outline of the thesis is as follows. In chapter 2 we review the definition of double categories and introduce the notions of *companion* and *conjoint*, due in this form to [DPP03]. Like the bicategorical notions of *equivalence* and *adjunction*, these notions play important roles in the study of both flavors of double category, but their meanings in the two cases are quite different. We then describe several notions of functor and transformation between double categories.

Chapter 3 is about ordinary, monoidal, and enriched abstract homotopy theory. We begin with general notions of homotopical category and derived functors, which generalize (and are simpler than) the more common notions of Quillen model category theory; this generalization will be especially important in the final chapter. We then present two different theorems on the functoriality of passage to derived functors, one using 2-categories and one using double categories, and apply this functoriality to study monoidal and enriched homotopical categories and functors between them. We end by lifting both functoriality statements to an enriched context.

Chapter 4 is about indexed monoidal categories, in the naive sense, and the corresponding homotopy theory. We first review the equivalence between indexed categories and fibrations, and prove analogous equivalences for indexed monoidal categories and monoidal fibrations. These equivalences are naturally expressed in the language of double categories. Then we study *homotopical* fibrations and monoidal fibrations, making use of our general results about derived monoidal functors from chapter 3.

In chapter 5 we introduce the definition of *framed bicategory*, which we regard as a different type of 'indexed monoidal structure.' We describe the bimodules construction, which produces new framed bicategories from old ones, and also show how to construct a framed bicategory from any well-behaved monoidal fibration. Combining these two constructions produces many important examples, such as enriched categories and distributors, internal categories and their distributors, and parametrized ring spectra and their bimodules.

Finally, chapter 6 is about the homotopy theory of enriched categories and distributors, a particular and tractable case of general indexed monoidal abstract homotopy theory. We explain how the two-sided bar construction produces derived tensor products in a quite general context, and then use this to build a well-behaved homotopy theory of enriched diagram categories and homotopy limits and colimits. We then use these techniques to construct a *derived framed bicategory* of enriched categories and distributors. We expect a similar construction to be possible for monoids in any suitably well-behaved framed bicategory.

Acknowledgements I am very grateful to my advisor, Peter May, for guidance and encouragement, and especially for reigning in my categorical flights of fancy and teaching me to write mathematics more coherently. (The remaining flights of fancy and incomprehensible sections of writing are, of course, solely my own fault.) I have also benefitted greatly from discussions with Tom Fiore, Joachim Kock, Gaunce Lewis, Phil Hirschhorn, Kate Ponto, and Stephan Stolz. Finally, I am immeasurably grateful to my family, for their invaluable love, support, and encouragement.

CHAPTER 2
DOUBLE CATEGORIES

In this chapter we introduce the notion of a *double category*, which will play a pivotal role in the entire thesis. As explained in the introduction, we have two uses for double categories and two corresponding flavors of double categories. Roughly speaking, the first flavor of double category is analogous to bicategories such as $\mathcal{C}at$ (which consists of categories, functors, and natural transformations), while the second flavor is analogous to bicategories such as $\mathcal{M}od$ (which consists of rings, bimodules, and bimodule maps). One formal difference between these bicategories is that $\mathcal{C}at$ is actually a strict 2-category while $\mathcal{M}od$ is only a bicategory; likewise the first flavor of double category is usually 'strict' while the second flavor is 'pseudo.'

2.1 Strict and pseudo double categories

Double categories were originally introduced by Ehresmann in [Ehr63]; see also [KS74]. A small **(strict) double category** $\mathbb{D}$ is defined to be an internal category in the category **Cat** of categories; thus it consists of two categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with source, target, identity, and composition functors

$$S, T\colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$$

$$I\colon \mathbb{D}_0 \to \mathbb{D}_1$$

$$C\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

satisfying suitable axioms. We call $\mathbb{D}_0$ the **vertical category** of $\mathbb{D}$ and think of it as drawn vertically, calling its objects **0-cells** or just **objects**, and its morphisms **vertical 1-cells** or **vertical arrows**. If f is an object of $\mathbb{D}_1$ with $S(f) = a$ and $T(f) = b$, we draw f as a horizontal arrow from a to b and call it a **horizontal**

1-cell. And if $\alpha\colon f \to g$ is a morphism of $\mathbb{D}_1$ with $S(\alpha) = h$ and $T(\alpha) = k$, where $h\colon a \to b$ and $k\colon c \to d$ are morphisms in $\mathbb{D}_0$, we draw α in a square

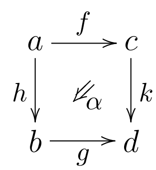
$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ {\scriptstyle h}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle k} \\ b & \xrightarrow[g]{} & d \end{array} \tag{2.1.1}$$

and call it a **2-cell**. In this chapter, we will write $\alpha \boxbar \beta$ for the horizontal composite of 2-cells and $\beta \boxminus \alpha$ for the vertical composite. Every object a has both a vertical identity 1_a and a horizontal identity 1^a, every vertical arrow $g\colon a \to b$ has an identity 2-cell 1^g, every horizontal arrow $f\colon a \to c$ has an identity 2-cell 1_f, and we have $1^{1_a} = 1_{1^a}$. We will often write identity arrows simply as equalities.

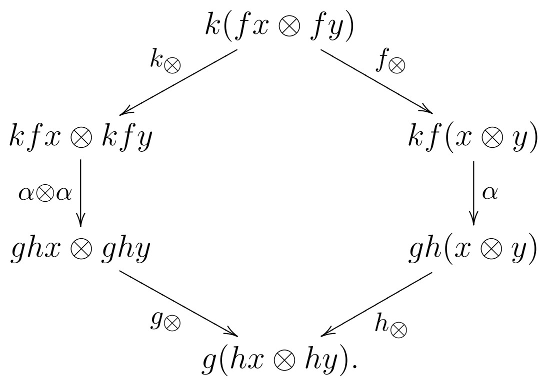
Example 2.1.2. There is a strict double category $\mathbb{C}\mathsf{at}$ whose objects are categories, whose vertical and horizontal 1-cells are functors, and whose 2-cells of the form (2.1.1) are natural transformations $\alpha\colon kf \to gh$. A similar double category can be constructed with any 2-category $\mathcal{K}$ replacing $\mathcal{C}at$; this is called the double category $\mathbb{S}\mathsf{q}(\mathcal{K})$ of **squares** in $\mathcal{K}$. (This was first considered by Ehresmann, who called it the double category of *quintets* in $\mathcal{K}$.) For example, from any monoidal category $\mathscr{V}$ we can construct a double category $\mathscr{V}$-$\mathbb{C}\mathsf{at}$ of $\mathscr{V}$-enriched categories, $\mathscr{V}$-functors, and $\mathscr{V}$-transformations.

Conversely, any strict double category has two underlying 2-categories. The objects, horizontal 1-cells, and 2-cells of the form $\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \| & \swarrow & \| \\ \cdot & \longrightarrow & \cdot \end{array}$ (which we call **h-globular**) form a 2-category called the **horizontal 2-category** $\mathcal{H}\mathbb{D}$ of $\mathbb{D}$. Similarly, we define **v-globular** 2-cells and the **vertical 2-category** $\mathcal{V}\mathbb{D}$.

