SPECTRAL SEQUENCES FOR LOCAL COEFFICIENTS

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Abstract. This is a small 60th birthday tribute to Bruce Williams, to whom the senior author is grateful for many years of congenial and illuminating mathematical conversations. We show that a description of local coefficients that Sammy Eilenberg introduced shortly before Bruce was born leads to spectral sequences for the computation of local homology and cohomology groups. As far as we know, although their construction is very elementary, these spectral sequences are new. We are interested mainly in an equivariant analogue, where the spectral sequences are certainly new and promise to be especially useful.

In applications of the Serre spectral sequence, one usually uses trivial local coefficients. This perhaps reflects the tendency of much of modern algebraic topology to steer away from the unstable world, where the fundamental group cannot be ignored. But it also perhaps reflects our lack of tools for computing homology and cohomology with local coefficients. We explain a simple universal coefficient spectral sequence that should have been known years ago but that the senior author, at least, cannot recall ever having seen. It starts from an old result of Eilenberg, popularized by Whitehead [7, VI§3] and more recently by Hatcher [2, App 3.H], that makes local coefficients especially familiar and tractable.

Let X be a path connected based space with universal cover $\tilde{X}$. Let $\pi = \pi_1(X)$ and let π act on the right of $\tilde{X}$ by deck transformations. Let M be a left and N be a right module over the group ring $R[\pi]$. Let C_* be the normalized singular chain complex functor with coefficients in R.

Definition 1. Define the homology of X with coefficients in M by

$$H_*(X; M) = H_*(C_*(\tilde{X}) \otimes_{R[\pi]} M).$$

Define the cohomology of X with coefficients in N by

$$H^*(X; N) = H^*(\mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}), N)).$$

Functoriality in M and N for fixed X is clear. For a based map $f\colon X \longrightarrow Y$, where $\pi_1(Y) = \rho$, and for a left $R[\rho]$-module P, we may regard P as a $R[\pi]$-module by pullback along $\pi_1(f)$, and then, using the standard functorial construction of the universal cover, we obtain

$$f_*\colon H_*(X; f^*P) \longrightarrow H_*(Y; P).$$

Cohomological functoriality is similar. The definition goes back to Eilenberg [1], a paper submitted for publication in May, 1946, and published a year later. This was well before he and others had assimilated Leray's cryptic 1946 announcement [3] of what we now call spectral sequences. The definition deserves more emphasis than it has previously been given because it leads very easily to spectral sequences for the calculation of homology and cohomology with local coefficients, as we shall explain.

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The definition has both the homology of spaces and the homology of groups as special cases.

Example 2. If π acts trivially on M and N, then $H_*(X;M)$ and $H^*(X;N)$ are the usual homology and cohomology groups of X with coefficients in M and N. We can identify $C_*(X)$ with $C_*(\tilde{X})/IC_*(\tilde{X}) \cong C_*(\tilde{X})\otimes_{R[\pi]} R$, where $R[\pi]$ acts trivially on R and $I \subset R[\pi]$ is the augmentation ideal. This implies the identifications

$$C_*(\tilde{X}) \otimes_{R[\pi]} M \cong C_*(X;M) \quad \text{and} \quad \mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}), N) \cong C^*(X;N).$$

Example 3. If $X = K(\pi,1)$, then $H_*(X;M)$ and $H^*(X;N)$ are the usual homology and cohomology groups of π with coefficients in M and N since $C_*(\tilde{X})$ is a $R[\pi]$-free resolution of R. That is,

$$H_*(K(\pi,1);M) = \mathrm{Tor}_*^{R[\pi]}(R,M) \quad \text{and} \quad H^*(K(\pi,1);M) = \mathrm{Ext}^*_{R[\pi]}(R,N).$$

Example 4. If $M = R[\pi]\otimes A$ and $N = \mathrm{Hom}(R[\pi],A)$ for an R-module A, where $\otimes = \otimes_R$ and $\mathrm{Hom} = \mathrm{Hom}_R$, then

$$H_*(X;M) \cong H_*(\tilde{X},A) \quad \text{and} \quad H^*(X;N) \cong H^*(\tilde{X};A).$$

Scholium 5. If we replace N by M (viewed as a right $R[\pi]$-module) in the cohomology case of the previous example, then we are forced to impose finiteness restrictions and consider cohomology with compact supports; compare [2, 3H.5].

