COMPARING COMPOSITES OF LEFT AND RIGHT DERIVED FUNCTORS

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Abstract. We introduce a new categorical framework for studying derived functors, and in particular for comparing composites of left and right derived functors. Our central observation is that model categories are the objects of a double category whose vertical and horizontal arrows are left and right Quillen functors, respectively, and that passage to derived functors is functorial at the level of this double category. The theory of conjunctions and mates in double categories, which generalizes the theory of adjunctions in 2-categories, then gives us canonical ways to compare composites of left and right derived functors.

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1. Introduction

Part of the general philosophy of category theory is that morphisms are often more important and subtler than objects. This applies also to categories and functors themselves, as well as to more complicated categorical structures, such as model categories and Quillen adjunctions. The passage from a model category to its homotopy category is well understood, but the passage from Quillen functors to derived functors seems more subtle and mysterious. In particular, the distinction between *left* and *right* derived functors is not well understood at a conceptual level.

For instance, it is well-known that taking derived functors of Quillen functors between model categories is pseudofunctorial—as long as all derived functors involved have the same 'handedness'. That is, we have coherent isomorphisms such as $\mathbf{L}(GF) \cong \mathbf{L}G \circ \mathbf{L}F$. However, not infrequently it happens that we want to compose a Quillen left adjoint with a Quillen right adjoint, and compare the result with another such composite. The aim of this paper is to provide a general categorical framework in which to speak about such comparisons.

We should stress at the outset that we do not give any general method to prove that two composites of left and right derived functors are isomorphic. Like the question of whether a given Quillen adjunction is a Quillen equivalence, the way to attack this question seems to depend a great deal on the particular situation. What we do give is a calculus describing the relationships between the natural transformations which compare such composites; this generalizes the calculus of 'mates' in 2-categories.

Any particular application of this calculus (and several have already appeared in print) tends to be fairly obvious, so the need for a general theory has perhaps not been greatly felt. The proofs are likewise quite trivial once the definitions are understood. However, we feel that our general framework is valuable, not just to avoid duplication of effort in particular cases, but because of the light it sheds on the distinction between left and right derived functors.

Many people have long felt uncomfortable with the category (or 2-category) of model categories and Quillen adjunctions. To define such a category, one must choose whether a Quillen adjunction goes in the direction of the right adjoint or the left adjoint, and either choice is asymmetrical and aesthetically unsatisfactory. We explain here that model categories actually form a *double category*, a categorical structure which includes both the left and right Quillen functors as different types of morphism. Quillen adjunctions then appear as 'conjunctions' in this double category; see Examples 5.3 and 5.8. The passage from Quillen functors to derived functors also naturally lives in the double-categorical context; see §6. This gives the right context for considering composites of left and right derived functors, about which standard model category theory has little to say.

Category theorists will be interested to see that there is also a formal analogy between, on the one hand, *left* Quillen functors and *oplax* monoidal functors (or oplax morphisms for any 2-monad), and on the other hand, between *right* Quillen functors and *lax* monoidal functors. A functor which is both left and right Quillen corresponds to a strong monoidal functor, while a Quillen adjunction corresponds to a 'doctrinal adjunction' or 'lax/oplax adjunction'.

The plan of this paper is as follows. Since many definitions and notations in the theory of double categories are not yet standard, and different authors take quite different points of view, we spend most of the paper developing those parts of this theory relevant for us. In §2 we define double categories and give our main examples, and in §3 we define the relevant notion of morphism between double categories, which we call a *double pseudofunctor*. Then in §§4–5 we describe the theory of *companions* and *conjoints*, which generalizes the calculus of adjunctions and mates in 2-categories to double categories; this appears in various forms in various papers, but our perspective on it is slightly different due to the examples we have in mind.

In §6 we finally focus on Quillen adjunctions, and prove that passage to derived functors defines a double pseudofunctor. Together with the theory of conjunctions described in §5, this implies the desired calculus of derived natural transformations relating composites of left and right derived functors. Then in §§7–8 we describe some applications of the theory.

2. Double categories

Double categories are a fundamental categorical structure, like ordinary categories, and as such they admit many different viewpoints and can play many different roles. These different roles also lead to variants of the definition, differing mostly in the level of strictness. A good reference for double categories from a point of view similar to ours, and also for the 2-categorical notions we will use, is [KS74].

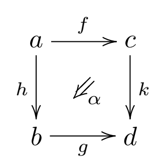
In this paper, **double category** will always mean a *strict* double category; this is defined to be an internal category in the category **Cat** of categories. Thus a double category $\mathbb{D}$ consists of two categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with source, target, identity, and composition functors:

$$S, T\colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$$
$$I\colon \mathbb{D}_0 \to \mathbb{D}_1$$
$$C\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$$

satisfying suitable axioms.

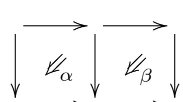
However, the definition is symmetric in a way which this description does not capture. To see this, we think of the category $\mathbb{D}_0$ as drawn vertically, calling its objects *0-cells* or just *objects* and its morphisms *vertical 1-cells*. If f is an object of $\mathbb{D}_1$ with $S(f) = a$ and $T(f) = b$, we draw f as a horizontal arrow from a to b and call it a *horizontal 1-cell*. And if $\alpha\colon f \to g$ is a morphism of $\mathbb{D}_1$ with $S(\alpha) = h$ and $T(\alpha) = k$, where $h\colon a \to b$ and $k\colon c \to d$ are morphisms in $\mathbb{D}_0$, we draw α in a square

(2.1)

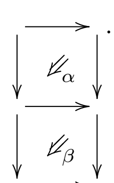


and call it a *2-cell*. We think of such an α as a morphism from 'the composite kf' to 'the composite gh', even though such composites do not actually exist.

The axioms of a double category tell us that the horizontal and vertical 1-cells each form ordinary categories, and that the 2-cells can be 'pasted' both horizontally and vertically. We write $\alpha \boxbar \beta$ for the horizontal composite of 2-cells



and $\beta \boxminus \alpha$ for the vertical composite



Every object a has both a vertical identity 1_a and a horizontal identity 1^a, every vertical arrow $g\colon a \to b$ has an identity 2-cell

$$\begin{array}{ccc} & \xrightarrow{1^a} & \\ g\big\downarrow & \Swarrow 1_g & \big\downarrow g \\ & \xrightarrow[1^b]{} & \end{array},$$

every horizontal arrow $f\colon a \to c$ has an identity 2-cell

$$\begin{array}{ccc} & \xrightarrow{f} & \\ 1_a\big\downarrow & \Swarrow 1_f & \big\downarrow 1_c \\ & \xrightarrow[f]{} & \end{array},$$

and we have

$$1^{1_a} = 1_{1^a}.$$

We will often write identity arrows simply as equalities.

The following example is fundamental.

Example 2.2. There is a double category $\mathbb{C}\mathbf{at}$ whose objects are categories, whose vertical and horizontal 1-cells are functors, and whose 2-cells of the form (2.1) are natural transformations $\alpha\colon kf \to gh$.

A similar double category can be constructed with any 2-category $\mathcal{K}$ replacing $\mathcal{C}at$. This is sometimes called the double category $\mathbb{Q}\mathbf{uin}(\mathcal{K})$ of **quintets** in $\mathcal{K}$, since a 2-cell in $\mathbb{Q}\mathbf{uin}(\mathcal{K})$ is defined by a quintet (f, g, h, k, α) where $\alpha\colon kf \to gh$ is a 2-cell in $\mathcal{K}$. For example, from any monoidal category $\mathscr{W}$ we can construct a double category $\mathscr{W}$-$\mathbb{C}\mathbf{at}$ of $\mathscr{W}$-categories, $\mathscr{W}$-functors, and $\mathscr{W}$-transformations.