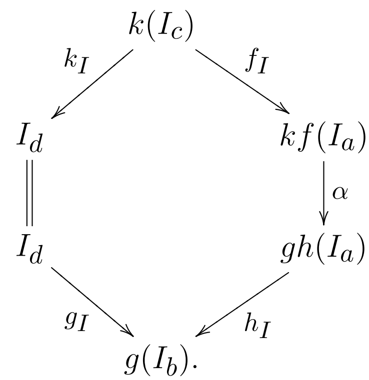
Example 2.1.3. There is a strict double category $\mathbb{M}\mathsf{onCat}$ whose objects are monoidal categories, whose horizontal arrows are *lax* monoidal functors, and whose vertical arrows are *oplax* monoidal functors. A 2-cell



is a natural transformation $\alpha\colon kf \to gh$ such that the following diagrams commute:



and



This double category was apparently first considered in [GP04]. Its horizontal 2-category $\mathcal{H}\mathbb{M}\mathsf{onCat}$ is the 2-category $\mathcal{M}onCat_\ell$ of monoidal categories, lax monoidal functors, and monoidal transformations, and dually for $\mathcal{V}\mathbb{M}\mathsf{onCat}$. Thus $\mathbb{M}\mathsf{onCat}$ packages information about both lax and oplax monoidal functors and how they relate to each other.

More generally, categories (or families of categories) equipped with almost any type of 'algebraic structure' form a double category analogous to $\mathbb{M}\mathsf{onCat}$. Braided and symmetric monoidal categories are two obvious examples. We will see many more examples in chapters 4 and 5. In fact, one can unify all of these examples by speaking about algebras for a *2-monad*, i.e. a **Cat**-enriched monad on a 2-category. For any such 2-monad T one can define T-algebras, lax and oplax T-morphisms, and T-transformations, which fit together into a strict double category T-$\mathbb{A}\mathsf{lg}$.

These double categories are all of the first flavor, which we informally call '$\mathbb{C}\mathsf{at}$-like.' We think of such double categories as similar to 2-categories, but with two types of 1-cells. Note that the two types of 1-cells play essentially symmetrical roles in the above examples. In the second flavor of double category, however, their roles are quite asymmetric: the vertical 1-cells are the ordinary 'morphisms' between the 0-cells, while the other type are a different sort of 'object' indexed by the 0-cells. This flavor of double category is also usually a *pseudo*, rather than a strict, double category.

The definition of a **pseudo double category** is similar to that of a strict double category: we still have categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with functors S, T, C, U, but now horizontal composition is only associative and unital up to coherent isomorphism, as in a bicategory. Formally speaking, it is a 'pseudo internal category' in the 2-category $\mathcal{C}at$. Note that the vertical composition remains strictly associative and unital. With more work, one can define double categories which are pseudo in both directions, as in [Ver92], but all the examples we care about will be vertically strict.

Example 2.1.4. There is a pseudo double category $\mathbb{M}\mathsf{od}$ defined as follows. Its objects are (not necessarily commutative) rings and its vertical morphisms are ring homomorphisms. A 1-cell $M\colon A \to B$ is an (A, B)-bimodule, and a 2-cell

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Swarrow_{\alpha} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

is an (f, g)-*equivariant map* $M \to N$. Horizontal composition is the tensor product. We may replace **Ab** with any cocomplete closed monoidal category $\mathscr{C}$, obtaining a double category $\mathbb{M}\mathsf{od}(\mathscr{C})$ of monoids and bimodules in $\mathscr{C}$.

$\mathbb{M}\mathsf{od}$ is the basic example of a double category of the second flavor; we will see many more in chapter 5. We refer to such double categories as '$\mathbb{M}\mathsf{od}$-like.'

Note that any pseudo double category has a horizontal *bicategory* $\mathcal{H}\mathbb{D}$, defined as in the strict case. With a bit of care, one can also define the vertical strict 2-category $\mathcal{V}\mathbb{D}$ of a pseudo double category, but we will not need to do so.

2.2 Companions and conjoints

We now introduce the notions of *companion pair* and *conjunction*, two notions whose roles in the theory of double categories are as important as the notions of equivalence and adjunction are in the theory of bicategories. However, the definitions seem to only recently have been isolated. The forms we give them in are due to [GP04, DPP]; our terminology is from [DPP].

The intuition is that although the horizontal and vertical 1-cells in a double category cannot be composed with each other, we can still speak about when a vertical 1-cell and a horizontal 1-cell are isomorphic, and when they are adjoint to each other. In fact, these two situations turn out to be dual to each other, in a precise sense. We begin with isomorphisms.

Definition 2.2.1. A **companion pair** in a double category $\mathbb{D}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $f'\colon a \to b$, and 2-cells

$$\begin{array}{ccc} a & \xrightarrow{f'} & b \\ {\scriptstyle f}\big\downarrow & \swArrow \varphi & \big\| \\ b & = & a \end{array} \qquad \text{and} \qquad \begin{array}{ccc} a & = & a \\ \big\| & \swArrow \psi & \big\downarrow{\scriptstyle f} \\ a & \xrightarrow[f']{} & b \end{array}$$

such that $\varphi \boxminus \psi = 1^f$ and $\psi \boxbar \varphi = 1_{f'}$. We say that f' is the **(horizontal) companion** of f and that f is the **(vertical) companion** of f', and write $f \approxeq f'$.

Remark 2.2.2. The above definition is only precisely correct when $\mathbb{D}$ is a strict double category; when $\mathbb{D}$ is a pseudo double category, we must insert unit isomorphisms for the second axiom to make sense.

Remark 2.2.3. A *folding* or *connection pair* on a double category, as considered in [BS76, BM99, Fio07], can be defined as a strictly functorial choice of a companion for each vertical 1-cell. Thus, this gives another proof of the result of [Fio07] that a folding on a double category is unique up to isomorphism. As observed below, companions are automatically pseudofunctorial, so an arbitrary choice of companions for each vertical 1-cell is the same as a *pseudo-folding* in the sense of [Fio07].

Example 2.2.4. In $\mathbb{C}\mathsf{at}$ or $\mathbb{S}\mathsf{q}(\mathcal{K})$, every 1-cell has a companion, which can be taken to be just itself, with φ and ψ both the identity in $\mathcal{K}$.

In general, however, not every 1-cell will have a companion.

Example 2.2.5. A 1-cell in $\mathbb{M}\mathsf{onCat}$ has a companion just when it is a *strong* monoidal functor. For the 2-cells φ and ψ show that f and f' are isomorphic as ordinary functors, and then the hexagon axioms in the definition of a 2-cell in $\mathbb{M}\mathsf{onCat}$ imply that the lax structure maps of f' are inverses to the oplax structure maps of f, so that both are strong. An analogous statement is true in T-$\mathbb{A}\mathsf{lg}$ for any 2-monad T.

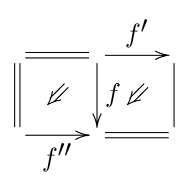
Generalizing this example, we say that a 1-cell in a general double category is **strong** if it has a companion.

