SPECTRAL SEQUENCES FOR LOCAL COEFFICIENTS

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ABSTRACT. This is a small 60th birthday tribute to Bruce Williams, to whom the senior author is grateful for many years of congenial and illuminating mathematical conversations. We first recall how a description of local coefficients that Eilenberg introduced shortly before Bruce was born leads to spectral sequences for the computation of local homology and cohomology groups. We then show how to construct new equivariant analogues of these spectral sequences and give a worked example of how to apply them.

In applications of the Serre spectral sequence, one usually uses trivial local coefficients. This perhaps reflects the tendency of much of modern algebraic topology to steer away from the unstable world, where the fundamental group cannot be ignored. But it also perhaps reflects our lack of tools for computing the relevant homology and cohomology with local coefficients. We recall a simple universal coefficient spectral sequence that seems to have long been folklore but, to our knowledge, first appeared in print in the form that we are concerned with in Goerss and Jardine [3, p. 340], although some analogues appeared earlier, for example in Brown [1, VII§7]. Even in [3], however, this spectral sequence is not connected up with the Serre spectral sequence, which is there only discussed for simply connected base spaces. Part of the point is that the definition of local coefficients that appears in the construction of the Serre spectral sequence is not tautologically the same as the definition that gives the cited universal coefficient spectral sequence.

The connection comes from an old result of Eilenberg [2], popularized by Whitehead [9, VI.3.4] and, more recently, by Hatcher [4, App3.H]. It identifies the local coefficients that appear in the context of fibrations with a more elementary definition in terms of the chains of the universal cover of the base space. The identification makes local coefficients especially familiar and tractable. We shall first say how this goes nonequivariantly and then explain the equivariant generalization.

Let X be a path connected based space with universal cover $\tilde{X}$. Let $\pi = \pi_1(X)$ and let π act on the right of $\tilde{X}$ by deck transformations. Let M be a left and N be a right module over the group ring $R[\pi]$. Let C_* be the normalized singular chain complex functor with coefficients in R.

Definition 1. Define the homology of X with coefficients in M by

$$H_*(X; M) = H_*(C_*(\tilde{X}) \otimes_{R[\pi]} M).$$

Define the cohomology of X with coefficients in N by

$$H^*(X; N) = H^*(\mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}), N)).$$

Functoriality in M and N for fixed X is clear. For a based map $f\colon X \longrightarrow Y$, where $\pi_1(Y) = \rho$, and for a left $R[\rho]$-module P, we may regard P as a $R[\pi]$-module

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by pullback along $\pi_1(f)$, and then, using the standard functorial construction of the universal cover, we obtain

$$f_*\colon H_*(X; f^*P) \longrightarrow H_*(Y;P).$$

Cohomological functoriality is similar. The definition goes back to Eilenberg [2], a paper submitted for publication in May, 1946, and published a year later. This was well before he and others had assimilated Leray's cryptic 1946 announcement [5] of what we now call spectral sequences. The definition deserves more emphasis than it is usually given because it implies spectral sequences for the calculation of homology and cohomology with local coefficients, as we shall recall. The definition has the homology of spaces and the homology of groups as special cases.

Example 2. If π acts trivially on M and N, then $H_*(X;M)$ and $H^*(X;N)$ are the usual homology and cohomology groups of X with coefficients in M and N. We can identify $C_*(X)$ with $C_*(\tilde{X})/IC_*(\tilde{X}) \cong C_*(\tilde{X}) \otimes_{R[\pi]} R$, where $R[\pi]$ acts trivially on R and $I \subset R[\pi]$ is the augmentation ideal. This implies the identifications

$$C_*(\tilde{X}) \otimes_{R[\pi]} M \cong C_*(X;M) \quad \text{and} \quad \mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}),N) \cong C^*(X;N).$$