We have universal coefficient spectral sequences that generalize the last two examples. When π acts trivially on M and N, they can be thought of as versions of the Serre spectral sequence of the evident fibration $\tilde{X} \longrightarrow X \longrightarrow K(\pi,1)$.

Theorem 6 (Universal Coefficients)**.** *There are spectral sequences*

$$E^2_{p,q} = \mathrm{Tor}^{R[\pi]}_{p,q}(H_*(\tilde{X}),M) \Longrightarrow H_{p+q}(X;M)$$

and

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{R[\pi]}(H_*(\tilde{X}),N) \Longrightarrow H^{p+q}(X;N).$$

Proof. In the E^2 and E_2 terms, p is the homological degree and q is the internal grading. Let $\varepsilon\colon P_* \longrightarrow M$ be an $R[\pi]$-projective resolution of M and form the bicomplex $C_*(\tilde{X})\otimes_{R[\pi]} P_*$. The quasi-isomorphism ε induces a quasi-isomorphism

$$id \otimes \varepsilon : C_*(\tilde{X}) \otimes_{R[\pi]} P_* \longrightarrow C_*(\tilde{X}) \otimes_{R[\pi]} M.$$

One way to see this is to filter $C_*(\tilde{X})\otimes_{R[\pi]} P_*$ by the degrees of $C_*(\tilde{X})$, giving a spectral sequence whose E^0-term has differential $\mathrm{id}\otimes d$. Since $C_*(\tilde{X})$ is $R[\pi]$-free, the resulting E^1-term is $C_*(\tilde{X})\otimes_{R[\pi]} M$, the resulting E^2-term is $H_*(X;M)$, and $E^2 = E^\infty$ with no extension problem. Filtering the other way, by the degrees of P_*, we obtain a spectral sequence whose E^0-term has differential $d\otimes\mathrm{id}$. The resulting E^1-term is $H_*(\tilde{X})\otimes_{R[\pi]} P_*$ and the resulting E^2-term is $\mathrm{Tor}^{R[\pi]}_{*,*}(H_*(\tilde{X}),M)$. The argument in cohomology is similar, starting from $\mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}),I^*)$ for an injective resolution $\eta\colon N \longrightarrow I^*$ of N. □

We record the most obvious example.

Corollary 7. *Let π be a finite group of order n and R be a field of characteristic prime to n. Then*

$$H_*(X;M) \cong H_*(\tilde{X})\otimes_{R[\pi]} M \quad \textit{and} \quad H^*(X;N) \cong \mathrm{Hom}_{R[\pi]}\left(H_*(\tilde{X}),N\right).$$

Proof. Since $R[\pi]$ is semi-simple, $E^2_{p,q} = 0$ and $E_2^{p,q} = 0$ for $p > 0$. Therefore the spectral sequences collapse to the claimed isomorphisms. □

Whitehead [7, VI.3.4 and 3.4*] (see also Hatcher [2, 3H.4]) proves the following result and ascribes it to Eilenberg [1].

Theorem 8. *For path connected spaces X and covariant and contravariant local coefficient systems $\mathscr{M}$ and $\mathscr{N}$ on X, the classical local homology and cohomology groups $H_*(X;\mathscr{M})$ and $H^*(X;\mathscr{N})$ are naturally isomorphic to the homology and cohomology groups $H_*(X;M)$ and $H^*(X;N)$, where M and N are the restrictions of $\mathscr{M}$ and $\mathscr{N}$ to π.*

Remark 9. When N is an $R[\pi]$-algebra, $H^*(X;N)$ is an R-algebra. An elaboration of Whitehead's proof shows that when $\mathscr{N}$ is R-algebra valued, $H^*(X;\mathscr{N})$ and $H^*(X;N)$ are isomorphic as R-algebras.

Therefore our Theorem 6 gives a universal coefficient theorem for the computation of homology and cohomology with local coefficients. If $p : E \longrightarrow X$ is a fibration with fiber F and path connected base space X, this gives a means to compute the local homology and cohomology groups that appear as

$$E^2_{*,*} = H_*(X;\mathscr{H}_*(F;R)) \quad \text{and} \quad E_2^{*,*} = H^*(X;\mathscr{H}^*(F;R))$$

of the Serre spectral sequences for the computation of $H_*(E;R)$ and $H^*(E;R)$. Even the trivial case when π is finite of order n and R is a field of characteristic prime to n often occurs in practice. Writing M_π for the coinvariants M/IM, where $I \subset R[\pi]$ is the augmentation ideal, and N^π for the invariants, or fixed points, Corollary 7 implies isomorphisms

$$E^2 \cong H_*(\tilde{X};H_*(F;R)_\pi) \quad \text{and} \quad E_2 \cong H^*(\tilde{X};H^*(F;R)^\pi)$$

(which should be well-known). In general, the spectral sequences of Theorem 6 should make the Serre spectral sequence more amenable to explicit calculation in the presence of non-trivial local coefficient systems.