Example 2.2.6. In $\mathbb{M}\mathsf{od}$, every vertical 1-cell is strong: a companion of a ring homomorphism $f\colon A \to B$ is given by the bimodule ${}_fB$; that is, B regarded as an A-B-bimodule via f on the left. Here φ is the identity function ${}_fB \to B$ and ψ is f regarded as a function $A \to {}_fB$. We will see in chapter 5 that this is typical for $\mathbb{M}\mathsf{od}$-like double categories.

When they exist, companions are unique up to globular isomorphism, and the composite of companions is a companion.

Proposition 2.2.7. *If f' and f'' are both horizontal companions of f, then there is a canonical isomorphism $f' \cong f''$ in $\mathcal{H}\mathbb{D}$, and similarly for vertical companions.*

Proof. An isomorphism is given by the following composite.



Its inverse is the obvious dual construction. □

Proposition 2.2.8. *If f and g have companions f' and g', then $g'f'$ is a companion of gf.*

Proof. It is straightforward to compose the 2-cells defining the companion pairs $f \approxeq f'$ and $g \approxeq g'$ to produce a companion pair $gf \approxeq g'f'$. ☐

In particular, the composite of strong 1-cells is strong. When $\mathbb{D}$ is a strict double category, we denote by $\mathcal{S}tr(\mathbb{D})$ the locally full sub-2-category of $\mathcal{V}\mathbb{D}$ containing all the objects but only the strong vertical arrows. (The choice of vertical arrows is just for definiteness; the results of §2.3 imply that we would obtain a biequivalent 2-category if we used the strong horizontal arrows instead.) For example, $\mathcal{S}tr(\mathbb{M}\mathsf{onCat})$ is the 2-category $\mathcal{M}on\mathcal{C}at$ of monoidal categories and strong monoidal functors, and likewise $\mathcal{S}tr(T\text{-}\mathbb{A}\mathsf{lg})$ is the 2-category $T\text{-}\mathcal{A}lg$.

Now, we obtain the definition of 'adjunction' between vertical and horizontal 1-cells, which we call a *conjunction* (following [DPP]), by simply reversing left and right in the notion of companion pair. That is, a conjunction in $\mathbb{D}$ is a companion pair in the 'horizontal opposite' of $\mathbb{D}$. Now the inverse isomorphisms φ and ψ become the unit and counit of the adjunction. Thus, one may say that "natural isomorphisms are dual to adjunctions."

Definition 2.2.9. A **conjunction** in a strict double category $\mathbb{D}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $g\colon b \to a$, and 2-cells

$$\begin{array}{ccc} a & = & a \\ {\scriptstyle f}\downarrow & \swArrow \eta & \| \\ b & \xrightarrow[g]{} & a \end{array} \qquad \text{and} \qquad \begin{array}{ccc} b & \xrightarrow{g} & a \\ \| & \swArrow \varepsilon & \downarrow{\scriptstyle f} \\ b & = & b \end{array}$$

(the **unit** and **counit**) such that $\varepsilon \boxbar \eta = 1_g$ and $\varepsilon \boxminus \eta = 1^f$. We say that f is the **left conjoint** and g is the **right conjoint**, and write $f \prec g$.

Example 2.2.10. A conjunction in $\mathbb{C}\mathsf{at}$ is simply an ordinary adjunction. Likewise, a conjunction in $\mathbb{S}\mathsf{q}(\mathcal{K})$ is simply an ordinary internal adjunction in $\mathcal{K}$, i.e. a pair of 1-cells $f\colon a \to b$ and $g\colon b \to a$ together with 2-cells $\eta\colon 1_a \to gf$ and $\varepsilon\colon fg \to 1_b$ satisfying the usual triangle identities.

Example 2.2.11. A conjunction in the double category $T\text{-}\mathbb{A}\mathsf{lg}$, for some 2-monad T (see Example 2.1.3), is precisely a *doctrinal adjunction* as studied in [Kel74]. This is an adjunction between T-algebras (such as monoidal categories) in which the left adjoint is oplax and the right adjoint is lax, and the oplax and lax structure maps are determined by each other via the adjunction (they are 'mates' in the sense of the next section). We will call such an adjunction a **T-conjunction**. In particular, we call a conjunction in $\mathbb{M}\mathsf{onCat}$ a **monoidal conjunction**.

Example 2.2.12. In $\mathbb{M}\mathsf{od}$, every vertical morphism has a conjoint. The conjoint of a ring homomorphism $f\colon A \to B$ is B_f; that is, B regarded as a B-A-bimodule via f on the right. The argument is precisely dual to the companion ${}_fB$. Again, we will see in chapter 5 that this is typical for $\mathbb{M}\mathsf{od}$-like double categories.

Like companions and ordinary adjoints, conjoints are unique up to unique isomorphism when they exist, and the composite of conjoints is a conjoint; the proofs are dual to those of Propositions 2.2.7 and 2.2.8.

2.3 Mates and comrades

One of the important properties of adjunctions in a 2-category is the *mate correspondence*. The most basic form of this says that if f and h are parallel 1-cells with right adjoints g and k, respectively, then there is a bijection between 2-cells $f \to h$ and 2-cells $k \to g$. A pair of 2-cells that correspond to each other under this bijection are called *mates* (or sometimes 'conjugates'). Explicitly, the mate of $\alpha\colon f \to h$ is the composite

$$k \xrightarrow{\eta k} gfk \xrightarrow{g\alpha k} ghk \xrightarrow{g\varepsilon} g.$$

where η is the unit of the adjunction $f \dashv g$ and ε is the counit of the adjunction $h \dashv k$. There is also a more general form of the mate correspondence in a 2-category that compares 2-cells $fi \to jh$ with 2-cells $ik \to gj$; see [KS74]. Mate correspondences provide a standard way to compare composites of left and right adjoints, such as in the study of various sorts of functors between monoidal categories and their interactions with closed structure (see, for instance, [FHM03]).

In this section we give a version of the mate correspondence that applies to conjunctions in any double category. When applied in $\mathbb{S}\mathsf{q}(\mathcal{K})$, this correspondence subsumes the general theory of mates in a 2-category. When applied in $\mathbb{M}\mathsf{onCat}$, it generalizes this theory to keep track of lax and oplax monoidal structures on functors. In chapter 3 we will apply it in a double category of homotopical categories to study mates of transformations between left and right derived functors. And in chapter 5 we will use it to describe base change in $\mathbb{M}\mathsf{od}$-like double categories.

Actually, we will first give a similar correspondence for companion pairs, and then dualize it to obtain the mate correspondence for conjunctions. A pair of 2-cells that correspond under a companion pair we call *comrades*. This 'comrade correspondence' also exists in a 2-category, but there it is too obvious to require comment (or a name): it simply says that if $f \cong f'$ and $g \cong g'$, then there is a bijection between 2-cells $f \to g$ and $f' \to g'$. In a double category, however, it takes on a more interesting form.

