SECOND READER'S REPORT ON MICHAEL SHULMAN'S THESIS

In his thesis, Michael Shulman introduces a new categorical structure called *framed bicategory*, provides a framework for composites of left and right derived functors, and develops the theory of enriched homotopy (co)limits.

Mike created the notion of *framed bicategory* with an immediate application in mind, namely situations involving base change. Modules over rings provide the simplest example. Between rings we have bimodules and ring homomorphisms, and these two types of morphism are linked via the operations of restriction of scalars, extension of scalars, and co-extension of scalars. How is one to encode all of this structure? One option is the "algebraic approach": specify coherence isomorphisms and required coherence diagrams, as this reader had done to encode the restriction of scalars. However, to additionally encode extension of scalars and co-extension of scalars, this explicit approach becomes impractical: the diagrams are too numerous and too complicated. Further, one would need to prove a complex coherence theorem which guarantees the commutativity of "all" diagrams given commutativity of a finite number of diagrams.

Mike solved the problem by combining an "algebraic" approach with a "non-algebraic" approach. Namely, the coherence isomorphisms and coherence diagrams for bimodule composition are specified, but the operations of restriction of scalars, extension of scalars, and co-extension of scalars are not chosen, instead they are characterized by *universal properties.* Instead of choosing a particular object, one merely requires that an object satifsying the property exists. Grothendieck's fibrations do precisely this. A *framed bicategory* then consists of an "algebraic part", namely categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with a weak composition, and a "non-algebraic part", namely the requirement that the source/target pair $(s,t) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a Grothendieck fibration. With this axiomatization, the coherence problems discussed above disapper.

The power of his notion lies in its utility. There are numerous examples of framed bicategories that are much more substantial than modules. Notably, the parametrized spectra of May-Sigurdsson. There, the two types of morphisms gives rise to two types of duality. This framed bicategory arises from a theorem of Mike's which constructs a framed bicategory from any well-behaved monoidal fibration. Many other framed bicategories arise as instances of this theorem. Given a framed bicategory $\mathbb{D}$ with local coequalizers, one may also consider monoids in $\mathbb{D}$. These also form a framed bicategory. The framed bicategory of modules arises in this way.

The dichotomy "non-algebraic vs. algebraic", "universal property vs. specified choices", is apparent in the equivalence between Grothendieck fibrations and pseudo functors into **Cat**. Given a Grothendieck fibration, one obtains a pseudo functor into **Cat** by assigning to each object its fiber and to each morphism a chosen pullback. From a pseudo functor into **Cat**, one obtains a Grothendieck fibration via the so-called Grothendieck construction, or category of elements. Mike extends this classical equivalence to a *double equivalence* and also proves the appropriate double equivalence for monoidal Grothendieck fibrations.

Mike treats composites of left and right derived functors by way of pseudo double categories without framing. As he points out, there are many interesting functors which are simply *not* part of a Quillen adjunction, the standard structure which guarantees an induced adjunction on the homotopy categories. Thus, instead of Quillen model structures, Mike uses the more general *homotopical categories* of Dwyer-Hirschhorn-Kan-Smith. Mike proves that there is a double category of such, and that passage to the homotopy category is a double pseudo functor. On the horizontal respectively vertical 1-categories, this double pseudo functor constructs the right respectively left derived functors. Most homotopy-theoretically interesting functors fit into this scheme. Further, Mike shows his theorem holds in the enriched setting as well. The compatibility of derived functors with monoidal structure is also worked out. Further, homotopical monoidal Grothendieck fibrations are also derived.

The last part of the thesis treats enriched homotopy (co)limits. Globally, one would like to say that the homotopy limit functor is a derived functor of the limit functor. Unfortunately, Quillen model structures just are not sufficient in many cases, and one must use the more general homotopical categories mentioned above. Locally, homotopy limits are given by the (two-sided) bar construction. One would like the global and local perspectives to agree, and various comparisons have been carried out classically. Mike extends this comparison to the enriched context. This is highly nontrivial, since enriched limits, called *weighted limits*, are themselves nontrivial. There are genuinely new kinds of limits that arise in the enriched context, as discovered by Borceux-Kelly in the 70's. Weighted limits were studied for several decades by the Australian school. Mike was the first to treat *weighted homotopy limits*. Mike proves directly that the enriched bar construction is a global homotopy limit. There are also some results about the homotopy framed bicategory of distributors and analogues of cofibrant replacements in homotopical categories of diagrams.

I strongly recommend acceptance of this thesis.

Thomas M. Fiore