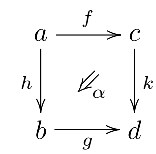
Conversely, any double category has two underlying 2-categories, defined as follows. We say that a 2-cell of the form

$$\begin{array}{ccc} & \longrightarrow & \\ \big\| & \Swarrow & \big\| \\ & \longrightarrow & \end{array}$$

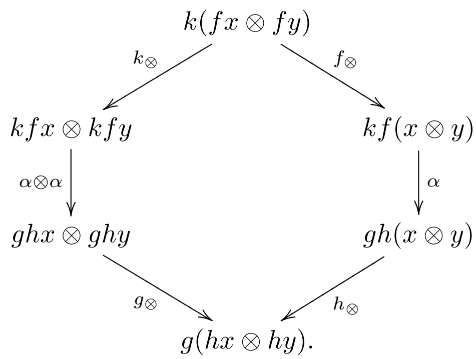
is **h-globular**. The objects, horizontal 1-cells, and h-globular 2-cells of a double category $\mathbb{D}$ form a 2-category called the **horizontal 2-category** $\mathcal{H}\mathbb{D}$ of $\mathbb{D}$. Similarly, we define **v-globular** 2-cells and the **vertical 2-category** $\mathcal{V}\mathbb{D}$.

Of course, much of the interest of double categories comes from the fact that the horizontal and vertical 1-cells can be different. In some applications, such as [Shu07a], the two play fundamentally different roles; these double categories tend to be 'weak' in one direction but not the other. Double categories which are weak in both directions were considered in [Ver92]. By contrast, in the double categories we are interested in here, which are strict in both directions, the two sorts of 1-cells play essentially symmetrical roles, representing two different types of morphisms between the 0-cells. The following two examples are paradigmatic for us.

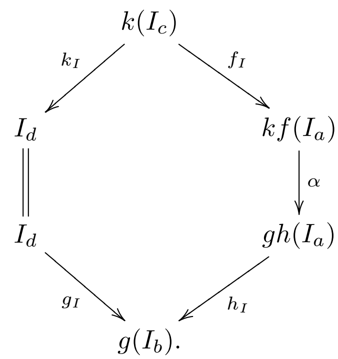
Example 2.3. There is a double category $\mathbb{M}\mathbf{onCat}$ whose objects are monoidal categories, whose horizontal arrows are *lax* monoidal functors, and whose vertical arrows are *oplax* monoidal functors. A 2-cell



is a natural transformation $\alpha\colon kf \to gh$ such that the following diagrams commute:



and



The horizontal 2-category $\mathcal{H}$**𝕄onCat** of **𝕄onCat** is the 2-category $\mathcal{M}onCat_\ell$ of monoidal categories, lax monoidal functors, and monoidal transformations, and dually for $\mathcal{V}$**𝕄onCat**. Thus **𝕄onCat** packages information about both lax and oplax monoidal functors and how they relate to each other.

More generally, we have a double category T-**𝔸lg** of T-algebras, lax and oplax T-morphisms, and T-transformations for any 2-monad T. The case when T is the 2-monad whose algebras are double categories was considered in [GP04].

Example 2.4. There is a double category **ℚ𝕄odel** whose objects are model categories, whose vertical arrows are left Quillen functors, whose horizontal arrows are right Quillen functors, and whose 2-cells are arbitrary natural transformations. Similarly, if $\mathscr{W}$ is a monoidal model category, we have a double category $\mathscr{W}$-**ℚ𝕄odel** of $\mathscr{W}$-model categories.

Our reference for model category theory is [Hov99]; in particular, we assume our model categories to be equipped with functorial factorizations. This is not strictly necessary, but it will make things easier.

We also have a double category **𝕄odel** whose objects are again model categories, but whose vertical arrows are the functors which preserve cofibrant objects and weak equivalences between cofibrant objects, and whose horizontal arrows are the functors which preserve fibrant objects and weak equivalences between fibrant objects. Again, the 2-cells are all natural transformations. By Ken Brown's lemma, **ℚ𝕄odel** is a sub-double-category of **𝕄odel**.

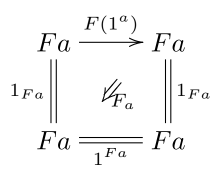
In §6 we will prove that 'deriving' defines a map from **𝕄odel** to **ℂat** which takes a model category $\mathscr{C}$ to its homotopy category $\mathrm{Ho}(\mathscr{C})$, a vertical arrow to its left derived functor, and a horizontal arrow to its right derived functor. We now define the sort of 'map' we are talking about.

3. Double pseudofunctors

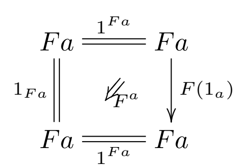
Although all our double categories are strict in both directions, the *maps* between them we need to consider are *weak* in both directions. The intuition is clear—what we call a *double pseudofunctor* must preserve composition and identities in both directions up to specified invertible 2-cells—but the precise definition turns out to be a little tricky. The reader who is uninterested in the details may skim this section.

Definition 3.1. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A **double pseudofunctor** $F\colon \mathbb{D} \to \mathbb{E}$ consists of the following structure and properties.

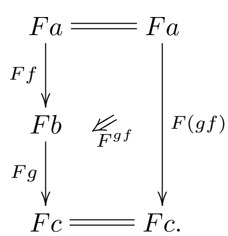
(i) Functions from the objects, vertical 1-cells, horizontal 1-cells, and 2-cells of $\mathbb{D}$ to those of $\mathbb{E}$, preserving sources, targets, and boundaries.
(ii) For each object a of $\mathbb{D}$, 2-cells



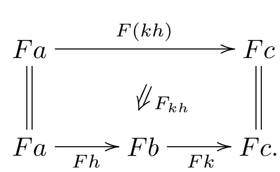
and



in $\mathbb{E}$, of which the first is an h-globular isomorphism and the second a v-globular isomorphism.
(iii) For each composable pair $a \xrightarrow{f} b \xrightarrow{g} c$ of vertical 1-cells in $\mathbb{D}$, a v-globular isomorphism



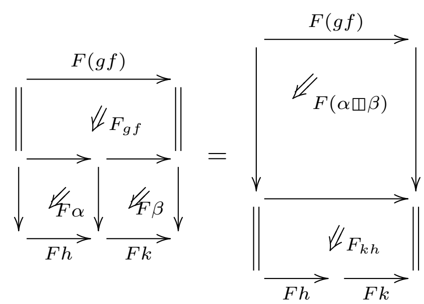
(iv) For each composable pair $a \xrightarrow{h} b \xrightarrow{k} c$ of horizontal 1-cells in $\mathbb{D}$, an h-globular isomorphism

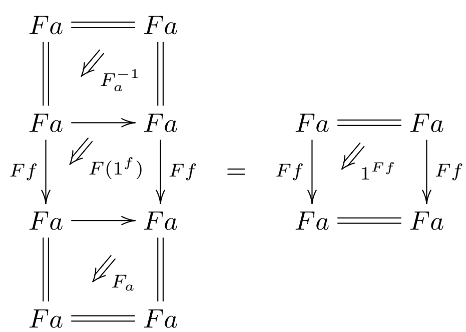


(v) The following coherence axioms hold (the usual coherence axioms for a pseudofunctor in both directions).

$$\begin{aligned} F^{h(gf)} \boxbar \left(1^{Fh} \boxminus F^{gf}\right) &= F^{(hg)f} \boxbar \left(F^{hg} \boxminus 1^{Ff}\right) \\ F^b \boxminus 1^{Ff} &= F^{1_b f} \\ 1^{Ff} \boxminus F^a &= F^{f1_a} \\ F_{h(gf)} \boxminus (F_{gf} \boxbar 1_{Fh}) &= F_{(hg)f} \boxminus (1_{Ff} \boxbar F_{hg}) \\ F_a \boxbar 1_{Ff} &= F_{f1_a} \\ 1_{Ff} \boxbar F_b &= F_{1_b f}. \end{aligned}$$

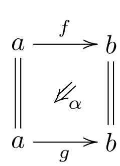
(vi) The following 'double naturality' axioms hold:





as do their transposes involving F^{gf} and F^a.

