FIRST READER'S REPORT ON MIKE SHULMAN'S THESIS

Shulman's thesis is an amalgam of three of his papers that were written earlier and separately. The amalgam is designed to make a more coherent theory, but it tends to obscure the more focused points of the separate papers. The last chapter of the thesis focuses on the first completed paper, "Homotopy limits and colimits and enriched homotopy theory". This is to my mind the best account of this fundamental part of homotopy theory. It is a subtle paper. Homotopy limits and colimits are usually seen as part of Quillen model category theory. However, Quillen model category theory is intrinsically unable to see the enriched kinds of limits and colimits that are intrinsic to many of the applications. Before Mike's work, there was no theory of enriched homotopy colimits and limits although, in the guise of two-sided bar constructions, there were myriads of applications on a pragmatic level. Mike developed the right conceptual framework, described enriched homotopy limits and colimits conceptually as derived functors, and proved that the classical concrete constructions agree with the conceptual concepts when both are defined.

Chapter 5, with preliminaries on categorical fibrations in Chapter 4, focuses on the completed paper "Framed bicategories and monoidal fibrations". That introduces a formal structure that appears whenever base change functors do. In a recent book of Sigurdsson and myself, "Parametrized homotopy theory", we found that certain types of bicategories were essential to a concrete understanding of two kinds of topological duality that we were encountering. However, we basically threw up our hands at the categorical problem of giving precise formulations of the structures we were seeing, although we knew we were seeing concrete examples of structures that category theorists had not yet defined. Mike took over and developed these structures rigorously with his notion of a framed bicategory. Again, this is seriously subtle. The categorical coherence, which resembles the (never fully worked out) categorical coherence that one sees in Grothendieck's six functor formalism, is made precise by use of the category theorist's notion of a fibration, so that the coherence is given by universal properties rather than by monstrous commutative diagrams. This conceptualization nicely formalizes things that one sees in nature in examples in algebraic topology, homological algebra, and algebraic geometry.

Chapter 3 places a short paper, "Comparing composites of left and right derived functors", in a more systematic framework that focuses on monoidal structures in homotopy theory in general. The paper addresses a major problem in homotopical algebra. It is possible to have isomorphisms of composites of functors, $F \circ G \cong J \circ H$ say, between categories with derived homotopy categories such that the derived functors exist, but the derived composites are far from isomorphic. There are even important examples where one of the derived composites is trivial and the other is an equivalence of categories. Mike's paper at least gives the right context for studying such problems. In Quillen model category theory, a map is usually defined, quite unsatisfactorily, as the left adjoint of a Quillen adjoint pair. Mike shows that the left and the right adjoints form the vertical and horizontal arrows of a double category. This automatically builds at least the right comparison maps between composites of left derived and right derived functors. It gives the conceptually right way of thinking about how maps between Quillen model categories assemble into a single categorical structure. In the thesis, the framework is tied in more thoroughly to the enriched category theory that one encounters in the applications.

A common theme in the thesis and the cited papers is the use of double categories, bicategories, and their common generalization to weak double bicategories. These are structures that appear throughout the algebraic and topological parts of mathematics, but that have never been studied systematically except in pieces that ignore one or another aspect of the structure that one actually sees in nature. It is really quite hard and illuminating mathematics to combine the different aspects of the structure into a coherent whole. This is the problem that Shulman's thesis addresses and in large part solves.

This is a highly sophisticated piece of mathematics, and acceptance is recommended.