Proposition 2.3.1. *If f and g have horizontal companions f' and g', then there is a canonical isomorphism*

$$\mathcal{V}\mathbb{D}(f,g) \cong \mathcal{H}\mathbb{D}(f',g'). \tag{2.3.2}$$

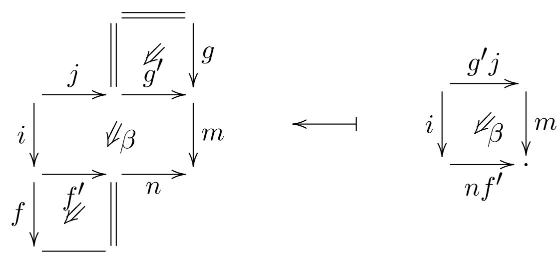
More generally, for any i, j, m, n there is a bijection between 2-cells with the following boundaries:

$$\begin{array}{ccc} & \xrightarrow{j} & \\ {\scriptstyle fi}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle mg} \\ & \xrightarrow[n]{} & \end{array} \quad \textit{and} \quad \begin{array}{ccc} & \xrightarrow{g'j} & \\ {\scriptstyle i}\downarrow & \Downarrow\beta & \downarrow{\scriptstyle m} \\ & \xrightarrow[nf']{} & \end{array}. \tag{2.3.3}$$

We say that a pair of 2-cells which correspond under (2.3.3) *are* ***comrades****.*

Proof. The bijection is given by the following correspondences.

$$\begin{array}{ccc} & \xrightarrow{j} & \\ {\scriptstyle fi}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle mg} \\ & \xrightarrow[n]{} & \end{array} \quad \longmapsto \quad \begin{array}{ccccccc} & & & \xrightarrow{j} & & \xrightarrow{g'} & \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle g}\downarrow & \Downarrow & \| \\ & = & & \Downarrow\alpha & & = & \\ \| & \Downarrow & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle m} & & \\ & \xrightarrow[f']{} & & \xrightarrow[n]{} & & & \end{array} \tag{2.3.4}$$



(2.3.5)

□

Dualizing this, we obtain the mate correspondence.

Proposition 2.3.6. *If $f \prec g$ and $h \prec k$ where $f, h\colon a \to b$, then we have a natural isomorphism*

$$\mathcal{V}\mathbb{D}(f,h) \cong \mathcal{H}\mathbb{D}(k,g), \tag{2.3.7}$$

under which isomorphisms $f \cong h$ correspond to isomorphisms $k \cong g$. More generally, for any i, j, m, n there is a bijection between 2-cells with the following boundaries:

(2.3.8)

We say that a pair of 2-cells corresponding to each other under (2.3.8) *are* ***mates****.* □

Putting composition together with mates, we obtain a 2-category $\mathcal{C}onj(\mathbb{D})$ whose objects are those of $\mathbb{D}$, whose 1-cells are the conjunctions in $\mathbb{D}$, and whose 2-cells are the mate-pairs of globular 2-cells in $\mathbb{D}$. Of course, to define $\mathcal{C}onj(\mathbb{D})$ we must choose whether to consider a conjunction as pointing in the direction of the left conjoint or the right conjoint.

As an immediate application of mates and comrades, we show that companion pairs 'mediate' between adjunctions and conjunctions.

Proposition 2.3.9. *Let $f\colon a \to b$ be a vertical 1-cell in $\mathbb{D}$ and let $f'\colon a \to b$ and $g\colon b \to a$ be horizontal 1-cells. Then any two of the following statements imply the third.*

(i) f' is a horizontal companion of f.

(ii) g is a right conjoint of f.

(iii) g is a right adjoint of f' in $\mathcal{H}\mathbb{D}$.

More precisely, any companion pair $f \approxeq f'$ and conjunction $f \prec g$ determine a unique horizontal adjunction $f' \dashv g$, and similarly in the other cases.

Proof. Assuming (i), the correspondence of Proposition 2.3.1 transforms a unit and counit for a conjunction $f \prec g$ into a unit and counit for a horizontal adjunction $f' \dashv g$, and vice versa. The other cases are similar. $\square$

Applying this to T-$\mathbb{A}\mathsf{lg}$, we recover part of one of the main results of [Kel74] about doctrinal adjunctions.

Corollary 2.3.10. *In a T-conjunction, the left adjoint is a strong T-morphism precisely when the conjunction is an adjunction in the 2-category T-$\mathcal{A}lg_\ell$.* $\square$

The other part is a special property of T-$\mathbb{A}\mathsf{lg}$ not shared by all double categories, but it can at least be stated concisely in our language.

Proposition 2.3.11 ([Kel74])**.** *If T is a 2-monad on the 2-category $\mathcal{K}$, then the forgetful double functor $U\colon T$-$\mathbb{A}\mathsf{lg} \to \mathbb{S}\mathsf{q}(\mathcal{K})$ lifts conjunctions. That is, if f is a vertical 1-cell in T-$\mathbb{A}\mathsf{lg}$, then any conjunction $Uf \prec g$ in $\mathbb{S}\mathsf{q}(\mathcal{K})$ (i.e. any adjunction $Uf \dashv g$ in $\mathcal{K}$) is the image of some conjunction $f \prec g'$ in T-$\mathbb{A}\mathsf{lg}$, and dually.*

Corollary 2.3.12. *Any left adjoint in T-$\mathcal{A}lg_\ell$ is strong.*

Proof. If $f \dashv g$ is an adjunction in T-$\mathcal{A}lg_\ell = \mathcal{H}T$-$\mathbb{A}\mathsf{lg}$, then we can regard $Uf \dashv Ug$ as a conjunction in $\mathbb{S}\mathsf{q}(\mathcal{K})$. Thus, by Proposition 2.3.11, we have a conjunction $f' \prec g$ in T-$\mathbb{A}\mathsf{lg}$. But then by Proposition 2.3.9, we have $f' \approxeq f$ and so f is strong. $\square$

2.4 Functors between double categories

In general, the correct sorts of functor and transformation between double categories depend greatly on the flavor of double categories under consideration. The most straightforward notion of functor between strict double categories is a **strict double functor**: it is just an internal functor in **Cat**. Thus, it consists of functions on objects, vertical and horizontal 1-cells, and squares, strictly preserving all sorts of composition and identities.

Strict double functors are often a useful sort of functor between strict $\mathbb{C}\mathsf{at}$-like double categories, just as strict 2-functors are often a useful sort of functor between $\mathcal{C}at$-like 2-categories. However, even between $\mathcal{C}at$-like strict 2-categories, one is sometimes forced to consider *pseudo* 2-functors, which preserve composition and identities only up to isomorphism. Furthermore, between $\mathcal{M}od$-like bicategories, the relevant functors are often only *lax* or *oplax*, meaning that composites and units are related only by a not-necessarily-invertible morphism in one direction or the other. In this section we define analogues of all these sorts of functors for double categories.

The easiest, and most well-known, case is the last one: lax and oplax morphisms between (pseudo) double categories. Since the vertical directions of our pseudo double categories are viewed as 'morphisms' rather than objects, it makes sense to ask vertical composition and identities to be preserved strictly; thus our functors will be lax only in the horizontal direction. The definition can be obtained abstractly by realizing pseudo double categories as algebras for a certain 2-monad, but we give an explicit version.

Definition 2.4.1. Let $\mathbb{D}$ and $\mathbb{E}$ be pseudo double categories. A **lax double functor** $F\colon \mathbb{D} \to \mathbb{E}$ consists of the following.