Heading towards an equivariant generalization, we recall part of the proof of Theorem 8. In Definition 1, we took M and N to be left and right modules over the group ring $R[\pi]$ and took $C_*(\tilde{X})$ to be the normalized singular chains on $\tilde{X}$. A module M over $R[\pi]$ is the same as a functor from π, viewed as a category with a single object, to the category of R-modules; the functor is covariant or contravariant depending on whether M is taken to be a left or a right module. Similarly, $\tilde{X}$ can be viewed as a contravariant functor from π to the category $\mathscr{U}$ of unbased topological spaces, and then $C_*(\tilde{X})$ is a functor from π to the category of chain complexes of R-modules.

A local coefficient system $\mathscr{M}$ on a space X is a functor (covariant or contravariant depending on context, corresponding to our left and right $R[\pi]$-module distinction above) from the fundamental groupoid $\Pi = \Pi X$ to the category of R-modules. When X is path connected with basepoint $*$, the category $\pi = \pi_1(X)$ with single object $*$ is a skeleton of Π. Therefore $\mathscr{M}$ is determined by its restriction M to π.

Rather than restricting to π, we could instead redefine $\tilde{X}$ to be the contravariant functor $\Pi \longrightarrow \mathscr{U}$ that sends a point $x \in X$ to the space $\tilde{X}(x)$ of equivalence classes of paths starting at x and sends a path γ from x to y to the map $\tilde{X}(y) \longrightarrow \tilde{X}(x)$ given by precomposition with γ. Since π is a skeleton of Π, the following definition is equivalent to Definition 1 when X is connected. By Theorem 8, there is no conflict with the classical notation for homology with local coefficients. Let Ch_R denote the category of chain complexes of R-modules.

Definition 10. Let $\mathscr{M} : \Pi \longrightarrow R$-mod and $\mathscr{N} : \Pi^{\mathrm{op}} \longrightarrow R$-mod be functors and let $C_*(\tilde{X}) : \Pi^{\mathrm{op}} \longrightarrow \mathscr{U} \longrightarrow \mathrm{Ch}_R$ be the composite of the universal cover functor with the functor C_*. Define the homology of X with coefficients in $\mathscr{M}$ to be

$$H_*(X;\mathscr{M}) = H_*(C_*(\tilde{X}) \otimes_\Pi \mathscr{M})$$

where $\otimes_\Pi$ is the tensor product of functors (which is given by an evident coequalizer diagram). Similarly, define

$$H^*(X;\mathscr{N}) = H^*\left(\mathrm{Hom}_\Pi(C_*(\tilde{X}), \mathscr{N})\right)$$

where Hom_Π is the hom of functors (which is given by an evident equalizer diagram).

Equivariant Generalizations

Note that our distinctions between left and right and between covariant and contravariant are unimportant above, since we are dealing with groups and groupoids. However, our motivation actually comes from (Bredon) equivariant homology and cohomology. Here the fundamental "groupoid" is only an EI-category (endomorphisms are isomorphisms) and the distinction is essential. There is an equivariant Serre spectral sequence, due to Moerdijk and Svensson [4], but it has not yet had significant calculational applications. The essential reason is the lack of a way to compute its E^2-terms. However, our observations above generalize nicely to Bredon homology and cohomology. In fact, the ideas here arose in work in progress by the second author on characteristic classes in Bredon cohomology, about which remarkably little is currently known.

Definition 10 generalizes directly to the equivariant case. From now on, let X be a G-space, where G is a finite group. Following tom Dieck [6], we can define the "fundamental EI-category"[1] $\Pi_G X$ to be the category whose objects are pairs (H, x), where $x \in X^H$; a morphism from (H, x) to (K, y) consists of a G-map $\alpha : G/H \longrightarrow G/K$, determined by $\alpha(eH) = gK$, together with a homotopy class rel endpoints $[\gamma]$ of paths from x to $\alpha^* y = gy$. Here $gy \in X^H$ when $y \in X^K$ since α is a G-map.