Note that in general, a double pseudofunctor does not preserve globularity of 2-cells, since it does not preserve either vertical or horizontal identities strictly. However, any h-globular 2-cell



in $\mathbb{D}$ gives rise to a canonical h-globular 2-cell

$$\begin{array}{ccccccc}
Fa & = & Fa & \xrightarrow{Ff} & Fb & = & Fb \\
\Vert & \Swarrow_{F^a} & \downarrow & \Swarrow_{F(\alpha)} & \downarrow & \Swarrow_{(F^a)^{-1}} & \Vert \\
Fa & = & Fa & \xrightarrow[Fg]{} & Fb & = & Fb
\end{array}$$

in $\mathbb{E}$, which we denote $\mathcal{H}F(\alpha)$. It is easy to check that this defines an ordinary pseudofunctor $\mathcal{H}F\colon \mathcal{H}\mathbb{D} \to \mathcal{H}\mathbb{E}$. Similarly, we have $\mathcal{V}F\colon \mathcal{V}\mathbb{D} \to \mathcal{V}\mathbb{E}$.

Examples 3.2. An ordinary pseudofunctor $F\colon \mathcal{K} \to \mathcal{L}$ gives rise to a double pseudofunctor $\mathbb{Q}\mathbf{uin}(F)\colon \mathbb{Q}\mathbf{uin}(\mathcal{K}) \to \mathbb{Q}\mathbf{uin}(\mathcal{L})$ in a fairly straightforward way. The only wrinkle is that if $\alpha\colon kf \to gh$ is a 2-cell in $\mathbb{Q}\mathbf{uin}(\mathcal{K})$, we must compose $F\alpha$ with the constraints of F on either side to obtain a 2-cell in $\mathbb{Q}\mathbf{uin}(\mathcal{L})$.

In particular, if $F\colon \mathcal{C}at \to \mathcal{C}at$ is a pseudofunctor, we obtain a double pseudofunctor $\mathbb{Q}\mathbf{uin}(F)\colon \mathbb{C}\mathbf{at} \to \mathbb{C}\mathbf{at}$, and some of the double pseudofunctors obtained in this way lift to $\mathbb{QM}\mathbf{odel}$. For instance, there is a double pseudofunctor $\mathbb{QM}\mathbf{odel} \to \mathbb{QM}\mathbf{odel}$ which takes a model category $\mathscr{C}$ to its pointed variant $\mathscr{C}_*$; see [Hov99, 1.1.8, 1.3.5].

Example 3.3. If we let $\mathbb{D}^\top$ denote the 'transpose' of a double category, in which the vertical and horizontal arrows are interchanged, then we have a double pseudofunctor $\mathbb{M}\mathbf{odel}^\top \to \mathbb{M}\mathbf{odel}$ which takes $\mathscr{C}$ to $\mathscr{C}^{op}$.

The composite $G \circ F$ of two double pseudofunctors is defined in an obvious way, with one minor wrinkle: since G need not preserve the globularity of the constraints for F, we need to compose with the unit constraints of G when defining the constraints of GF. For example, the composition constraint of GF is given by the composite

$$\begin{array}{ccccccc}
\cdot & = & \cdot & \xrightarrow{GF(gf)} & \cdot & = & \cdot \\
\Vert & \Swarrow_{G^{Fa}} & \downarrow & \Swarrow_{G(F_{gf})} & \downarrow & \Swarrow_{G^{Fc}} & \Vert \\
\cdot & = & \cdot & \longrightarrow & \cdot & = & \cdot \\
\Vert & & & \Swarrow_{G_{(Fg)(Ff)}} & & & \Vert \\
\cdot & & \xrightarrow[GFf]{} & \cdot & \xrightarrow[GFg]{} & & \cdot
\end{array}$$

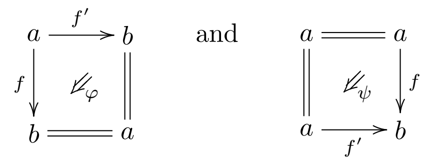
We thereby obtain a category $\mathfrak{Dbl}$ of double categories and double pseudofunctors. The operations $\mathcal{V}$ and $\mathcal{H}$ define functors from $\mathfrak{Dbl}$ to the category $\mathbf{2}\text{-}\mathfrak{Cat}$ of 2-categories and pseudofunctors. In the other direction, $\mathbb{Q}\mathbf{uin}$ defines a functor from $\mathbf{2}\text{-}\mathfrak{Cat}$ to $\mathfrak{Dbl}$.

Remark 3.4. Although we are deliberately ignoring size issues, it is worth pointing out that they can get rather severe. Since most model categories are themselves large, the double category $\mathbb{M}\mathbf{odel}$ must be 'very large', and if it is to be an object of $\mathfrak{Dbl}$, then the latter must be 'extremely large'. Of course, all the formal results we prove will be true in particular cases independent of whether such 'global' categories as $\mathfrak{Dbl}$ actually 'exist'.

4. Companions in double categories

In the double category $\mathbb{C}\mathbf{at}$, every vertical 1-cell has a corresponding horizontal 1-cell which is the same functor, and vice versa. The fact that these 1-cells are 'the same' can be expressed within $\mathbb{C}\mathbf{at}$, even though one is vertical and the other is horizontal. The following terminology is from [GP04, PPD].

Definition 4.1. A **companion pair** in a double category $\mathbb{D}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $f'\colon a \to b$, and 2-cells

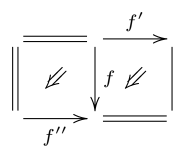


such that $\varphi \boxminus \psi = 1^f$ and $\psi \boxbar \varphi = 1_{f'}$. We say that f' is the **(horizontal) companion** of f and that f is the **(vertical) companion** of f', and write $f \approxeq f'$.

We think of a companion pair as exhibiting an 'isomorphism' between f and f'. When they exist, companions are unique up to globular isomorphism.

Proposition 4.2. *If f' and f'' are both horizontal companions of f, then there is a canonical isomorphism $f' \cong f''$ in $\mathcal{H}\mathbb{D}$, and similarly for vertical companions.*

Proof. An isomorphism is given by the following composite.



Its inverse is the obvious dual construction. □

Of course, in $\mathbb{C}\mathbf{at}$, every 1-cell has a companion, which can be taken to be just the same functor, with φ and ψ both the identity natural transformation. The same is true in any double category of quintets. But in general, not every 1-cell will have a companion.

Example 4.3. A 1-cell in $\mathbb{M}\mathbf{onCat}$ has a companion just when it is a *strong* monoidal functor. More generally, a 1-cell in T-$\mathbb{A}\mathbf{lg}$ has a companion just when it is a strong T-morphism.

Example 4.4. A 1-cell in $\mathbb{QM}\mathbf{odel}$ has a companion just when it is both a left and a right Quillen functor. Thus it is both a left and a right adjoint, and preserves cofibrations, trivial cofibrations, fibrations, and trivial fibrations. Since every weak equivalence factors as a trivial cofibration followed by a trivial fibration, such a functor also preserves all weak equivalences.