Example 3. If $X = K(\pi,1)$, then $H_*(X;M)$ and $H^*(X;N)$ are the usual homology and cohomology groups of π with coefficients in M and N since $C_*(\tilde{X})$ is an $R[\pi]$-free resolution of R. That is,

$$H_*(K(\pi,1);M) = \mathrm{Tor}_*^{R[\pi]}(R,M) \quad \text{and} \quad H^*(K(\pi,1);M) = \mathrm{Ext}^*_{R[\pi]}(R,N).$$

Example 4. If $M = R[\pi] \otimes A$ and $N = \mathrm{Hom}(R[\pi],A)$ for an R-module A, where $\otimes = \otimes_R$ and $\mathrm{Hom} = \mathrm{Hom}_R$, then

$$H_*(X;M) \cong H_*(\tilde{X},A) \quad \text{and} \quad H^*(X;N) \cong H^*(\tilde{X};A).$$

Scholium 5. If we replace N by M (viewed as a right $R[\pi]$-module) in the cohomology case of the previous example, then we are forced to impose finiteness restrictions and consider cohomology with compact supports; compare [4, 3H.5].

We have universal coefficient spectral sequences that generalize the last two examples. When π acts trivially on M and N, they can be thought of as versions of the Serre spectral sequence of the evident fibration $\tilde{X} \longrightarrow X \longrightarrow K(\pi,1)$.

Theorem 6 (Universal Coefficients)**.** *There are spectral sequences*

$$E^2_{p,q} = \mathrm{Tor}^{R[\pi]}_{p,q}(H_*(\tilde{X}),M) \Longrightarrow H_{p+q}(X;M)$$

and

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{R[\pi]}(H_*(\tilde{X}),N) \Longrightarrow H^{p+q}(X;N).$$

Proof. In the E^2 and E_2 terms, p is the homological degree and q is the internal grading. Let $\varepsilon\colon P_* \longrightarrow M$ be an $R[\pi]$-projective resolution of M and form the bicomplex $C_*(\tilde{X}) \otimes_{R[\pi]} P_*$. The quasi-isomorphism ε induces a quasi-isomorphism

$$id \otimes \varepsilon : C_*(\tilde{X}) \otimes_{R[\pi]} P_* \longrightarrow C_*(\tilde{X}) \otimes_{R[\pi]} M.$$

One way to see this is to filter $C_*(\tilde{X}) \otimes_{R[\pi]} P_*$ by the degrees of $C_*(\tilde{X})$, giving a spectral sequence whose E^0-term has differential $\mathrm{id} \otimes d$. Since $C_*(\tilde{X})$ is $R[\pi]$-free, the resulting E^1-term is $C_*(\tilde{X}) \otimes_{R[\pi]} M$, the resulting E^2-term is $H_*(X;M)$, and $E^2 = E^\infty$ with no extension problem. Filtering the other way, by the degrees of

P_*, we obtain a spectral sequence whose E^0-term has differential $d\otimes \mathrm{id}$. The resulting E^1-term is $H_*(\tilde{X}) \otimes_{R[\pi]} P_*$ and the resulting E^2-term is $\mathrm{Tor}^{R[\pi]}_{*,*}(H_*(\tilde{X}), M)$. The argument in cohomology is similar, starting from $\mathrm{Hom}_{R[\pi]}(C_*(\tilde{X}), I^*)$ for an injective resolution $\eta\colon N \longrightarrow I^*$ of N. □

We record the most obvious example.

Corollary 7. *Let π be a finite group of order n and R be a field of characteristic prime to n. Then*

$$H_*(X;M) \cong H_*(\tilde{X}) \otimes_{R[\pi]} M \quad \text{and} \quad H^*(X;N) \cong \mathrm{Hom}_{R[\pi]}\Big(H_*(\tilde{X}), N\Big).$$

Proof. Since $R[\pi]$ is semi-simple, $E^2_{p,q} = 0$ and $E_2^{p,q} = 0$ for $p > 0$. Therefore the spectral sequences collapse to the claimed isomorphisms. □

Whitehead [9, VI.3.4 and 3.4*] (see also Hatcher [4, 3H.4]) proves the following result and ascribes it to Eilenberg [2]. We will say a little about the proof below.

