

PROJECT DESCRIPTION

MICHAEL A. SHULMAN

1. INTRODUCTION

Category theory is increasingly important in modern mathematics. Many modern theorems would be cumbersome or impossible to state and prove without categorical language. Categorical notions also provide a framework to connect different fields of mathematics, and they can lead to new constructions and questions even in established fields. However, it is trickier than it may appear to find the most appropriate categorical structure for any particular application. Sometimes the best structure is ‘obvious’ to category theorists but has never been effectively communicated to other mathematicians. Other times it is not yet known, because no one with a sufficiently categorical mindset has studied the question.

Thus, there is a need for category theorists who are also willing and able to mediate between category theory and the rest of mathematics, by communicating categorical ideas to other mathematicians and helping them attack their problems with a categorical mindset. My goal is to be such a mathematician myself, and to train others to do it as well. Category theory has a lot to offer other fields of mathematics, both in its current state and its potential future, but only if such a symbiosis can be created and maintained. Category theory will benefit from this as well, of course: many or most interesting categorical ideas were first suggested by mathematics outside of pure category theory.

Algebraic topology is one fertile ground for this synergy, and much of my work so far has been prompted by ideas in algebraic topology, although many parts of it have wider potential application. Because my work so far has been in several fairly unrelated areas, as are my intentions for future research, I will discuss each subject in its own section; what they have in common are the goals and perspective described above.

2. ENRICHED HOMOTOPY LIMITS AND COLIMITS

There are two points of view on homotopy limits: a traditional one based on explicit constructions, and a modern abstract one based on derived functors. The former has calculational advantages and is often used in practice, while the latter has better formal properties. That the two agree is well-known to experts, but not as widely appreciated as it should be. In the first, expository, part of my paper [Shu06] I gave a proof of this equivalence, emphasizing the categorical bar construction. Then in the second part, I extended this result to *enriched* homotopy limits and colimits, which are very important in practice, but had never really been considered before from the modern abstract viewpoint.

In the process of proving the enriched comparison, it turned out that the now-standard formalism for abstract homotopy theory—Quillen model categories—was inadequate, because suitable model category structures are not always inherited by enriched diagram categories. Thus, a substantial part of [Shu06] was spent extending the more general formalism of [DHKS04], based on *homotopical categories*, to the enriched context. This new formalism

also enabled me to give an abstract account of the ‘middle derived functors’ used in [MS06], and I expect it to find further applications in the future.

3. FRAMED BICATEGORIES

It is well-known by now that an appropriate abstract framework for duality theory is a symmetric monoidal category. Much of this theory extends easily to a *bicategory*, such as the bicategory $\mathcal{M}od$ of noncommutative rings and bimodules. Duality in a bicategory is essential to the parametrized duality of [MS06], but the authors of [MS06] realized that something important was missing: the action of *base change functors*, such as the restriction, extension, and coextension of scalars functors associated to a ring homomorphism. Moreover, many classical bicategorical notions, such as biadjunctions and biequivalences, are ill-suited to bicategories such as $\mathcal{M}od$.

In [Shu07b] I described a structure called a *framed bicategory*, which is a bicategory equipped with a second type of morphisms (e.g. ring homomorphisms), which act on the 1-cells (e.g. bimodules) through base change functors. Framed bicategories turn out to be ubiquitous in mathematics: in addition to rings and bimodules, and spaces and parametrized spectra, there are framed bicategories of DGAs and complexes of bimodules, structured ring spectra and bimodules, categories and distributors, sets and relations, and manifolds and cobordisms, to name a few. This perspective may also be useful for similar base change situations in algebraic geometry.

Framed bicategories themselves are the objects of a 2-category, so the powerful methods of 2-category theory developed by Australian category theorists can be applied to them. For instance, they come automatically with notions of *framed equivalence*, *framed adjunction*, and *monoidal framed bicategory*. In fact, a structure equivalent to a framed bicategory, called an ‘equipment’, has also been studied by category theorists (see [Woo82, Woo85, Str81, Ver92]). The advantage of framed bicategories is that their base change functors are defined in a ‘nonalgebraic’ way, so that an object such as f^*X is characterized by a universal property; this simplifies the theory by reducing the necessary coherence axioms.

I am currently working on several extensions and applications of framed bicategories. Kate Ponto and I are incorporating into the framed context the *bicategorical shadows* used in [Pon07] to define traces in bicategories. (See [Shu08b] for a talk I gave about these traces.) These shadows are actually a higher-dimensional version of traces—a fact which is known to category theorists, but not to those for whom it would be most useful. This is a good example of the need for category theory to engage with the rest of mathematics.