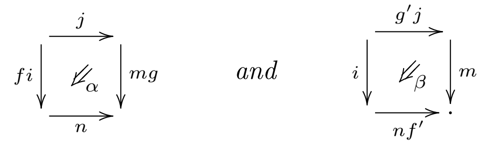
Motivated by these examples, we say that a 1-cell in a double category is **strong** if it has a companion. The following proposition says that 2-cells between strong 1-cells 'behave like quintets'; that is, they don't care whether we use a strong 1-cell or its companion.

Proposition 4.5. *If f and g have horizontal companions f' and g', then there is a canonical isomorphism*

$$\mathcal{V}\mathbb{D}(f,g) \cong \mathcal{H}\mathbb{D}(f',g'). \tag{4.6}$$

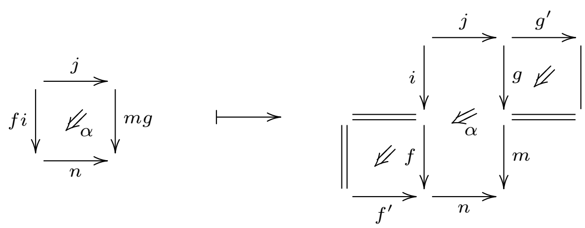
More generally, for any i, j, m, n there is a bijection between 2-cells with the following boundaries:

(4.7)

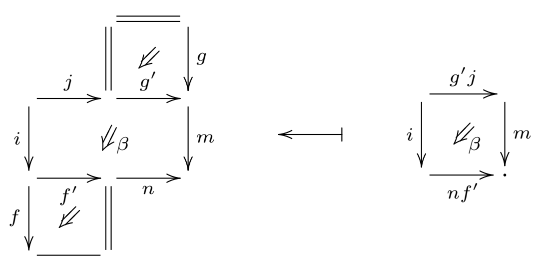


Proof. The bijection is given by the following correspondences.

(4.8)



(4.9)



□

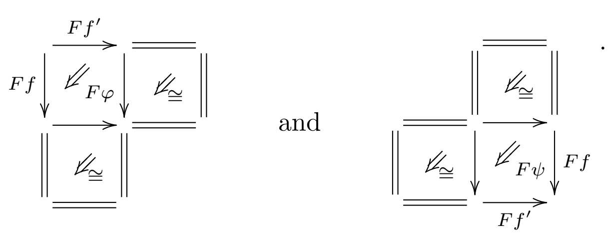
We say that a pair of 2-cells which correspond under (4.7) are **comrades**.

It is easy to see that if f and g have companions f' and g', then $g'f'$ is a companion for gf. Thus, we have a 2-category $\mathcal{S}tr(\mathbb{D})$ whose 0-cells are the 0-cells of $\mathbb{D}$, whose 1-cells are companion pairs in $\mathbb{D}$, and whose 2-cells are comrade-pairs of globular 2-cells in $\mathbb{D}$. We have canonical 2-functors

$$\mathcal{S}tr(\mathbb{D}) \hookrightarrow \mathcal{V}\mathbb{D}$$
$$\mathcal{S}tr(\mathbb{D}) \hookrightarrow \mathcal{H}\mathbb{D}$$

which are full and faithful on hom-categories.

Now suppose that f' is a horizontal companion of f in $\mathbb{D}$ and $F\colon \mathbb{D} \to \mathbb{E}$ is a double pseudofunctor. Then $F(f')$ is a horizontal companion of $F(f)$ in $\mathbb{E}$ when equipped with the 2-cells



This makes $\mathcal{S}tr$ into another functor from $\mathfrak{Dbl}$ to $\mathbf{2}\text{-}\mathfrak{Cat}$. Moreover, by Proposition 4.5, we have a natural inclusion

$$\mathbb{Q}\mathbf{uin}(\mathcal{S}tr(\mathbb{D})) \hookrightarrow \mathbb{D}$$

which is bijective on 2-cells with given boundary. It is easy to check that this is the counit of an adjunction $\mathbb{Q}\mathbf{uin} \dashv \mathcal{S}tr$.

5. Conjunctions in double categories

Adjunctions are inarguably one of the most important concepts in category theory. It is well-known that the concept of adjunction can be fruitfully interpreted in any 2-category, but it is an important and fairly recent discovery (see [GP04, PPD]) that there is a more general notion of adjunction in a *double* category. This notion includes naturally-occurring examples in which the left and the right adjoint are morphisms of different 'types', and may not even be composable! Nevertheless, it is still meaningful to talk about an adjunction. In [GP04] this notion was called an *orthogonal adjunction*; we follow [PPD] in using a different term, to distinguish it from an ordinary adjunction in $\mathcal{V}\mathbb{D}$ or $\mathcal{H}\mathbb{D}$.

Definition 5.1. A **conjunction** in a double category $\mathbb{D}$ consists of a vertical 1-cell $f\colon a \to b$, a horizontal 1-cell $g\colon b \to a$, and 2-cells

$$\begin{array}{ccc} a & = & a \\ {\scriptstyle f}\downarrow & \swarrow\eta & \| \\ b & \xrightarrow[g]{} & a \end{array} \qquad \text{and} \qquad \begin{array}{ccc} b & \xrightarrow{g} & a \\ \| & \swarrow\varepsilon & \downarrow{\scriptstyle f} \\ b & = & b \end{array}$$

(the **unit** and **counit**) such that $\varepsilon \boxbar \eta = 1_g$ and $\varepsilon \boxminus \eta = 1^f$. We say that f is the **left conjoint** and g is the **right conjoint**, and write $f \prec g$.

Clearly a conjunction is 'dual' to a companion pair in a precise sense. A conjunction in $\mathbb{C}\mathbf{at}$ is simply an ordinary adjunction, and similarly in any double category of quintets. However, in other cases it can be significantly more interesting.

Example 5.2. A conjunction in the double category T-$\mathbb{A}\mathbf{lg}$ is precisely a *doctrinal adjunction* [Kel74]. This is an adjunction between T-algebras (such as monoidal categories) in which the left adjoint is oplax and the right adjoint is lax, and the oplax and lax structure maps are mates under the adjunction.

Example 5.3. A conjunction in $\mathbb{QM}\mathbf{odel}$ is precisely a Quillen adjunction; in particular, every 1-cell in $\mathbb{QM}\mathbf{odel}$ has a conjoint. A conjunction in $\mathbb{M}\mathbf{odel}$ is a special case of a *deformable adjunction* in the sense of [DHKS04].

Conjunctions have many of the same good formal properties as adjunctions. The following results are proven in the same way as the corresponding results for companions.

Proposition 5.4. *If $f \prec g$ and $h \prec k$, where $f\colon a \to b$ and $h\colon b \to c$, then $hf \prec gk$.* □

Proposition 5.5. *If $f \prec g$ and $h \prec k$ where $f, h\colon a \to b$, then we have a natural isomorphism*

$$\mathcal{V}\mathbb{D}(f, h) \cong \mathcal{H}\mathbb{D}(k, g), \tag{5.6}$$

under which isomorphisms $f \cong h$ correspond to isomorphisms $k \cong g$. More generally, for any i, j, m, n there is a bijection between 2-cells with the following boundaries:

$$\begin{array}{ccc} \cdot & \xrightarrow{i} & \cdot \\ {\scriptstyle mh}\downarrow & & \downarrow{\scriptstyle fj} \\ \cdot & \xrightarrow[n]{} & \cdot \end{array} \quad \textit{and} \quad \begin{array}{ccc} \cdot & \xrightarrow{ik} & \cdot \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle j} \\ \cdot & \xrightarrow[gn]{} & \cdot \end{array} \quad . \tag{5.7}$$

□

A pair of 2-cells corresponding to each other under these isomorphisms are called **mates**. Putting composition together with mates, we obtain a 2-category $\mathcal{C}onj(\mathbb{D})$ whose objects are those of $\mathbb{D}$, whose 1-cells are the conjunctions in $\mathbb{D}$, and whose 2-cells are the mate-pairs of globular 2-cells in $\mathbb{D}$. Of course, to define $\mathcal{C}onj(\mathbb{D})$ we must choose whether to consider a conjunction as pointing in the direction of the left conjoint or the right conjoint; it is precisely this arbitrariness which the double-categorical context avoids.