Theorem 8. *For path connected spaces X and covariant and contravariant local coefficient systems $\mathscr{M}$ and $\mathscr{N}$ on X, the classical local homology and cohomology groups $H_*(X;\mathscr{M})$ and $H^*(X;\mathscr{N})$ are naturally isomorphic to the homology and cohomology groups $H_*(X;M)$ and $H^*(X;N)$, where M and N are the restrictions of $\mathscr{M}$ and $\mathscr{N}$ to π.*

Remark 9. When N is an $R[\pi]$-algebra, $H^*(X;N)$ is an R-algebra. An elaboration of Whitehead's proof shows that when $\mathscr{N}$ is R-algebra valued, $H^*(X;\mathscr{N})$ and $H^*(X;N)$ are isomorphic as R-algebras.

Therefore our Theorem 6 gives a universal coefficient theorem for the computation of homology and cohomology with local coefficients. If $p : E \longrightarrow X$ is a fibration with fiber F and path connected base space X, this gives a means to compute the local homology and cohomology groups that appear as

$$E^2_{*,*} = H_*(X;\mathscr{H}_*(F;R)) \quad \text{and} \quad E_2^{*,*} = H^*(X;\mathscr{H}^*(F;R))$$

of the Serre spectral sequences for the computation of $H_*(E;R)$ and $H^*(E;R)$. Even the trivial case when π is finite of order n and R is a field of characteristic prime to n often occurs in practice. Writing M_π for the coinvariants M/IM, where $I \subset R[\pi]$ is the augmentation ideal, and N^π for the invariants, or fixed points, Corollary 7 implies isomorphisms

$$E^2 \cong H_*(\tilde{X}; H_*(F;R)_\pi) \quad \text{and} \quad E_2 \cong H^*(\tilde{X}; H^*(F;R)^\pi).$$

In general, the spectral sequences of Theorem 6 help make the Serre spectral sequence amenable to explicit calculation in the presence of non-trivial local coefficient systems.

Heading towards an equivariant generalization, we recall part of the proof of Theorem 8. In Definition 1, we took M and N to be left and right modules over the group ring $R[\pi]$ and took $C_*(\tilde{X})$ to be the normalized singular chains of $\tilde{X}$. A (left or right) $R[\pi]$-module M is the same as a (covariant or contravariant) functor from π, viewed as a category with a single object, to the category of R-modules. Similarly, $\tilde{X}$ can be viewed as a contravariant functor from π to the category $\mathscr{U}$ of unbased topological spaces, and then $C_*(\tilde{X})$ is a functor from π to the category of chain complexes of R-modules.

A local coefficient system $\mathscr{M}$ on a space X is a functor (covariant or contravariant depending on context, corresponding to our left and right $R[\pi]$-module distinction above) from the fundamental groupoid $\Pi = \Pi X$ to the category of R-modules. When X is path connected with basepoint $*$, the category $\pi = \pi_1(X)$ with single object $*$ is a skeleton of Π. Therefore $\mathscr{M}$ is determined by its restriction M to π.

Rather than restricting to π, we could instead redefine $\tilde{X}$ to be the contravariant functor $\Pi \longrightarrow \mathscr{U}$ that sends a point $x \in X$ to the space $\tilde{X}(x)$ of equivalence classes of paths starting at x and sends a path γ from x to y to the map $\tilde{X}(y) \longrightarrow \tilde{X}(x)$ given by precomposition with γ. Since π is a skeleton of Π, the following definition is equivalent to Definition 1 when X is connected. By Theorem 8, there is no conflict with the classical notation for homology with local coefficients. Let Ch_R denote the category of chain complexes of R-modules.