I also see a need for a *homotopical* theory of framed bicategories. This may include a notion of *model framed bicategory*, but must also go beyond that. For example, the framed bicategory of parametrized spectra used in [MS06] should be the ‘homotopy framed bicategory’ of a point-set level one; but the latter cannot be expected to be a model framed bicategory, because its composition is a composite of left and right Quillen functors.

Framed bicategories also provide a natural framework in which to define generalized multicategories, or ‘operads with many objects’. These are a categorical structure with operations codified by a monad, which have recently become important in algebraic topology and higher category theory. So far, different authors have defined generalized multicategories in very different settings; I plan to show that all these approaches can be unified and simplified by the notion of a *framed monad*.

4. DOUBLE CATEGORIES IN HOMOTOPY THEORY

One important puzzle in homotopy theory is the difference between *left* and *right* derived functors. Both extract homotopical information from a point-set level functor, but the importance of the ‘leftness’ or ‘rightness’ is unclear, especially when a given functor is both a left and a right adjoint. The related question of commutation relations between left and right derived functors is a serious mathematical concern with real depth; it arose in [MS06] and is also important for Grothendieck’s ‘six-functor formalism’ in algebraic geometry.

In [Shu07a] I pointed out that *double categories* provide a useful framework in which to talk about comparisons between left and right derived functors. A double category is like a 2-category, except that it has two different classes of 1-cells, which we call ‘horizontal’ and ‘vertical’. Double categories were originally defined by Ehresmann in [Ehr63, BE74] but have not become well-known outside the categorical community until fairly recently.

Framed bicategories are actually an ‘asymmetric’ type of double category, in which the horizontal 1-cells are the 1-cells from the bicategory (such as bimodules, which are not ‘morphisms’ in the usual sense) and the vertical 1-cells are the new morphisms (such as ring homomorphisms). However, there are other sorts of double categories in which vertical and horizontal 1-cells are ‘dual’ and should be treated on an equal footing. In one such double category, $\mathbb{Q}\mathbf{Model}$, the objects are Quillen model categories, the vertical 1-cells are *left* Quillen functors, and the horizontal 1-cells are *right* Quillen functors.

In [Shu07a] I showed that the operation of taking derived functors, which is well-known to define a pseudofunctor on the 2-category of model categories and Quillen adjunctions, actually extends to a ‘double pseudofunctor’ defined on $\mathbb{Q}\mathbf{Model}$. Since the 2-cells in $\mathbb{Q}\mathbf{Model}$ are natural transformations relating composites of left and right Quillen functors, this double pseudofunctor gives information about the relationship between left and right derived functors. In particular, it shows that the calculus of *mates* descends along derived functors. This can then be used to prove commutation relations between left and right derived functors.

5. LOCALLY CONSTANT HOMOTOPY SHEAVES

Recently there has been much interest in homotopy theory over a base space B . The existing approaches can be classified into those based on *spaces* with a map to B (or ‘parametrized spaces’), such as [MS06], and those based on *homotopy sheaves* (or ‘stacks’) of spaces on B , such as [Jar87, Lur07]. The former is perhaps more natural to a classical homotopy theorist, while the second may seem more natural to an algebraic geometer.

In [Shu08a] I showed that parametrized spaces may be identified with the ‘sub-homotopy theory’ of homotopy sheaves determined by those sheaves of spaces which are ‘homotopically locally constant’. The crucial intermediaries are a model structure on parametrized spaces constructed in [LJ02], and a ‘mixed’ version of the classical model structure, whose existence follows from [Col06]. To show that the base change functors often also agree, I used a special case of my result in [Shu07a] to compare a pair of left and right derived functors.

This may be regarded as a homotopical version of the classical fact that covering spaces over B can be identified with locally constant sheaves of *sets* on B . It may also prove useful in the study of parametrized spaces, since the derived right adjoint f_* (the parametrized version of cohomology) is easier to define in the sheaf context. It has already been used, in an undistributed preprint, to relate the approach of [EKMM97] to unit and Thom spectra to the quasi-categorical techniques of [Lur07].

I plan to pursue two generalizations of this. First, there is the equivariant context, in which a fixed group G acts on the base and total spaces. Even the correct equivariant notion of sheaf is not entirely clear and needs to be investigated (see, for example, [BL94]). Secondly, one hopes for an analogous relationship between parametrized *spectra* and sheaves of spectra, which is even more important for the analysis of cohomology; the theory of spectra is one of the main motivations for both [MS06] and [Lur07].

6. LIMITS FOR LAX MORPHISMS

By ‘categorification’ is meant the lifting of ordinary structures defined on sets to analogous structures defined on categories, such as the passage from a monoid to a monoidal category. Sometimes this is straightforward, but other times new phenomena emerge. For example, there is usually more than one type of morphism between categorified structures: in addition to the *strong* morphisms, which preserve the structure up to isomorphism, there are *lax* (and *colax*) ones, which preserve it only up to a non-invertible map in one direction (or the other). Attention often focuses on the strong morphisms, but the lax ones are also ubiquitous and important: lax monoidal functors are at least as common as strong ones, and in fact any functor at all between categories with (say) finite colimits is an example of a lax morphism (which is strong if it preserves the colimits).