Examples 5.8. Of course, $\mathcal{C}onj(\mathbb{C}\mathbf{at})$ is the usual 2-category of categories and adjunctions. Similarly, $\mathcal{C}onj(\mathbb{QM}\mathbf{odel})$ is the 2-category of model categories and Quillen adjunctions.

As is the case for companions, if we have a conjunction $f \prec g$ in $\mathbb{D}$ and a double pseudofunctor $f \colon \mathbb{D} \to \mathbb{E}$, we obtain a conjunction $Ff \prec Fg$ by composing the unit and counit with suitable unit constraints of F. Thus, $\mathcal{C}onj$ defines another functor from $\mathfrak{Dbl}$ to $\mathbf{2}\text{-}\mathfrak{Cat}$. It is important to note that the mate correspondence (5.7) is also preserved by double pseudofunctors.

We also observe that companions mediate between conjunctions and adjunctions.

Proposition 5.9. *Let $f \colon a \to b$ be a vertical 1-cell in $\mathbb{D}$ and let $f' \colon a \to b$ and $g \colon b \to a$ be horizontal 1-cells. Then any two of the following statements imply the third.*

(i) *f' is a horizontal companion of f.*
(ii) *g is a right conjoint of f.*
(iii) *g is a right adjoint of f' in $\mathcal{H}\mathbb{D}$.*

More precisely, any companion pair $f \approxeq f'$ and conjunction $f \prec g$ determine a unique horizontal adjunction $f' \dashv g$, and similarly in the other cases.

Proof. Assuming (i), the correspondence of Proposition 4.5 transforms a unit and counit for a conjunction $f \prec g$ into a unit and counit for a horizontal adjunction $f' \dashv g$, and vice versa. The other cases are similar. □

Example 5.10. Applied to T-$\mathbb{A}\mathbf{lg}$, this says that given a doctrinal adjunction, the left adjoint is strong precisely when the adjunction lifts to the 2-category T-$\mathcal{A}lg_\ell$, and dually. This is part of one of the main results of [Kel74]. (The other part says that any left adjoint in T-$\mathcal{A}lg_\ell$ is automatically strong; this is clearly a special property of T-$\mathbb{A}\mathbf{lg}$ not shared by all double categories.)

Remark 5.11. The *framed bicategories* studied in [Shu07a] are (pseudo) double categories such that every vertical 1-cell $f \colon A \to B$ has both a horizontal companion, there called ${}_fB$, and a right conjoint, there called B_f. The resulting horizontal adjunction ${}_fB \dashv B_f$ is there called the *base change dual pair.*

Similarly, a *folding* or *connection pair* on a double category, as in [BS76, BM99, Fio06], is a functorial choice of a companion for each vertical 1-cell. Of course, in the double categories we are interested in here, not every vertical 1-cell has a companion or conjoint.

6. Derived functors

In this section we construct the homotopy double pseudofunctor. We first recall the definitions of left and right derived functors. Suppose that $f\colon \mathscr{C} \to \mathscr{D}$ is a functor between model categories which preserves weak equivalences between cofibrant objects; for instance, f might be a left Quillen functor. Then the composite $f \circ Q$ preserves all weak equivalences, hence descends to a functor $\mathbf{L}f\colon \operatorname{Ho}(\mathscr{C}) \to \operatorname{Ho}(\mathscr{D})$ which we call the **left derived functor** of f. Dually, if $g\colon \mathscr{D} \to \mathscr{C}$ preserves weak equivalences between fibrant objects, it has a **right derived functor** $\mathbf{R}g\colon \operatorname{Ho}(\mathscr{D}) \to \operatorname{Ho}(\mathscr{C})$ respresented on the point-set level by $g \circ R$.

If $f\colon \mathscr{C} \to \mathscr{D}$ and $h\colon \mathscr{D} \to \mathscr{E}$ both preserve weak equivalences between cofibrant objects, and moreover f preserves cofibrant objects, then the transformation

$$hQfQx \longrightarrow hfQx$$

is a weak equivalence, since fQx is cofibrant and h preserves weak equivalences between cofibrant objects. Thus it represents an isomorphism

$$\mathbf{L}h \circ \mathbf{L}f \xrightarrow{\cong} \mathbf{L}(hf).$$

More trivially, the identity functor $\operatorname{Id}\colon \mathscr{C} \to \mathscr{C}$ preserves all weak equivalences, and we have $\mathbf{L}\operatorname{Id} = Q$; thus the weak equivalence $Q \to \operatorname{Id}$ represents an isomorphism

$$\mathbf{L}\operatorname{Id} \cong \operatorname{Id}.$$

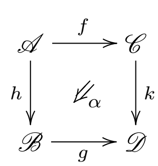
Straightforward verification (see [Hov99, 1.3.7]) shows that these isomorphisms make 'left deriving' into a pseudofunctor

$$\mathbf{L}\colon \mathcal{V}\mathbb{M}\mathbf{odel} \longrightarrow \mathcal{C}at. \tag{6.1}$$

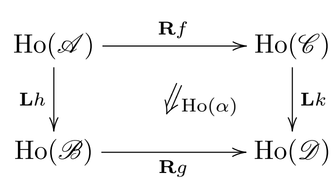
Dually, right deriving is a pseudofunctor

$$\mathbf{R}\colon \mathcal{H}\mathbb{M}\mathbf{odel} \longrightarrow \mathcal{C}at. \tag{6.2}$$

Given the theory we have already developed, it is natural to expect these pseudofunctors to be induced by a double pseudofunctor $\mathbb{M}\mathbf{odel} \to \mathbb{C}\mathbf{at}$, and this is indeed the case. Given a 2-cell



in $\mathbb{M}\mathbf{odel}$, we define its derived 2-cell



to be represented by the composite of the following zigzag:

$$kQfR \xleftarrow{\sim} kQfQR \longrightarrow kfQR \xrightarrow{\alpha} ghQR \longrightarrow gRhQR \xleftarrow{\sim} gRhQ.$$

Note that the backwards maps are weak equivalences, hence represent isomorphisms in $\mathrm{Ho}(\mathscr{D})$, so this makes sense. (Here we assume that the functor Q is chosen so as to preserve fibrant objects.)

Remark 6.3. We can express this more simply as follows. Assume that $X \in \mathscr{C}$ is both cofibrant and fibrant. Then $\mathbf{R}fX \cong fX$ and $\mathbf{L}hX \cong hX$, and modulo these isomorphisms $\mathrm{Ho}(\alpha)_X$ is represented by

$$kQfX \longrightarrow kfX \xrightarrow{\alpha} ghX \longrightarrow gRhX.$$

This suffices to determine $\mathrm{Ho}(\alpha)$, since every object is isomorphic in $\mathrm{Ho}(\mathscr{C})$ to a cofibrant and fibrant one.