Definition 10. Let $\mathscr{M} : \Pi \longrightarrow R$-mod and $\mathscr{N} : \Pi^{\mathrm{op}} \longrightarrow R$-mod be functors and let $C_*(\tilde{X}) : \Pi^{\mathrm{op}} \longrightarrow \mathscr{U} \longrightarrow \mathrm{Ch}_R$ be the composite of the universal cover functor with the functor C_*. Define the homology of X with coefficients in $\mathscr{M}$ to be

$$H_*(X;\mathscr{M}) = H_*(C_*(\tilde{X}) \otimes_\Pi \mathscr{M})$$

where $\otimes_\Pi$ is the tensor product of functors (which is given by an evident coequalizer diagram). Similarly, define

$$H^*(X;\mathscr{N}) = H^*\left(\mathrm{Hom}_\Pi(C_*(\tilde{X}), \mathscr{N})\right)$$

where Hom_Π is the hom of functors (which is given by an evident equalizer diagram).

Equivariant Generalizations

Note that our distinctions between left and right and between covariant and contravariant are unimportant above, since we are dealing with groups and groupoids. However, our motivation actually comes from (Bredon) equivariant homology and cohomology. Here the fundamental "groupoid" is only an EI-category (endomorphisms are isomorphisms) and the distinction is essential. There is an equivariant Serre spectral sequence, due to Moerdijk and Svensson [6], but it has not yet had significant calculational applications. The essential reason is the lack of a way to compute its E^2-terms. However, our observations above generalize nicely to Bredon homology and cohomology. In fact, the ideas here arose in work in progress by the second author on characteristic classes in Bredon cohomology, about which remarkably little is currently known.

Definition 10 generalizes directly to the equivariant case. From now on, let X be a G-space, where G is a discrete group[1]. Following tom Dieck [8], we can define the fundamental group category[2] $\Pi_G X$ to be the category whose objects are pairs (H, x), where $x \in X^H$; a morphism from (H, x) to (K, y) consists of a G-map $\alpha : G/H \longrightarrow G/K$, determined by $\alpha(eH) = gK$, together with a homotopy class rel endpoints $[\gamma]$ of paths from x to $\alpha^*(y) = gy$. Here $\alpha^* \colon X^K \longrightarrow X^H$ is the map given by $\alpha^*(z) = gz$, which makes sense since $g^{-1}Hg \subset K$.

Likewise, tom Dieck defines the equivariant universal cover $\tilde{X}$ to be the functor $\tilde{X} : (\Pi_G X)^{\mathrm{op}} \longrightarrow \mathscr{U}$ which sends (H, x) to $\widetilde{X^H}(x)$, the universal cover of X^H viewed as having basepoint x. For a morphism $(\alpha, [\gamma]) : (H, x) \longrightarrow (K, y)$,

[1] With a little more detail, we could generalize to topological groups.

[2] It might better be called the fundamental groupoid or EI-category.

$\tilde{X}(\alpha, [\gamma]) : \tilde{X}(K, y) \longrightarrow \tilde{X}(H, x)$ takes a class of paths $[\beta]$ starting at $y \in X^K$ to the class of the composition $(\alpha^*\beta) * \gamma$.

We can now define equivariant cohomology with local coefficients. In fact, abbreviating notation by letting $\Pi = \Pi_G X$, Definition 10 applies verbatim: we need only add G to the notations. We repeat the definition for emphasis.

Definition 11. Let X be a G-space and write $\Pi = \Pi_G X$. Let $\mathscr{M} : \Pi \longrightarrow R$-mod and $\mathscr{N} : \Pi^{\text{op}} \longrightarrow R$-mod be functors and let $C_*^G(\tilde{X}) : \Pi^{\text{op}} \longrightarrow \mathscr{U} \longrightarrow \text{Ch}_R$ be the composite of the universal cover functor with the functor C_*. Define the homology of X with coefficients in $\mathscr{M}$ to be

$$H_*^G(X; \mathscr{M}) = H_*(C_*^G(\tilde{X}) \otimes_\Pi \mathscr{M})$$

and

$$H_G^*(X; \mathscr{N}) = H^* \left(\text{Hom}_\Pi(C_*^G(\tilde{X}), \mathscr{N}) \right).$$

Mutatis mutandis[3], the proofs in Whitehead or Hatcher [9, 4] apply to show that this definition of Bredon (co)homology with local coefficients is naturally isomorphic to the Bredon (co)homology with local coefficients, as defined in Mukherjee and Pandey [7], which they in turn show is naturally isomorphic to the (co)homology with local coefficients, as defined and used by Moerdijk and Svensson in [6] to construct the equivariant Serre spectral sequence of a G-fibration $p \colon E \longrightarrow B$.