However, lax morphisms are less well-behaved formally. For example, all suitably categorical limit-constructions, such as products and comma categories, are inherited by categories of strong morphisms, but not by categories of lax morphisms. In recent unpublished work, Steve Lack and I (building on his paper [Lac05]) have characterized exactly those limits which do lift to categories of lax morphisms. Steve spoke about our work at the International Category Theory Conference 2008 [Lac08]. In future work, we plan to extend this to limit-constructions that combine strong, lax, and colax morphisms.

These results are useful in the study of basic categorical structures such as monoidal categories, fibered categories, and categories with limits, but they may also have applications in homotopy theory. There are indications (see [Gar]) that it may be possible to ‘algebraicize’ model categories so that right and left Quillen functors are the lax and oplax morphisms. If so, then the limit-constructions admitted by such functors will provide new ways of constructing model categories, which is frequently a difficult problem.

7. ENRICHED HIGHER CATEGORY THEORY

While much progress has been made in higher category theory in recent years, most work has focused on the unenriched case. Since in practice, many categories are enriched, we expect to eventually need enriched higher categories as well. From this point of view, one promising notion of higher category is the iterated operadic approach proposed by Trimble and May (summarized in [Lei02]), but so far little has been done to relate this approach to others (a notable exception being [Che]). Even the operadic notion of $(\infty, 1)$ -category, usually called an A_∞ -category, has not yet been connected into the web of Quillen equivalences that relates most other known definitions of $(\infty, 1)$ -category (see [Ber]). This is especially unfortunate because A_∞ -categories enriched over chain complexes are already used in several areas of algebra and geometry, such as the theory of Fukaya categories.

Peter May has suggested a way of comparing (topological) A_∞ -categories to Segal categories, based on the methods used in [MT78, Tho79] to compare delooping machines; I hope

to work with him in implementing this idea. If successful, this comparison may also shed light on the relationship between Trimble-May n -categories and other types of n -categories such as Tamsamani-Simpson. A lofty goal would be a complete homotopy theory for operadic categories, including A_∞ -categories as well as ordinary enriched categories. There is, as yet, no general theorem giving a model structure for enriched categories, although piecemeal results are known for specific enrichments (see [Ber07, Tab05, Lac02]).

8. CHOICE OF SPONSOR AND HOST

The University of Chicago has one of the strongest category theory communities of any university in the United States, in a tradition stretching back to Saunders MacLane and now led by Peter May. Moreover, the algebraic topology community, also led by Peter May, is especially open to new categorical ideas (see [MS06, Pon07, Joh]). Niles Johnson, currently a graduate student at the University of Chicago, is studying bicategorical Morita theory, and Kate Ponto, with whom I am working on shadows and traces in framed bicategories, is nearby at the University of Notre Dame. Moreover, the University of Chicago is also well-connected to the larger Midwestern algebraic topology community, so that new ideas can be dispersed more widely.

The University of Chicago is also now attracting noticeable numbers of graduate students interested in studying category theory, and who will especially benefit from the presence of someone more experienced. I am already acting as co-advisor for two second-year graduate students, Claire Tomesch and Emily Riehl, with probably a third joining us next year.

9. EDUCATION

The success of mathematics requires that we not only develop new ideas, but effectively communicate existing ones to students, both to those who may eventually become mathematicians and those who will not. I have always had a strong interest in education. I have taught for six years at a summer program for high school students called Mathcamp, and coordinated its academic program for two years. I have taught college classes at a number of different levels, ranging from precalculus to one-variable real analysis, and won a departmental teaching award. I have participated in and served on the steering committee for the Directed Reading Program, a part of Chicago's VIGRE program which pairs undergraduates with graduate students to do independent study in mathematics. I have mentored in and helped to organize the summer VIGRE REU program for Chicago undergraduates. I have assisted for two quarters with the SESAME program, which trains mathematics teachers for the Chicago Public Schools. I have taught at the graduate student 'warm-up program' which prepares incoming graduate students for the first-year graduate classes. And I have given numerous seminar talks for graduate students on diverse subjects. By remaining at the University of Chicago, I would be able to continue to contribute to its already strong graduate and undergraduate math education programs, in these and other ways.

In a way, the problem of math education is parallel to the problem of category theory, on a larger and far more important scale. We, as mathematicians, have developed theories which are beautiful and useful to the rest of the world. However, if we don't communicate those theories effectively, not only will we have no successors to continue developing our theories, but our theories will fail to find their most effective uses, and we will fail to discover the interesting mathematics which arises from questions outside of mathematics.