Remark 6.4. Even if α is a natural isomorphism $kf \cong hg$, $\mathrm{Ho}(\alpha)$ need not be an isomorphism. This is not a problem for functoriality, however, since the fact that α is an isomorphism in $\mathcal{C}at$ is not visible to the double category $\mathbb{M}\mathbf{odel}$.

Theorem 6.5. *The above constructions define a double pseudofunctor*

$$\mathrm{Ho}\colon \mathbb{M}\mathbf{odel} \longrightarrow \mathbb{C}\mathbf{at} \tag{6.6}$$

such that $\mathcal{V}\,\mathrm{Ho} = \mathbf{L}$ *and* $\mathcal{H}\,\mathrm{Ho} = \mathbf{R}$.

Proof. We have all the data for a double pseudofunctor: we send a model category $\mathscr{C}$ to $\mathrm{Ho}(\mathscr{C})$, a vertical arrow f to $\mathbf{L}f$, a horizontal arrow g to $\mathbf{R}g$, and a 2-cell α to $\mathrm{Ho}(\alpha)$ as defined above. We take the constraint 2-cells to be those of the pseudofunctors $\mathbf{L}$ and $\mathbf{R}$ defined above; the ordinary pseudofunctor coherence axioms then follow from those of $\mathbf{L}$ and $\mathbf{R}$.

Proving the double-naturality axioms and verifying that $\mathcal{V}\,\mathrm{Ho} = \mathbf{L}$ and $\mathcal{H}\,\mathrm{Ho} = \mathbf{R}$ are exercises in filling up big diagrams with lots of naturality squares. The only slightly less trivial input is the fact that if f preserves weak equivalences between cofibrant objects and $Q \xrightarrow{q} \mathrm{Id}$ is a cofibrant replacement functor equipped with a natural weak equivalence to the identity, then $f(qQ)$ and $f(Qq)$ represent the same map in the homotopy category. This is because naturality gives us $q \circ qQ = q \circ Qq$, and at least for cofibrant objects, $f(q)$ becomes an isomorphism in the homotopy category. □

Since $\mathbb{QM}\mathbf{odel} \subset \mathbb{M}\mathbf{odel}$, the double pseudofunctor Ho also restricts to $\mathbb{QM}\mathbf{odel}$. Applying it to companions and conjunctions, respectively, we obtain the following well-known corollaries.

Corollary 6.7. *If* $F\colon \mathscr{C} \to \mathscr{D}$ *is both left and right Quillen (with respect to the same model structures), or more generally preserves weak equivalences both between fibrant objects and between cofibrant objects, then* $\mathbf{L}F \cong \mathbf{R}F$. □

Corollary 6.8. *If* $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : G$ *is a Quillen adjunction, or more generally an adjunction in which* F *preserves weak equivalences between cofibrant objects and* G *preserves weak equivalences between fibrant objects, then we have a derived adjunction* $\mathbf{L}F\colon \mathrm{Ho}(\mathscr{C}) \rightleftarrows \mathrm{Ho}(\mathscr{D}) : \mathbf{R}G$.

In fact, applying the functor $\mathcal{C}onj$ to the map Ho in $\mathfrak{Dbl}$, we obtain the ordinary homotopy pseudofunctor

$$\mathcal{C}onj(\mathrm{Ho})\colon \mathcal{C}onj(\mathbb{QM}\mathbf{odel}) \longrightarrow \mathcal{C}onj(\mathbb{C}\mathbf{at}) = \mathcal{A}dj(\mathcal{C}at)$$

defined in [Hov99, 1.4.3], which sends a Quillen adjunction to its derived adjunction.

However, the double pseudofunctor Ho contains more information than this. In particular, it tells us that *if α and β are mates under a pair of Quillen adjunctions, then* $\mathrm{Ho}(\alpha)$ *and* $\mathrm{Ho}(\beta)$ *are mates under the derived adjunctions.* This gives us canonical ways to compare left and right derived functors; we will consider some examples in §7.

Remark 6.9. Theorem 6.5 admits various generalizations. For example, if $\mathscr{W}$ is a monoidal model category, there is an enriched version

$$\mathrm{Ho}\colon \mathscr{W}\text{-}\mathbb{QM}\mathbf{odel} \longrightarrow \mathrm{Ho}(\mathscr{W})\text{-}\mathbb{C}\mathbf{at}.$$

Generalizing in another direction, instead of passing all the way down to homotopy categories, we can lift the codomain of Ho to the double category $\mathbb{QC}\mathbf{at}$ of quasicategories (see [Joy, Lur07]). We define $\mathbb{QC}\mathbf{at}$ to be the double category of quintets of the 2-category of quasicategories described in [Joy]; its objects are quasicategories (that is, simplicial sets satisfying the inner Kan condition), its vertical and horizontal 1-cells are simplicial maps, and its 2-cells are morphisms in the fundamental category of the simplicial mapping space.

We can also generalize the domain to contain more general *deformable functors* in the sense of [DHKS04]. The only problem is that the composite of two left deformable functors may not in general be left deformable; this is why we required the 1-cells in $\mathbb{M}\mathbf{odel}$ to preserve cofibrant or fibrant objects, in addition to weak equivalences between such. However, insofar as such compositions exist, the results proven above for $\mathbb{M}\mathbf{odel}$ remain valid.

7. Applications to base change

A common situation in which we need to compare left and right derived functors is when dealing with 'base change'. In this case we have a family of model categories $\mathscr{C}_B$ indexed by the objects B of some other category $\mathscr{B}$, and for each map $f\colon A \to B$ in $\mathscr{B}$ we have a Quillen adjunction such as

$$f_!\colon \mathscr{C}_A \rightleftarrows \mathscr{C}_B \;:f^* \qquad \text{or}$$
$$f^*\colon \mathscr{C}_B \rightleftarrows \mathscr{C}_A \;:f_*.$$

These adjunctions depend pseudofunctorially on f, so we have isomorphisms such as $\mathbf{R}(gf)^* \cong \mathbf{R}f^* \circ \mathbf{R}g^*$, but we are often also interested in what we can say about composites such as $\mathbf{L}f_! \circ \mathbf{R}g^*$.

Examples 7.1. $\mathscr{B}$ could be a category of topological spaces, and $\mathscr{C}_B$ could be a category of spaces or spectra parametrized over B, as in [MS06], or a category of 'homotopy sheaves' on B, as in [Jar87, Lur07, Shu07b]. Algebraically, $\mathscr{B}$ could be a category of rings, or DGAs, and $\mathscr{C}_B$ could be a category of chain complexes of B-modules. In a brave new world, $\mathscr{B}$ could be a category of ring spectra, and $\mathscr{C}_B$ the category of B-modules. More exotically, $\mathscr{B}$ could be a category of operads in some monoidal model category, and $\mathscr{C}_B$ the category of B-algebras, as in [BM03].