We quickly review the homological algebra needed for the equivariant generalization of Theorem 6. Since R-mod is an Abelian category, the categories $[\Pi, R\text{-mod}]$ and $[\Pi^{\text{op}}, R\text{-mod}]$ of functors from Π to R-mod are also Abelian, with kernels and cokernels defined levelwise. These categories have enough projectives and injectives. Specifically, by the Yoneda lemma the covariant and contravariant represented functors $R\Pi((H, x), -)$ and $R\Pi(-, (H, x))$ (where R denotes the free R-module functor) are projective for each (H, x). Therefore, given a functor $\mathscr{M}$, we can construct an epimorphism $\mathscr{P} \longrightarrow \mathscr{M}$ with $\mathscr{P}$ projective by taking $\mathscr{P}$ to be a direct sum of representables, one for each element of each R-module $\mathscr{M}(H, x)$. Similarly, the contravariant functors $\text{Hom}(R\Pi((H, x), -)\,, A)$ are injective when A is an injective R-module. Given any $\mathscr{N}$, we can construct a monomorphism $\mathscr{N} \longrightarrow \mathscr{I}$ by taking $\mathscr{I}$ to be a product of such injective functors, one for each R-module $\mathscr{N}(H, x)$. We define $\text{Tor}^\Pi(\mathscr{N}, \mathscr{M})$ in the obvious way. It is the homology of the complex of R-modules that is obtained by tensoring the functor $\mathscr{N}$ with a projective resolution of the functor $\mathscr{M}$. We define $\text{Ext}_\Pi(\mathscr{N}_1, \mathscr{N}_2)$ similarly.

The following equivariant analogue of the nonequivariant statement that $C_*(\tilde{X})$ is a free $R[G]$-module should be a standard first observation in equivariant homology theory, but we have not seen it in the literature. While the nonequivariant assertion is obvious, it is the crux of the proof of Theorem 6.

Theorem 12. *With $\Pi = \Pi_G X$, each functor $C_n^G(\tilde{X}) \colon \Pi^{op} \longrightarrow R$-mod is a direct sum of representable functors of the form $R\Pi(-, (H, x))$.*

Granting this result, we can prove the equivariant generalization of Theorem 6.

Theorem 13. *With $\Pi = \Pi_G X$, there are spectral sequences*

$$E^2_{p,q} = \text{Tor}^\Pi_{p,q}(\mathscr{H}_*(\tilde{X}), \mathscr{M}) \Longrightarrow H^G_{p+q}(X; \mathscr{M})$$

[3]Changing that which must be changed, essentially nothing in our case.

and

$$E_2^{p,q} = \mathrm{Ext}_\Pi^{p,q}(\mathscr{H}_*(\tilde{X}), \mathscr{N}) \Longrightarrow H_G^{p+q}(X; \mathscr{N}).$$

Here the functor $\mathscr{H}_*(\tilde{X}) : \Pi^{\mathrm{op}} \longrightarrow R$-mod is the homology of the chain complex functor $C_*^G(\tilde{X})$; that is, $\mathscr{H}_*(\tilde{X})(H,x)$ is the homology of the chain complex $C_*(\tilde{X})(H,x)$.