- Functors $F_i\colon \mathbb{D}_i \to \mathbb{E}_i$, $i = 0, 1$ such that $S \circ F_1 = F_0 \circ S$ and $T \circ F_1 = F_0 \circ T$.
- Natural transformations $F_{\boxbar}\colon (F_1 g)(F_1 f) \to F_1(gf)$ and $F_U\colon 1^{F_0 a} \to F_1(1^a)$ (where a is an object of $\mathbb{D}$ and f, g are horizontal 1-cells in $\mathbb{D}$), whose components are h-globular, and which satisfy the usual coherence axioms for a lax monoidal functor (see, for example, [ML98, §XI.2]).

Dually, we have the definition of an **oplax double functor**, for which $F_{\boxbar}$ and F_U go in the opposite direction. A **strong double functor** is a lax double functor for which $F_{\boxbar}$ and F_U are (h-globular) isomorphisms. If just F_U is an isomorphism, we say that F is **normal**.

We occasionally abuse notation by writing just F for either F_0 or F_1. Like the constraints $\mathfrak{a}, \mathfrak{l}, \mathfrak{r}$ for a pseudo double category, the maps $F_{\boxbar}$ and F_U are h-globular, but must be natural with respect to all 2-cells, not only globular ones.

Example 2.4.2. If $\mathscr{C}$ and $\mathscr{D}$ are cocomplete closed monoidal categories, any lax monoidal functor $F\colon \mathscr{C} \to \mathscr{D}$ induces a normal lax double functor

$$\mathbb{M}\mathsf{od}(F)\colon \mathbb{M}\mathsf{od}(\mathscr{C}) \to \mathbb{M}\mathsf{od}(\mathscr{D}).$$

If F is strong and preserves coequalizers, then $\mathbb{M}\mathsf{od}(F)$ is also strong. This is a special case of a general result we will see in §5.5.

Just as for monoidal categories (and algebras for other 2-monads), lax and oplax double functors fit together into a strict *double* category, and we can recover the strong double functors as the 1-cells with companions. This was apparently first observed in [GP04].

Definition 2.4.3. Let $F\colon \mathbb{D} \to \mathbb{F}$ and $G\colon \mathbb{E} \to \mathbb{G}$ be lax double functors and $H\colon \mathbb{D} \to \mathbb{E}$ and $K\colon \mathbb{F} \to \mathbb{G}$ be oplax double functors. A **2-cell**

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{F} & \mathbb{F} \\ {\scriptstyle H}\downarrow & \swArrow_{\alpha} & \downarrow{\scriptstyle K} \\ \mathbb{E} & \xrightarrow[G]{} & \mathbb{G} \end{array}$$

consists of the following structure and properties.

(i) For each object a of $\mathbb{D}$, a vertical arrow $\alpha_a\colon KF(a) \to GH(a)$.

(ii) For each horizontal 1-cell $f\colon a \to b$ of $\mathbb{D}$, a 2-cell

$$\begin{array}{ccc} KFa & \xrightarrow{KFf} & KFb \\ {\scriptstyle\alpha_a}\big\downarrow & \Downarrow\alpha_f & \big\downarrow{\scriptstyle\alpha_b} \\ GHa & \xrightarrow[GHf]{} & GHb \end{array}$$

(iii) The following composites are equal:

$$\begin{array}{ccccc} KFa & & \xrightarrow{\quad K(Fg\circ Ff)\quad} & & KFC \\ \Vert & & \Downarrow & & \Vert \\ KFa & \xrightarrow{KFf} & KFB & \xrightarrow{KFg} & KFC \\ {\scriptstyle\alpha_a}\big\downarrow & \Downarrow\alpha_f & \big\downarrow{\scriptstyle\alpha_B} & \Downarrow\alpha_g & \big\downarrow{\scriptstyle\alpha_C} \\ GHa & \xrightarrow[GHf]{} & GHB & \xrightarrow[GHg]{} & GHC \\ \Vert & & \Downarrow & & \Vert \\ GHa & & \xrightarrow[\quad G(Hg\circ Hf)\quad]{} & & GHC \end{array} \quad = \quad \begin{array}{ccc} KFa & \xrightarrow{\quad K(Fg\circ Ff)\quad} & KFC \\ \Vert & \Downarrow & \Vert \\ KFa & \xrightarrow{\quad KF(gf)\quad} & KFC \\ {\scriptstyle\alpha_a}\big\downarrow & \Downarrow\alpha_{gf} & \big\downarrow{\scriptstyle\alpha_C} \\ GHa & \xrightarrow[\quad GH(gf)\quad]{} & GHC \\ \Vert & \Downarrow & \Vert \\ GHa) & \xrightarrow[\quad G(Hg\circ Hf\quad]{} & GHC \end{array}$$

and

$$\begin{array}{ccc} KFa & \xrightarrow{K(1^{Fa})} & KFa \\ \Vert & \Downarrow & \Vert \\ KFa & \xrightarrow{1^{KFa}} & KFa \\ {\scriptstyle\alpha_a}\big\downarrow & \Downarrow 1^{\alpha_a} & \big\downarrow{\scriptstyle\alpha_a} \\ GHa & \xrightarrow[1^{GHa}]{} & GHa \\ \Vert & \Downarrow & \Vert \\ GHa & \xrightarrow[G(1^{Ha})]{} & GHa \end{array} \quad = \quad \begin{array}{ccc} KFa & \xrightarrow{K(1^{Fa})} & KFa \\ \Vert & \Downarrow & \Vert \\ KFa & \xrightarrow{KF(1^{a})} & KFa \\ {\scriptstyle\alpha_a}\big\downarrow & \Downarrow\alpha_{1^a} & \big\downarrow{\scriptstyle\alpha_a} \\ GHa & \xrightarrow[GH(1^{a})]{} & GHa \\ \Vert & \Downarrow & \Vert \\ GHa & \xrightarrow[G(1^{Ha})]{} & GHa \end{array}$$

Theorem 2.4.4. *There is a strict double category* $\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl}$ *whose objects are pseudo double categories, whose horizontal arrows are* lax *double functors, whose vertical arrows are* oplax *double functors, and whose 2-cells are as defined above. The strong morphisms in* $\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl}$*, as defined in §2.2, are the strong double functors.*

The case of Definition 2.4.3 which arises most in practice is when H and K are identities; in this case α is often called a *vertical transformation* between the lax double functors F and G. Of course, we have a 2-category $\mathcal{P}s\mathcal{D}bl_\ell = \mathcal{H}(\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl})$ of pseudo double categories, lax double functors, and vertical transformations. Similarly, we have $\mathcal{P}s\mathcal{D}bl_{op\ell} = \mathcal{V}(\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl})$ and $\mathcal{P}s\mathcal{D}bl = \mathcal{S}tr(\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl})$.

The existence of $\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl}$ and these 2-categories, and in particular of their 2-cells, is the main advantage of pseudo double categories over $\mathcal{M}od$-like bicategories. In chapter 5 we will return to consideration of $\mathbb{P}\mathsf{s}\mathbb{D}\mathsf{bl}$, and in particular of a certain full subcategory of it spanned by a class of pseudo double categories called 'framed bicategories.' However, we now turn to morphisms between $\mathbb{C}\mathsf{at}$-like strict double categories.