Suppose we have such a base change situation, and suppose for the sake of argument that the Quillen adjunctions we have are $f_! \dashv f^*$. Then any commutative square in $\mathscr{B}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array} \tag{7.2}$$

gives rise to a square of functors

$$\begin{array}{ccc} \mathscr{C}_A & \xleftarrow{f^*} & \mathscr{C}_B \\ {\scriptstyle h^*}\uparrow & \Swarrow\cong & \uparrow{\scriptstyle k^*} \\ \mathscr{C}_C & \xleftarrow[g^*]{} & \mathscr{C}_D \end{array} \tag{7.3}$$

which commutes up to natural isomorphism. By ordinary pseudofunctoriality, it follows that the derived square

$$\begin{array}{ccc} \mathrm{Ho}(\mathscr{C}_A) & \xleftarrow{\mathbf{R}f^*} & \mathrm{Ho}(\mathscr{C}_B) \\ {\scriptstyle \mathbf{R}h^*}\uparrow & \Swarrow\cong & \uparrow{\scriptstyle \mathbf{R}k^*} \\ \mathrm{Ho}(\mathscr{C}_C) & \xleftarrow[\mathbf{R}g^*]{} & \mathrm{Ho}(\mathscr{C}_D) \end{array} \tag{7.4}$$

also commutes up to isomorphism. However, the isomorphism (7.3) also has a mate

$$\begin{array}{ccc} \mathscr{C}_A & \xrightarrow{f_!} & \mathscr{C}_B \\ {\scriptstyle h^*}\uparrow & \Searrow & \uparrow{\scriptstyle k^*} \\ \mathscr{C}_C & \xrightarrow[g_!]{} & \mathscr{C}_D. \end{array} \tag{7.5}$$

When this mate is an isomorphism, the square (7.2) is said to satisfy the 'Beck-Chevalley condition' on the point-set level.

However, we are usually more interested in the *derived* Beck-Chevalley condition, meaning instead that the mate of (7.4) is an isomorphism. Theorem 6.5 implies that in fact, the mate of (7.4) is equal to the derived natural transformation of (7.5). This gives us an explicit formula which we can then analyze to determine whether it is an isomorphism. An argument of this sort is implicit in several places in [MS06], for example.

Theorem 6.5 is also useful when comparing base change functors in different homotopy theories, or different models for the same homotopy theory. For instance, suppose we have two base change situations $\mathscr{C}$ and $\mathscr{D}$ indexed on the same category $\mathscr{B}$, both with Quillen adjunctions $f^* \dashv f_*$, and also a collection of Quillen adjunctions $\iota^\star \colon \mathscr{C}_B \rightleftarrows \mathscr{D}_B : \iota_\star$ comparing the two (which might be Quillen equivalences). There are then six transformations relating the base change functors in the two situations which are related by mate correspondences.

$$\begin{aligned} \iota^\star \circ f^* &\longrightarrow f^* \circ \iota^\star & \qquad f^* \circ \iota^\star &\longrightarrow \iota^\star \circ f^* \\ f^* \circ \iota_\star &\longrightarrow \iota_\star \circ f^* & \qquad \iota^\star \circ f_* &\longrightarrow f_* \circ \iota^\star \\ \iota_\star \circ f_* &\longrightarrow f_* \circ \iota_\star & \qquad f_* \circ \iota_\star &\longrightarrow \iota_\star \circ f^* \end{aligned}$$

Of these, each determines the others in the same column by the ordinary mate correspondence, and each transformation in the first row is an isomorphism if and only if the one below it in the third row is. (The other two possible transformations have no mates.) Theorem 6.5 then tells us that this remains true for the corresponding derived natural transformations.

As another example, [Shu07b] used a special case of Theorem 6.5 to compare two base change situations where one had a Quillen adjunction $f_! \dashv f^*$ and the other had a Quillen adjunction $f^* \dashv f_*$.

8. Applications to monoidal functors

Another common situation in which composites of left and right adjoints arise is when considering functors between monoidal categories. In this section, we suppose that $\mathscr{C}$ and $\mathscr{D}$ are *monoidal model categories* (see [Hov99, Ch. 4]). For simplicity, we assume that they are symmetric and that their units are cofibrant. Being a monoidal model category then means only that the tensor product is a left Quillen bifunctor, and therefore preserves pairs of weak equivalences between pairs of cofibrant objects. This implies that there is a derived tensor product defined by $X \otimes^{\mathbf{L}} Y = QX \otimes QY$, which makes the homotopy category monoidal.

We suppose first that $f_! \colon \mathscr{C} \to \mathscr{D}$ is a left Quillen functor which is also lax monoidal, so that we have comparison maps

$$I_{\mathscr{D}} \longrightarrow f_!(I_{\mathscr{C}}) \tag{8.1}$$

$$f_! X \otimes f_! Y \longrightarrow f_!(X \otimes Y). \tag{8.2}$$

When both of these are isomorphisms, $f_!$ is said to be *strong* monoidal. Since $f_!$ and $\otimes$ are both left Quillen and the units are cofibrant, ordinary pseudofunctoriality tells us that the left derived functor $\mathbf{L}f_!$ is lax monoidal, and strong monoidal whenever $f_!$ is. The same holds if $f_!$ is oplax monoidal (having transformations in the other direction).

Now consider the right adjoint f^* of $f_!$. If $f_!$ is oplax (or strong) monoidal, f^* is automatically lax monoidal; its comparison map

$$f^* X \otimes f^* Y \longrightarrow f^*(X \otimes Y) \tag{8.3}$$

is the mate of the oplax comparison map

$$f_!(X \otimes Y) \longrightarrow f_! X \otimes f_! Y \tag{8.4}$$

under the adjunctions $f_! \dashv f^*$ and $(f_! \times f_!) \dashv (f^* \times f^*)$. We can view (8.3) as a 2-cell in $\mathbb{M}$**odel**:

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} & \xrightarrow{f^* \times f^*} & \mathscr{D} \times \mathscr{D} \\ {\scriptstyle \otimes}\downarrow & \swArrow & \downarrow{\scriptstyle \otimes} \\ \mathscr{C} & \xrightarrow[f^*]{} & \mathscr{D} \end{array}$$

which is the mate of (8.4) under appropriate conjunctions. It then follows from Theorem 6.5 that the lax structure induced on $\mathbf{R}f^*$ from the oplax structure on $\mathbf{L}f_!$ is the same as the *derived* transformation of the lax structure on f^*. Explicitly, this is represented by the following composite.

$$\begin{aligned}
\mathbf{R}f^*X \otimes^{\mathbf{L}} \mathbf{R}f^*Y &= Q(f^*RX) \otimes Q(f^*RY) \\
&\xleftarrow{\sim} Q(f^*QRX) \otimes Q(f^*QRY) \\
&\longrightarrow f^*QRX \otimes f^*QRY \\
&\longrightarrow f^*(QRX \otimes QRY) \\
&\longrightarrow f^*R(QRX \otimes QRY) \\
&\xleftarrow{\sim} f^*R(QX \otimes QY) \\
&= \mathbf{R}f^*(X \otimes^{\mathbf{L}} Y).
\end{aligned}$$

In particular, when f^* is right Quillen and lax monoidal, then $\mathbf{R}f^*$ is also lax monoidal. However, it need not be strong, even if f^* is so.

Now, the mate of (8.3) under the adjunctions

$$\begin{aligned}
(f^*X \otimes -) &\dashv \operatorname{Hom}(f^*X, -) \quad \text{and} \\
(X \otimes -) &\dashv \operatorname{Hom}(X, -)
\end{aligned}$$

is a transformation

$$f^* \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(f^*X, f^*Y). \tag{8.5}$$

When (8.5) is an isomorphism, f^* is said to be a **closed monoidal** functor. However, we are usually interested more in whether $\mathbf{R}f^*$ is closed, and Theorem 6.5 also gives us a way to attack this question.