Proof. Let $\varepsilon : \mathscr{P}_* \longrightarrow \mathscr{M}$ be a projective resolution of $\mathscr{M}$. As in the nonequivariant theorem, form the bicomplex of R-modules $C_*(\tilde{X}) \otimes_\Pi \mathscr{P}_*$. Recall that the tensor product of a functor with a representable functor is given by evaluation, $R\Pi(-,(H,x)) \otimes_\Pi \mathscr{M} \cong \mathscr{M}(H,x)$. It follows that tensoring with such projective modules is exact.

In particular, if we filter our bicomplex by degrees of $C_*(\tilde{X})$, we get a spectral sequence with E^1-term $C_*(\tilde{X}) \otimes_\Pi \mathscr{M}$, so the resulting $E^2 = E^\infty$ term is $H_*(X, \mathscr{M})$, exactly as in the nonequivariant case.

If we instead filter by degrees of $\mathscr{P}_*$, $d^0 = d \otimes \mathrm{id}$, then the E^1 term is $\mathscr{H}_*(\tilde{X}) \otimes_\Pi, \mathscr{P}_*$ and the E^2 term is $\mathrm{Tor}^\Pi_{*,*}(\mathscr{H}_*(\tilde{X}), \mathscr{M})$, as desired.

The construction of the second spectral sequence is similar, starting from an injective resolution $\eta : \mathscr{N} \longrightarrow \mathscr{I}^*$. □

Proof of Theorem 12. The proof is analogous to that of the nonequivariant result, but more involved. We may identify $C_n^G(\tilde{X})(H,x)$ with the free R-module on generators given by the nondegenerate singular n-simplices $\sigma : \Delta_n \longrightarrow \tilde{X}(H,x)$. We must show that these free R-modules piece together appropriately into a free functor. More specifically, by the Yoneda lemma, each $\sigma : \Delta_n \longrightarrow \tilde{X}(K,y)$ determines a natural transformation

$$\iota_\sigma : R\Pi(-,(K,y)) \longrightarrow C_n^G(\tilde{X})$$

that takes $\mathrm{id} \in \Pi((K,y),(K,y))$ to σ. We thus obtain a natural transformation

$$\bigoplus_{(K,y)} \bigoplus_{\{\tau\}} R\Pi(-,(K,y)) \longrightarrow C_n^G(\tilde{X})$$

from any set of sets of nondegenerate n-simplices $\{\tau : \Delta_n \longrightarrow \tilde{X}(K,y)\}$, one set for each object (K,y) in Π. We must show that there is a set of choices of sets $\{\tau\}$ such that the resulting natural transformation is a natural isomorphism, that is, a levelwise isomorphism. This amounts to showing that the following statements hold for our choice of generators τ and each object (H,x). First, for any arrows $(\alpha_1,[\gamma_1])$ and $(\alpha_2,[\gamma_2])$ in Π with source (H,x) and any generators τ_1 and τ_2, $(\alpha_1,[\gamma_1])^*\tau_1 = (\alpha_2,[\gamma_2])^*\tau_2$ must imply that both $(\alpha_1,[\gamma_1]) = (\alpha_2,[\gamma_2])$ and $\tau_1 = \tau_2$. Second, for every $\sigma : \Delta_n \longrightarrow \tilde{X}(H,x)$, there must be a generator τ and an arrow $(\alpha,[\gamma])$ such that $\sigma = (\alpha,[\gamma])^*\tau$.

Fixing n, define the generating set for varying (K,y) as follows. Regard the initial vertex v of Δ_n as a basepoint. Recall that $\tilde{X}(K,y)$ is the universal cover of X^K defined with respect to the basepoint $y \in X^K$, so that the equivalence class of the constant path $c_{K,y}$ at y is the basepoint of $\widetilde{X^K}$. In choosing our generating sets, we restrict attention to based maps $\sigma : \Delta_n \longrightarrow \tilde{X}(K,y)$ that are non-degenerate n-simplices of $\widetilde{X^K}$. Such maps σ are in bijective correspondence with based nondegenerate n-simplices $\sigma_0 \colon \Delta_n \longrightarrow X^K$. The correspondence sends σ to its composite with the end-point evaluation map $p \colon \tilde{X}(K,y) \longrightarrow X^K$ and sends σ_0 to the map $\sigma\colon \Delta_n \longrightarrow \tilde{X}(K,y)$ that sends a point $a \in \Delta_n$ to the image under σ_0 of the straight-line path from v to a. Restrict further to those σ that cannot be written as a composite