Likewise, tom Dieck defines the equivariant universal cover $\tilde{X}$ to be the functor $\tilde{X} : (\Pi_G X)^{\mathrm{op}} \longrightarrow \mathscr{U}$ which sends (H, x) to $\widetilde{X^H}(x)$, the universal cover of X^H viewed as having basepoint x. For a morphism $(\alpha, [\gamma]) : (H, x) \to (K, y)$, $\tilde{X}(\alpha, [\gamma]) : \tilde{X}(K, y) \longrightarrow \tilde{X}(H, x)$ takes a class of paths $[\beta]$ starting at $y \in X^K$ to the class of the composition $(\alpha^*\beta) * \gamma$.

We can now define equivariant cohomology with local coefficients. In fact, abbreviating notation by letting $\Pi = \Pi_G X$, Definition 10 applies verbatim: we need only add G to the notations. We repeat the definition for emphasis.

Definition 11. Let X be a G-space and write $\Pi = \Pi_G X$. Let $\mathscr{M} : \Pi \longrightarrow R$-mod and $\mathscr{N} : \Pi^{\mathrm{op}} \longrightarrow R$-mod be functors and let $C_*^G(\tilde{X}) : \Pi^{\mathrm{op}} \longrightarrow \mathscr{U} \longrightarrow \mathrm{Ch}_R$ be the

[1] Tom Dieck calls it the fundamental group category; it is often called the equivariant fundamental groupoid, by abuse of language. We restrict to finite groups for simplicity, but with sufficient care the definitions can be extended to all topological groups.

composite of the universal cover functor with the functor C_*. Define the homology of X with coefficients in $\mathscr{M}$ to be

$$H^G_*(X;\mathscr{M}) = H_*(C^G_*(\tilde{X}) \otimes_\Pi \mathscr{M})$$

and

$$H_G^*(X;\mathscr{N}) = H^*\left(\mathrm{Hom}_\Pi(C^G_*(\tilde{X}),\mathscr{N})\right).$$

Mutatis mutandis[2] the proofs in Whitehead or Hatcher [7, 2] go through to show that this definition of Bredon (co)homology with local coefficients is naturally isomorphic to the Bredon (co)homology with local coefficients as defined in Mukherjee and Pandey [5], which they in turn show is naturally isomorphic to the (co)homology with local coefficients defined and used by Moerdijk and Svensson in [4] to construct the equivariant Serre spectral sequence of a G-fibration $p\colon E \longrightarrow B$.

We quickly review the homological algebra needed for the equivariant generalization of Theorem 6. Since R-mod is an Abelian category, the categories $[\Pi, R\text{-mod}]$ and $[\Pi^{\mathrm{op}}, R\text{-mod}]$ of functors from Π to R-mod are also Abelian, with kernels and cokernels defined levelwise. These categories have enough projectives and injectives. Specifically, by the Yoneda lemma the covariant represented functors $R\Pi((H,x),-)$ (where R denotes the free R-module functor) are projective for each (H,x). Therefore, given a functor $\mathscr{M}$, we can construct an epimorphism $\mathscr{P} \longrightarrow \mathscr{M}$ with $\mathscr{P}$ projective by taking $\mathscr{P}$ to be a direct sum of representables, one for each element of each R-module $\mathscr{M}(H,x)$. Similarly, the contravariant functors $\mathrm{Hom}(R\Pi((H,x),-)\,,A)$ $\mathrm{Hom}(R\Pi(-,(H,x))\,,A)$ are injective when A is an injective R-module. Given any $\mathscr{N}$, we can construct a monomorphism $\mathscr{N} \longrightarrow \mathscr{I}$ by taking $\mathscr{I}$ to be a product of such injective functors, one for each R-module $\mathscr{N}(H,x)$. We define $\mathrm{Tor}^\Pi(\mathscr{N},\mathscr{M})$ in the obvious way. It is the homology of the complex of R-modules that is obtained by tensoring the functor $\mathscr{N}$ with a projective resolution of the functor $\mathscr{M}$. We define $\mathrm{Ext}_\Pi(\mathscr{N}_1,\mathscr{N}_2)$ similarly.