Suppose that X is fibrant and cofibrant. Then $(X \otimes -)$ and $(Q(f^*X) \otimes -)$ are both left Quillen functors, with isomorphisms

$$\mathbf{L}\big(X \otimes -\big) \cong \big(X \otimes^{\mathbf{L}} -\big) \quad \text{and} \tag{8.6}$$

$$\mathbf{L}\big(Q(f^*X) \otimes -\big) \cong \big(\mathbf{R}f^*X \otimes^{\mathbf{L}} -\big). \tag{8.7}$$

The composite

$$Q(f^*X) \otimes f^*Y \longrightarrow f^*X \otimes f^*Y \longrightarrow f^*(X \otimes Y)$$

gives us a 2-cell in $\mathbb{M}$**odel**:

$$\begin{array}{ccc}
\mathscr{C} & \xrightarrow{f^*} & \mathscr{D} \\
{\scriptstyle X\otimes -}\downarrow & \swarrow & \downarrow{\scriptstyle Q(f^*X)\otimes -} \\
\mathscr{C} & \xrightarrow[f^*]{} & \mathscr{D}.
\end{array} \tag{8.8}$$

The derived natural transformation of (8.8) is represented by the composite

$$\begin{aligned} Q(f^*X) \otimes Q(f^*RY) &\xleftarrow{\sim} Q(f^*X) \otimes Q(f^*QRY) \\ &\longrightarrow Q(f^*X) \otimes f^*QRY \\ &\longrightarrow f^*X \otimes f^*QRY \\ &\longrightarrow f^*(X \otimes QRY) \\ &\longrightarrow f^*R(X \otimes QRY) \\ &\xleftarrow{\sim} f^*R(X \otimes QY). \end{aligned}$$

Modulo the isomorphisms (8.6) and (8.7), this is clearly equal to the above derived lax structure map of $\mathbf{R}f^*$.

Now, (8.8) has a mate under the conjunctions

$$\begin{aligned} (Qf^*X \otimes -) &\prec \mathrm{Hom}(Qf^*X, -) \quad \text{and} \\ (X \otimes -) &\prec \mathrm{Hom}(X, -) \end{aligned}$$

which is a transformation

$$f^* \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(Qf^*X, f^*Y). \tag{8.9}$$

It is easy to check that this is simply the composite

$$f^* \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(f^*X, f^*Y) \longrightarrow \mathrm{Hom}(Qf^*X, f^*Y). \tag{8.10}$$

Theorem 6.5 then tells us that the transformation

$$\mathbf{R}f^*\big(\mathbf{R}\,\mathrm{Hom}(X, Y)\big) \longrightarrow \mathbf{R}\,\mathrm{Hom}\,\big(\mathbf{R}f^*X, \mathbf{R}f^*Y\big), \tag{8.11}$$

defined to be the mate of the lax structure map of $\mathbf{R}f^*$, is equal to the derived natural transformation of (8.9). Therefore, $\mathbf{R}f^*$ is a closed monoidal functor just when the derived natural transformation of (8.9) is an isomorphism. However, (8.9) is a transformation between composites of right Quillen functors only, so ordinary pseudofunctoriality applies. For example, we have the following.

Proposition 8.12. *If f^* is a right Quillen functor which is closed monoidal and preserves cofibrant objects, then $\mathbf{R}f^*$ is also closed monoidal.*

Proof. By ordinary pseudofunctoriality, the derived natural transformation of (8.9) is represented by (8.9) applied at a fibrant Y. In this case, the first map in the composite (8.10) is an isomorphism because f^* is closed, and the second is a weak equivalence because f^*X is already cofibrant. □

In many cases, an easier way to check that a functor is closed monoidal is to use an equivalent adjoint condition. The transformation

$$f^* \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(f^*X, f^*Y) \tag{8.13}$$

also has a mate

$$f_!(f^*X \otimes Z) \longrightarrow X \otimes f_!Z \tag{8.14}$$

under the composite adjunctions

$$\begin{aligned} f_!(f^*X \otimes -) &\dashv \mathrm{Hom}(f^*X, f^*-) \quad \text{and} \\ (X \otimes f_!-) &\dashv f^*(\mathrm{Hom}(X, -)). \end{aligned}$$

For a fixed X, (8.13) is h-globular in the double category $\mathbb{C}\mathbf{at}$, and (8.14) is v-globular; thus one is an isomorphism if and only if the other is. In particular, to show that f^* is closed monoidal it suffices to show that (8.14) is an isomorphism.

Now suppose again that X is fibrant and cofibrant. Then the mate of (8.9), which is h-globular in $\mathbb{M}\mathbf{odel}$, is the composite

$$f_!(Q(f^*X) \otimes Z) \longrightarrow f_!(f^*X \otimes Z) \longrightarrow X \otimes f_!Z. \tag{8.15}$$

which is v-globular in $\mathbb{M}\mathbf{odel}$. Theorem 6.5 now tells us that the transformation

$$\mathbf{L}f_!(\mathbf{R}f^*X \otimes^{\mathbf{L}} Z) \longrightarrow X \otimes^{\mathbf{L}} \mathbf{L}f_!Z, \tag{8.16}$$

defined to be the mate of (8.11), is equal to the derived natural transformation of (8.15). If f^* itself is closed, the second map in (8.15) is an isomorphism, so it remains to analyze the first map in any particular case.

There is much more to say about derived functors between monoidal model categories. For example, we might be in a situation where $f^* \colon \mathscr{D} \to \mathscr{C}$ is *oplax* monoidal and *left* Quillen, with a (lax monoidal) right adjoint f_*. In this case the oplax structure map

$$f^*(X \otimes Y) \longrightarrow f^*X \otimes f^*Y \tag{8.17}$$

has a mate

$$f_* \operatorname{Hom}(f^*X, Z) \longrightarrow \operatorname{Hom}(X, f_*Z)$$

which is an isomorphism if and only if (8.17) is—that is, if and only if f^* is strong monoidal. This correspondence descends to homotopy categories by ordinary pseudofunctoriality. However, the lax structure map of f_* also has a mate

$$Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X), \tag{8.18}$$

which may or may not be an isomorphism; when it does, one says the *projection formula* holds (see [FHM03]). Choosing X fibrant and cofibrant, we have a 2-cell

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{f_*} & \mathscr{D} \\ {\scriptstyle -\otimes X}\downarrow & \swarrow & \downarrow{\scriptstyle -\otimes Qf_*X} \\ \mathscr{C} & \xrightarrow[f_*]{} & \mathscr{D} \end{array}$$

in $\mathbb{M}\mathbf{odel}$ given by the composite

$$f_*Y \otimes Qf_*X \longrightarrow f_*Y \otimes f_*X \longrightarrow f_*(Y \otimes X), \tag{8.19}$$

whose mate under the conjunction $f^* \prec f_*$ is the 2-cell

$$\begin{array}{ccc} \mathscr{D} & = & \mathscr{D} \\ {\scriptstyle (f^*-)\otimes X}\downarrow & \swarrow & \downarrow{\scriptstyle -\otimes Qf_*X} \\ \mathscr{C} & \xrightarrow[f_*]{} & \mathscr{D} \end{array}$$

given by the composite

$$Y \otimes Qf_*X \longrightarrow Y \otimes f_*X \longrightarrow f_*(f^*Y \otimes X). \tag{8.20}$$

The derived natural transformation of (8.19) is clearly the lax structure map of $\mathbf{R}f_*$ once again. Theorem 6.5 then tells us that the canonical map

$$Y \otimes^{\mathbf{L}} \mathbf{R}f_*X \longrightarrow \mathbf{R}f_*\big(\mathbf{L}f^*Y \otimes^{\mathbf{L}} X\big),$$

defined to be the mate of the lax structure map of $\mathbf{R}f_*$, is equal to the derived natural transformation of the composite (8.20). As always, we then have an explicit formula which we can analyze to see whether the derived projection formula holds.

The paper [FHM03] goes further than this, considering adjoint strings of three functors such as $f_! \dashv f^* \dashv f_*$ or $f^* \dashv f_* \dashv f^!$ and various transformations that may exist between their composites. The calculus of derived natural transformations continues to be useful in such contexts, whenever we want to relate point-set level transformations to derived ones.

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