Since any strict double category is a pseudo double category, we can consider the above notions of functor, but since the two types of 1-cells in a $\mathbb{C}\mathsf{at}$-like double category play symmetrical roles, we should expect the relevant notions of functor and transformation to treat them symmetrically. Of the notions introduced so far, only *strict* double functors satisfy this requirement. The resulting category **Dbl** of strict double categories and strict double functors is analogous to the category **2-Cat** of strict 2-categories and strict 2-functors: it is a useful starting point, but frequently it is not sufficient. We now define two different improvements on **Dbl**.

(i) A category $\mathfrak{Dbl}$ of strict double categories and *double pseudofunctors* which preserve composition and units up to isomorphism in both directions. This is analogous to the category **2-$\mathfrak{Cat}$** of strict 2-categories and ordinary pseudofunctors.

(ii) A 2-category $\mathcal{D}bl$ of strict double categories, strict double functors, and *pointwise-strong transformations*. This is analogous to the 2-category 2-$\mathcal{C}at$ of strict 2-categories, strict 2-functors, and strict 2-natural transformations.

Logically, it would be natural to combine these two sorts of improvement into a *tri-category* of double categories, double pseudofunctors, pointwise-strong pseudonatural transformations, and modifications. However, this would be a good deal more work, and is unnecessary for our purposes.

We begin with the definition of double pseudofunctor.

Definition 2.4.5. Let $\mathbb{D}$ and $\mathbb{E}$ be strict double categories. A **double pseudofunctor** $F\colon \mathbb{D} \to \mathbb{E}$ consists of the following structure and properties.

(i) Functions from the objects, vertical 1-cells, horizontal 1-cells, and 2-cells of $\mathbb{D}$ to those of $\mathbb{E}$, preserving sources, targets, and boundaries.

(ii) For each object a of $\mathbb{D}$, 2-cells

$$\begin{array}{ccc} Fa & \xrightarrow{F(1^a)} & Fa \\ {\scriptstyle 1_{Fa}}\Big\| & \swarrow F_a & \Big\|{\scriptstyle 1_{Fa}} \\ Fa & \underset{1^{Fa}}{=\!=\!=} & Fa \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Fa & \overset{1^{Fa}}{=\!=\!=} & Fa \\ {\scriptstyle 1_{Fa}}\Big\| & \swarrow F^a & \Big\downarrow{\scriptstyle F(1_a)} \\ Fa & \underset{1^{Fa}}{=\!=\!=} & Fa \end{array}$$

in $\mathbb{E}$, of which the first is an h-globular isomorphism and the second a v-globular isomorphism.

(iii) For each composable pair $a \xrightarrow{f} b \xrightarrow{g} c$ of vertical 1-cells in $\mathbb{D}$, a v-globular isomorphism

$$\begin{array}{ccc} Fa & =\!=\!= & Fa \\ {\scriptstyle Ff}\Big\downarrow & & \\ Fb & \swarrow F_{gf} & \Big\downarrow{\scriptstyle F(gf)} \\ {\scriptstyle Fg}\Big\downarrow & & \\ Fc & =\!=\!= & Fc. \end{array}$$

(iv) For each composable pair $a \xrightarrow{h} b \xrightarrow{k} c$ of horizontal 1-cells in $\mathbb{D}$, an h-globular isomorphism

$$\begin{array}{ccccc} Fa & & \xrightarrow{\quad F(kh) \quad} & & Fc \\ \Big\| & & \Downarrow F_{kh} & & \Big\| \\ Fa & \xrightarrow[Fh]{} & Fb & \xrightarrow[Fk]{} & Fc. \end{array}$$

(v) The following coherence axioms hold (the usual coherence axioms for a pseudofunctor in both directions).

$$F^{h(gf)} \boxbar \left(1^{Fh} \boxminus F^{gf}\right) = F^{(hg)f} \boxbar \left(F^{hg} \boxminus 1^{Ff}\right)$$

$$F^{b} \boxminus 1^{Ff} = F^{1_b f}$$

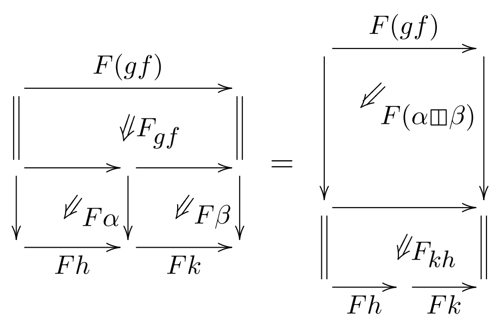
$$1^{Ff} \boxminus F^{a} = F^{f 1_a}$$

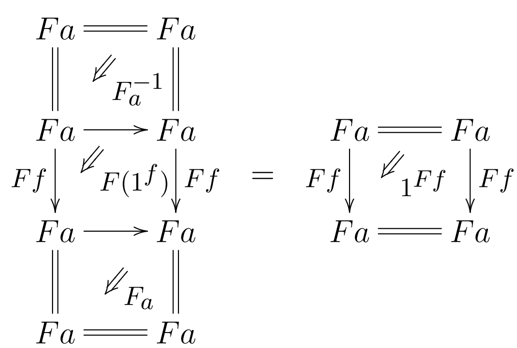
$$F_{h(gf)} \boxminus \left(F_{gf} \boxbar 1_{Fh}\right) = F_{(hg)f} \boxminus \left(1_{Ff} \boxbar F_{hg}\right)$$

$$F_{a} \boxbar 1_{Ff} = F_{f 1_a}$$

$$1_{Ff} \boxbar F_{b} = F_{1_b f}.$$

(vi) The following 'double naturality' axioms hold:





as do their transposes involving F^{gf} and F^{a}.

Note that in general, a double pseudofunctor does not preserve globularity of 2-cells, since it does not preserve either vertical or horizontal identities strictly. However, any h-globular 2-cell

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \| & \Swarrow\alpha & \| \\ a & \xrightarrow[g]{} & b \end{array}$$

in $\mathbb{D}$ gives rise to a canonical h-globular 2-cell

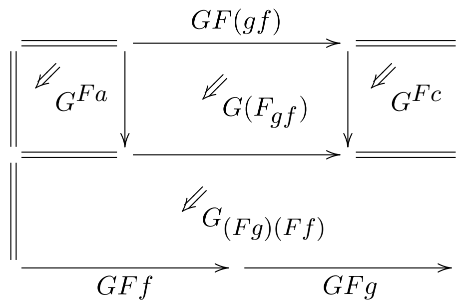
$$\begin{array}{ccccccc} Fa & = & Fa & \xrightarrow{Ff} & Fb & = & Fb \\ \| & \Swarrow F^a & \downarrow & \Swarrow F(\alpha) & \downarrow & \Swarrow (F^a)^{-1} & \| \\ Fa & = & Fa & \xrightarrow[Fg]{} & Fb & = & Fb \end{array}$$

in $\mathbb{E}$, which we denote $\mathcal{H}F(\alpha)$. It is easy to check that this defines an ordinary pseudofunctor $\mathcal{H}F\colon \mathcal{H}\mathbb{D} \to \mathcal{H}\mathbb{E}$. Similarly, we have $\mathcal{V}F\colon \mathcal{V}\mathbb{D} \to \mathcal{V}\mathbb{E}$.