Lemma 12. *Each functor $C^G_n(\tilde{X})$ is a direct sum of representable functors of the form $R\Pi(-,(H,x))$.*

Proof. The proof is analogous to that of the nonequivariant result. Observe that $C^G_n(\tilde{X})(H,x)$ is the free R-module on generators given by maps $\sigma : \Delta_n \longrightarrow \tilde{X}(H,x)$. Let v denote the initial vertex of Δ_n. Restrict attention for the moment to those maps σ such that $\sigma(v) = c_{H,x}$, the constant path at x in $\tilde{X}(H,x)$. Such σ are in bijective correspondence with the maps $\Delta_n \longrightarrow X^H$ that take v to x. Restrict further to those σ that are not the image of any $\tau : \Delta_n \longrightarrow \tilde{X}(K,y)$ where H is properly subconjugate to K; that is, σ cannot be written as a composite

$$\Delta_n \longrightarrow \tilde{X}(K,y) \longrightarrow \tilde{X}(H,x),$$

where the second map is induced by some $(\alpha,[\gamma]) \in \Pi((H,x),(K,y))$.

For such a singular simplex σ, the Yoneda lemma gives a corresponding natural transformation $\iota_\sigma\colon R\Pi(-,(H,x)) \longrightarrow C^G_n(\tilde{X})$ that takes $\mathrm{id} \in \Pi((H,x),(H,x))$ to σ. We claim [SIC?] that the sum of the ι_σ specifies the required isomorphism of $C^G_n(\tilde{X})$ with a direct sum of represented functors. We must first show that each ι_σ is an injection on hom sets. For any map $(\alpha,[\gamma]) \in \Pi((K,y),(H,x))$, we have an induced map

$$(\alpha,[\gamma])^*\sigma : \Delta_n \longrightarrow \tilde{X}(H,x) \longrightarrow \tilde{X}(K,y).$$

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[2]Changing that which must be changed, essentially nothing in our case.

If $(\alpha_1, [\gamma_1])^*\sigma = (\alpha_2, [\gamma_2])^*\sigma$ for morphisms $(\alpha_1, [\gamma_1])$ and $(\alpha_2, [\gamma_2])$, then, since $\sigma(v) = c_{(H,x)}$, we have $(\alpha_1, [\gamma_1])^*\sigma(v) = [\gamma_1]$ and $(\alpha_2, [\gamma_2])^*\sigma(v) = [\gamma_2]$ and thus $[\gamma_1] = [\gamma_2]$. Since $(\alpha, [\gamma]) = (\alpha, [c_{\alpha^* x}]) \circ (\mathrm{id}, [\gamma])$, and $(\mathrm{id}, [\gamma])$ is an isomorphism, this means that $(\alpha_1, [c_{\alpha_1^* x}])^*\sigma = (\alpha_2, [c_{\alpha_2^* x}])^*\sigma$. Also, for each $t \in \Delta_n$, we must have $(\alpha_1, [c_{\alpha_1^* x}])^*\sigma(t) = (\alpha_2, [c_{\alpha_2^* x}])^*\sigma(t)$. Thus these paths must have the same endpoints $\alpha_1^*(\sigma(t)(1)) = \alpha_2^*(\sigma(t)(1))$. Let $g_1, g_2 \in G$ define $\alpha_1, \alpha_2 : G/K \longrightarrow G/H$; then $g_1(\sigma(t)(1)) = g_2(\sigma(t)(1))$ for all t. Thus $g_2^{-1}g_1$ [FUZZY] fixes the image of $\Delta_n \longrightarrow \tilde{X}(H,x) \longrightarrow X^H$ and hence $\Delta_n \longrightarrow \tilde{X}(H,x)$. By our second assumption above, this means that $g_2^{-1}g_1 \in H$, hence $g_2H = g_1H$, henc $\alpha_1 = \alpha_2$ and $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$. This shows that our natural transformation $R\Pi(-, (H,x))$ is injective on hom sets.

For each $\sigma : \Delta_n \longrightarrow \tilde{X}(H,x)$ as above and invertible map $\alpha : G/K \longrightarrow G/H$, $(\alpha, c_{\alpha^* x})^*\sigma$ meets the same criteria, so we must choose one generator for $C_n(\tilde{X})$ from each such equivalence class. Conversely, if $(\alpha_1, [\gamma_1])^*\sigma_1 = (\alpha_2, [\gamma_2])^*\sigma_2$, then the fact that each $\sigma_i(a_0)$ is a constant path again implies that $[\gamma_1] = [\gamma_2]$ and thus $(\alpha_1, \mathrm{id})^*\sigma_1 = (\alpha_2, \mathrm{id})^*\sigma_2$. Again writing g_i for the group elements which determine $\alpha_i : G/J \longrightarrow G/H_i$, this means that $g_1\sigma_1(t) = g_2\sigma_2(t)$ for all $t \in \Delta_n$. Note that, since σ_i has image in X^{H_i}, $g_i\sigma_i$ must have image in the $g_iH_ig_i^{-1}$ fixed points. Since these two images agree, they must land in X^L, where L is the subgroup generated by the $g_iH_ig_i^{-1}$. But we assumed that the σ_i were not images of simplices landing in more highly fixed subspaces of X, so the only possibility is that $L = g_1H_1g_1^{-1} = g_2H_2g_2^{-1}$, and so H_1 and H_2 are conjugate.