REFERENCES

- [BE74] A. Bastiani and C. Ehresmann. Multiple functors I. Limits relative to double categories. *Cahiers de Topologie et Geometrie Differentielle Categoriqes*, 15:215–292, 1974.
- [Ber] Julia E. Bergner. A survey of $(\infty, 1)$ -categories. To appear in the proceedings of the IMA workshop on n -categories, arXiv:math/0610239.
- [Ber07] Julia E. Bergner. A model category structure on the category of simplicial categories. *Trans. Amer. Math. Soc.*, 359(5):2043–2058 (electronic), 2007, arXiv:math.AT/0406507.
- [BL94] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.
- [Che] Eugenia Cheng. Comparing operadic theories of n -category. arXiv:0809.2070.
- [Col06] Michael Cole. Mixing model structures. *Topology Appl.*, 153(7):1016–1032, 2006.
- [DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy Limit Functors on Model Categories and Homotopical Categories*, volume 113 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2004.
- [Ehr63] Charles Ehresmann. Catégories structurées. *Ann. Sci. École Norm. Sup. (3)*, 80:349–426, 1963.
- [EKMM97] A. D. Elmendorf, I. Kris, M. A. Mandell, and J. P. May. *Rings, Modules, and Algebras in Stable Homotopy Theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997. With an appendix by M. Cole.
- [Gar] Richard Garner. Understanding the small object argument. arXiv:0712.0724.
- [IJ02] Michele Intermont and Mark W. Johnson. Model structures on the category of ex-spaces. *Topology Appl.*, 119(3):325–353, 2002.
- [Jar87] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.
- [Joh] Niles Johnson. Morita theory for derived categories: A bicategorical perspective. arXiv:0805.3673.
- [Lac02] Stephen Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.
- [Lac05] Stephen Lack. Limits for lax morphisms. *Appl. Categ. Structures*, 13(3):189–203, 2005.
- [Lac08] Stephen Lack. Strict morphisms and weak morphisms. Talk at International Category Theory Conference 2008. Slides available at <http://www-lmpa.univ-littoral.fr/CT08/slides/Lack.pdf>, June 2008.
- [Lei02] Tom Leinster. A survey of definitions of n -category. *Theory Appl. Categ.*, 10:1–70 (electronic), 2002.
- [Lur07] Jacob Lurie. *Higher Topos Theory*. 2007, arXiv:math.CT/0608040.
- [MS06] J. P. May and J. Sigurdsson. *Parametrized Homotopy Theory*, volume 132 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., 2006, arXiv:math.AT/0411656.
- [MT78] J. P. May and R. Thomason. The uniqueness of infinite loop space machines. *Topology*, 17(3):205–224, 1978.
- [Pon07] Kathleen Ponto. *Fixed point theory and trace for bicategories*. PhD thesis, University of Chicago, 2007.
- [Shu06] Michael A. Shulman. Homotopy limits and colimits and enriched homotopy theory. Submitted, 2006, arXiv:math.CT/0610194.
- [Shu07a] Michael A. Shulman. Comparing composites of left and right derived functors. Submitted, 2007, arXiv:0706.2868.
- [Shu07b] Michael A. Shulman. Framed bicategories and monoidal fibrations. Submitted, 2007, arXiv:0706.1286.
- [Shu08a] Michael A. Shulman. Parametrized spaces model locally constant homotopy sheaves. *Topology and its Applications*, 155(5):412–432, 2008, arXiv:0706.2874.
- [Shu08b] Michael A. Shulman. Traces in monoidal categories and bicategories. Talk at CCR La Jolla Slides available at <http://www.math.uchicago.edu/~shulman/research/ccrtraces.pdf>, August 2008.
- [Str81] Ross Street. Cauchy characterization of enriched categories. *Rend. Sem. Mat. Fis. Milano*, 51:217–233 (1983), 1981. Reprinted as Repr. Theory Appl. Categ. 4:1–16, 2004.

- [Tab05] Goncalo Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.
- [Tho79] R. W. Thomason. Uniqueness of delooping machines. *Duke Math. J.*, 46(2):217–252, 1979.
- [Ver92] Dominic Verity. *Enriched categories, internal categories, and change of base*. PhD thesis, University of Cambridge, April 1992. Macquarie Mathematics Report No. 93-123.
- [Woo82] R. J. Wood. Abstract proarrows. I. *Cahiers Topologie Géom. Différentielle*, 23(3):279–290, 1982.
- [Woo85] R. J. Wood. Proarrows. II. *Cahiers Topologie Géom. Différentielle Catég.*, 26(2):135–168, 1985.