Examples 2.4.6. An ordinary pseudofunctor $F\colon \mathcal{K} \to \mathcal{L}$ gives rise to a double pseudofunctor $\mathbb{S}\mathsf{q}(F)\colon \mathbb{S}\mathsf{q}(\mathcal{K}) \to \mathbb{S}\mathsf{q}(\mathcal{L})$ in a fairly straightforward way. The only wrinkle is that if $\alpha\colon kf \to gh$ is a 2-cell in $\mathbb{S}\mathsf{q}(\mathcal{K})$, we must compose $F\alpha$ with the constraints of F on either side to obtain a 2-cell in $\mathbb{S}\mathsf{q}(\mathcal{L})$.

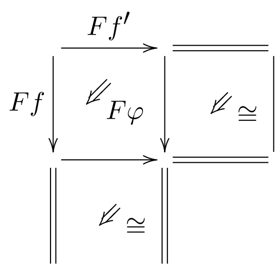
In particular, if $F\colon \mathcal{C}at \to \mathcal{C}at$ is a pseudofunctor, we obtain a double pseudofunctor $\mathbb{S}\mathsf{q}(F)\colon \mathbb{C}\mathsf{at} \to \mathbb{C}\mathsf{at}$. Similarly, if we let $\mathbb{D}^\top$ denote the 'transpose' of a double category, in which the vertical and horizontal arrows are interchanged (this is a double-categorical analogue of reversing the 2-cells in a 2-category), then we have a double pseudofunctor $\mathbb{C}\mathsf{at}^\top \to \mathbb{C}\mathsf{at}$ which takes $\mathscr{C}$ to $\mathscr{C}^{op}$.

The composite $G \circ F$ of two double pseudofunctors is defined in an obvious way, with one minor wrinkle: since G need not preserve the globularity of the constraints for F, we need to compose with the unit constraints of G when defining the constraints of GF. For example, the composition constraint of GF is given by the composite

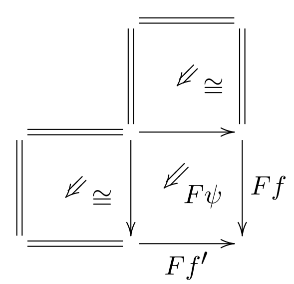


We thereby obtain a category $\mathfrak{Dbl}$ of double categories and double pseudofunctors. The operations $\mathcal{V}$ and $\mathcal{H}$ define functors from $\mathfrak{Dbl}$ to the category $\mathbf{2}$-$\mathfrak{Cat}$ of 2-categories and pseudofunctors. In the other direction, $\mathbb{S}\mathfrak{q}$ defines a functor from $\mathbf{2}$-$\mathfrak{Cat}$ to $\mathfrak{Dbl}$.

It is evident that strict double functors preserve companion pairs and conjunctions. This is also true for double pseudofunctors: if f' is a companion of f in $\mathbb{D}$ and $F\colon \mathbb{D} \to \mathbb{E}$ is a double pseudofunctor, then $F(f')$ is a companion of $F(f)$ in $\mathbb{E}$ when equipped with the 2-cells



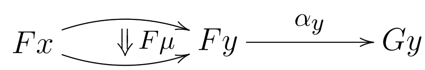
and



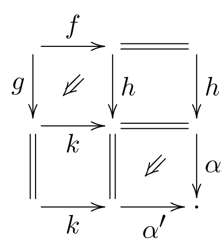
It follows that $\mathcal{S}tr$ and $\mathcal{C}onj$, like $\mathcal{V}$ and $\mathcal{H}$, define functors from $\mathfrak{Dbl}$ to $\mathbf{2}$-$\mathfrak{Cat}$.

It is easy to verify that mate and comrade correspondences are also preserved by double pseudofunctors. In other words, if $F\colon \mathbb{D} \to \mathbb{E}$ is a double pseudofunctor and α and β are mates or comrades in $\mathbb{D}$, then so are $F(\alpha)$ and $F(\beta)$ in $\mathbb{E}$. This will be crucial in our applications of mate correspondences for derived functors in chapter 3.

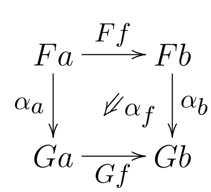
We now turn to the 2-category $\mathcal{D}bl$, which entails giving a definition of an appropriate sort of transformation between strict double functors. Since these transformations should treat both types of 1-cell symmetrically, it is natural to require that their components should be not just 1-cells but companion pairs. This can also be motivated by considering the 2-cells in the 2-category 2-$\mathcal{C}at$ that we are emulating: a *2-natural transformation* $\alpha \colon F \to G$ consists of components $\alpha_X \colon Fx \to Gx$ which are strictly natural on the underlying categories ($\alpha_y \circ F(f) = G(f) \circ \alpha_x$) and such that for any 2-cell $\mu \colon f \to g$, we also have $\alpha_y \circ F(\mu) = G(\mu) \circ \alpha_x$. Here $\alpha_y \circ F(\mu)$ is the 'whiskering' composite



and likewise for $G(\mu) \circ \alpha_x$. Now in general, there is no way to 'whisker' a 2-cell in a double category by either a horizontal or a vertical arrow, but it *is* possible to whisker by a companion pair $\alpha \approxeq \alpha'$: we can form the composite



In looking for the right axioms, we observe that an equivalent definition of a 2-natural transformation is obtained by observing that **2-Cat** is cartesian closed, and defining a 2-natural transformation to be a morphism (1-cell) in the the internal-hom $[\mathcal{K}, \mathcal{L}]$. Now the category **Dbl** is also cartesian closed: its internal-hom $[\![\mathbb{D}, \mathbb{E}]\!]$ consists of strict double functors, vertical transformations, horizontal transformations, and a suitable notion of 2-cell. Here a *vertical transformation* is the specialization of Definition 2.4.3 to the case when H and K are identities and F and G are strict double functors; explicitly it consists of vertical 1-cells $\alpha_X \colon FX \to GX$ which are natural with respect to vertical composition of vertical 1-cells, along with 2-cells



which are natural with respect to vertical composition of 2-cells, and such that $\alpha_f \boxbar \alpha_g = \alpha_{gf}$ and $\alpha_{1^a} = 1^{\alpha_a}$. The definition of **horizontal transformation** is dual, and a 2-cell

$$\begin{array}{ccc} F & \xrightarrow{\gamma} & G \\ {\scriptstyle\alpha}\big\downarrow & \swarrow\!\!\!\!\!\!\!\!\not\,\mu & \big\downarrow{\scriptstyle\beta} \\ H & \xrightarrow[\delta]{} & K \end{array}$$

in $[\![\mathbb{D},\mathbb{E}]\!]$ consists of components

$$\begin{array}{ccc} Fa & \xrightarrow{\gamma_a} & Ga \\ {\scriptstyle\alpha_a}\big\downarrow & \Swarrow\,\mu_a & \big\downarrow{\scriptstyle\beta_a} \\ Ha & \xrightarrow[\delta_a]{} & Ka \end{array}$$