It remains only to show that we have accounted for all elements of $C_n(\tilde{X})$. If $\sigma : \Delta_n \longrightarrow \tilde{X}(H,x)$ is any equivariant singular simplex, then $\sigma(a_0)$ is a homotopy class of paths from (H,x) to (H,x') in X^H. Call this class $[\gamma]$. Then $(\mathrm{id}, [\gamma])$ is an isomorphism with inverse $(\mathrm{id}, [\gamma^{-1}])$, and $\sigma' = (\mathrm{id}, [\gamma^{-1}])^*\sigma$ takes a_0 to the homotopy class of the constant path at (H,x'); it follows that $\sigma = (\mathrm{id}, [\gamma])^*\sigma'$. Similarly, if σ factors through $\tilde{X}(K,y)$ for some K properly containing a conjugate of H, then by definition $\sigma = (\alpha, [\gamma])^*\sigma''$ for some $\sigma'' : \Delta_n \longrightarrow \tilde{X}(K,y)$. □

We can now prove a generalization of Theorem 6.

Theorem 13. *There are spectral sequences*

$$E^2_{p,q} = \mathrm{Tor}^{\Pi}_{p,q}(\mathscr{H}_*(\tilde{X}), \mathscr{M}) \Longrightarrow H^G_{p+q}(X; \mathscr{M})$$

and

$$E_2^{p,q} = \mathrm{Ext}_{\Pi}^{p,q}(\mathscr{H}_*(\tilde{X}), \mathscr{N}) \Longrightarrow H_G^{p+q}(X; \mathscr{N}).$$

Here $\mathrm{Tor}^{\Pi}(\mathscr{N}, \mathscr{M})$ is defined in the obvious way. It is the homology of the complex of R-modules that is obtained by tensoring the functor $\mathscr{N}$ with a projective resolution of the functor $\mathscr{M}$; $\mathrm{Ext}_{\Pi}(\mathscr{N}_1, \mathscr{N}_2)$ is defined similarly. The functor $\mathscr{H}_*(\tilde{X}) : \Pi^{\mathrm{op}} \longrightarrow R$-mod is the homology of the chain complex functor $C^G_*(\tilde{X})$; that is, $\mathscr{H}_*(\tilde{X})(H,x)$ is the homology of the chain complex $C_*(\tilde{X})(H,x)$.

Proof. Let $\varepsilon : \mathscr{P}_* \longrightarrow \mathscr{M}$ be a projective resolution of $\mathscr{M}$. As in the nonequivariant theorem, form the bicomplex of R-modules $C_*(\tilde{X}) \otimes_{\Pi} \mathscr{P}_*$. Recall that the tensor product of a functor with a representable functor is given by evaluation, $R\Pi(-, (H,x)) \otimes_{\Pi} \mathscr{M} = \mathscr{M}(H,x)$. It follows that tensoring with such projective modules is exact.

In particular, if we filter our bicomplex by degrees of $C_*(\tilde{X})$, we get a spectral sequence with E^1-term $C_*(\tilde{X}) \otimes_\Pi \mathscr{M}$, so the resulting $E^2 = E^\infty$ term is $H_*(X, \mathscr{M})$, exactly as in the nonequivariant case.

If we instead filter by degrees of $\mathscr{P}_*$, $d^0 = d \otimes \mathrm{id}$, the E^1 term is $\mathscr{H}_*(\tilde{X}) \otimes_\Pi, \mathscr{P}_*$ and the E^2 term is $\mathrm{Tor}^\Pi_{*,*}(\mathscr{H}_*(\tilde{X}), \mathscr{M})$, as desired.

The construction of the second spectral sequence is similar, starting from an injective resolution $\eta : \mathscr{N} \longrightarrow \mathscr{I}^*$. □

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