such that

$$\begin{array}{ccccc} Fa & \xrightarrow{\gamma_a} & Ga & \xrightarrow{Gf} & Gb \\ {\scriptstyle\alpha_a}\big\downarrow & \Swarrow\,\mu_a & \big\downarrow{\scriptstyle\beta_a} & \Swarrow\,\beta_f & \big\downarrow{\scriptstyle\beta_b} \\ Ha & \xrightarrow[\delta_a]{} & Ka & \xrightarrow[Kf]{} & Kb \end{array} \quad = \quad \begin{array}{ccccc} Fa & \xrightarrow{Ff} & Fb & \xrightarrow{\gamma_b} & Gb \\ {\scriptstyle\alpha_a}\big\downarrow & \Swarrow\,\alpha_f & \big\downarrow{\scriptstyle\alpha_b} & \Swarrow\,\mu_b & \big\downarrow{\scriptstyle\beta_b} \\ Ha & \xrightarrow[Hf]{} & Hb & \xrightarrow[\delta_b]{} & Kb \end{array} \tag{2.4.7}$$

and dually (here the top and bottom boundaries agree by ordinary naturality of γ and δ). The following result shows that as for 2-functors, the two definitions of transformation between double functors agree.

Proposition 2.4.8. *Given strict double functors $F, G\colon \mathbb{D} \to \mathbb{E}$, the following data are equivalent.*

(i) A companion pair $(\alpha, \alpha')\colon F \to G$ in $[\![\mathbb{D},\mathbb{E}]\!]$.

(ii) For each $a \in \mathbb{D}$, a companion pair $(\alpha_a, \alpha'_a)\colon Fa \to Ga$ such that

(a) the α_a are natural with respect to vertical 1-cells,

(b) the α'_a are natural with respect to horizontal 1-cells, and

(c) for any 2-cell
$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\Big\downarrow & \Swarrow_{\theta} & \Big\downarrow{\scriptstyle h} \\ c & \xrightarrow[k]{} & d \end{array}$$
in $\mathbb{D}$, the 'whiskering' composites

$$\begin{array}{ccccccc}
Fa & \xrightarrow{Ff} & Fb & = & Fb & = & Fb \\
{\scriptstyle Fg}\Big\downarrow & \Swarrow_{F\theta} & \Big\downarrow{\scriptstyle Fh} & & \Big\downarrow{\scriptstyle Fh} & & \Big\downarrow{\scriptstyle \alpha_b} \\
Fc & \xrightarrow[Fk]{} & Fd & = & Fd & & Gb \\
\Vert & & \Vert & \Swarrow & \Big\downarrow{\scriptstyle \alpha_d} & & \Big\downarrow{\scriptstyle Gh} \\
Fc & \xrightarrow[Fk]{} & Fd & \xrightarrow[\alpha'_d]{} & Gd & = & Gd \\
\Vert & & & & \Vert & & \Vert \\
Fc & \xrightarrow[\alpha'_c]{} & Gc & \xrightarrow[Gk]{} & Gd & = & Gd
\end{array}$$

and

$$\begin{array}{ccccccc}
Fa & = & Fa & \xrightarrow{Ff} & Fb & \xrightarrow{\alpha'_b} & Gb \\
\Vert & & \Vert & & & & \Vert \\
Fa & = & Fa & \xrightarrow{\alpha'_a} & Ga & \xrightarrow{Gf} & Gb \\
{\scriptstyle Fg}\Big\downarrow & & {\scriptstyle \alpha_a}\Big\downarrow & \Swarrow & \Vert & & \Vert \\
Fc & & Ga & = & Ga & \xrightarrow{Gf} & Gb \\
{\scriptstyle \alpha_c}\Big\downarrow & & {\scriptstyle Gg}\Big\downarrow & & {\scriptstyle Gg}\Big\downarrow & \Swarrow_{G\theta} & \Big\downarrow{\scriptstyle Gh} \\
Gc & = & Gc & = & Gc & \xrightarrow[Gk]{} & Gd
\end{array}$$

are comrades under the companion pairs $\alpha_b \approxeq \alpha'_b$ and $\alpha_c \approxeq \alpha'_c$.

Proof. Clearly the data (i) include, in particular, companion pairs $(\alpha_a, \alpha'_a)\colon Fa \to Ga$ such that (ii)a and (ii)b hold, and it is easy to verify that (2.4.7) applied to the 2-cells exhibiting the companion pair $\alpha \approxeq \alpha'$ implies (ii)c. Conversely, supposing we are given the data (ii), we define

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ {\scriptstyle \alpha_a}\Big\downarrow & \Swarrow_{\alpha_f} & \Big\downarrow{\scriptstyle \alpha_b} \\ Ga & \xrightarrow[Gf]{} & Gb \end{array}$$

to be the comrade of the equality $\alpha'_b \circ Ff = Gf \circ \alpha'_a$ (which is true since α' is assumed natural), and dually for the 2-cell components of α'. The assumption (ii)c is precisely what is needed to show that these 2-cell components are natural with respect to vertical composition of 2-cells. Finally, the axioms of a companion pair imply that $\alpha_f \boxbar \alpha_g = \alpha_{gf}$ and $\alpha_{1^a} = 1^{\alpha_a}$ (so that α is a vertical transformation, and dually α' is a horizontal transformation), and that the 2-cells exhibiting the companion pairs $\alpha_a \approxeq \alpha'_a$ satisfy (2.4.7). $\square$

We call such data a **pointwise-strong transformation**, and we write $\mathcal{D}bl$ for the 2-category of strict double categories, strict double functors, and pointwise-strong transformations.

We can then apply various 2-categorical notions inside $\mathcal{D}bl$. For instance, we define a **double equivalence** to be an internal equivalence in $\mathcal{D}bl$. Thus, it consists of strict double functors $F\colon \mathbb{D} \to \mathbb{E}$ and $G\colon \mathbb{E} \to \mathbb{D}$ together with pointwise-strong natural isomorphisms $GF \cong \mathrm{Id}_{\mathbb{D}}$ and $FG \cong \mathrm{Id}_{\mathbb{E}}$. This notion of equivalence between double categories will be relevant when we come to compare different types of indexed monoidal structure in chapter 4.

We also observe that $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{S}tr$ (though not $\mathcal{C}onj$) are all strict 2-functors from $\mathcal{D}bl$ to 2-$\mathcal{C}at$. In other words, a pointwise-strong transformation induces both a vertical, horizontal, and a strong 2-natural transformation. It follows that any double equivalence $\mathbb{D} \simeq \mathbb{E}$ induces 2-equivalences $\mathcal{H}\mathbb{D} \simeq \mathcal{H}\mathbb{E}$, $\mathcal{V}\mathbb{D} \simeq \mathcal{V}\mathbb{E}$, and $\mathcal{S}tr\mathbb{D} \simeq \mathcal{S}tr\mathbb{E}$. If we were to define a tricategory containing $\mathfrak{Dbl}$ and $\mathcal{D}bl$, we would expect these three functors to also extend to this tricategory.

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