$$\Delta_n \xrightarrow{\rho} \tilde{X}(K',y') \xrightarrow{(\alpha,\gamma)^*} \tilde{X}(K,y)$$

for any morphism $(\alpha,\gamma)\colon (K',y') \longrightarrow (K,y)$ in Π. For each such σ, we can obtain another such σ by composing with the isomorphism $(\xi,\delta)^*$ induced by an isomorphism (ξ,δ) in Π. We say that the resulting maps σ are equivalent, and we choose one τ in each equivalence class of such based singular n-simplices σ.

It remains to verify that the natural transformation defined by these sets $\{\tau\}$ is an isomorphism. This is straightforward but somewhat tedious and technical.

For the injectivity, suppose that $(\alpha_1,[\gamma_1])^*\tau_1 = (\alpha_2,[\gamma_2])^*\tau_2$, where τ_1,τ_2 are in our generating set and

$$(\alpha_1,[\gamma_1]) \in \Pi((H,x),(K_1,y_1))\,, \quad \tau_1 : \Delta_n \longrightarrow \tilde{X}(K_1,y_1)$$
$$(\alpha_2,[\gamma_2]) \in \Pi((H,x),(K_2,y_2))\,, \quad \tau_2 : \Delta_n \longrightarrow \tilde{X}(K_2,y_2).$$

Since $\tau_i(v) = c_{(K_i,y_i)}$ for $i = 1,2$, we see that $(\alpha_i,[\gamma_i])^*\tau_i$ must take v to $[\gamma_i]$. Since $(\alpha_1,[\gamma_1])^*\tau_1 = (\alpha_2,[\gamma_2])^*\tau_2$, this means that $[\gamma_1] = [\gamma_2]$; call this path class $[\gamma]$. In turn, this implies that $\alpha_1^*y_1 = \alpha_2^*y_2$; call this point $z \in X^H$, so that $[\gamma]$ is a path from x to z. Since $(\alpha_i,[\gamma_i]) = (\alpha_i,[c_z]) \circ (\mathrm{id},[\gamma])$ and $(\mathrm{id},[\gamma])$ is an isomorphism in Π, we must have

$$(\alpha_1,[c_z])^*\tau_1 = (\alpha_2,[c_z])^*\tau_2.$$

In particular, if we compose each side of this equation with p, we obtain

$$p \circ (\alpha_1,[c_z])^*\tau_1 = p \circ (\alpha_2,[c_z])^*\tau_2$$

as maps $\Delta_n \longrightarrow X^H$. Since we have commutative diagrams

$$\begin{array}{ccccc} \Delta_n & \xrightarrow{\tau_i} & \tilde{X}(K_i,y_i) & \xrightarrow{(\alpha,[c_z])^*} & \tilde{X}(H,z) \\ & & \big\downarrow p & & \big\downarrow p \\ & & X^{K_i} & \xrightarrow{\alpha_i^*} & X^H \end{array}$$

for each i, this implies that we have a commutative square

$$\begin{array}{ccc} \Delta_n & \xrightarrow{p\circ\tau_1} & X^{K_1} \\ {\scriptstyle p\circ\tau_2}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_1^*} \\ X^{K_2} & \xrightarrow{\alpha_2^*} & X^H \end{array}$$

If the maps $\alpha_i : G/H \longrightarrow G/K_i$ are defined by elements $g_i \in G$, this implies that the common composite $\Delta_n \longrightarrow X^H$ factors through the fixed-point sets $X^{g_iK_ig_i^{-1}}$ for each i, and hence through X^L, where L is the smallest subgroup containing $g_1K_1g_1^{-1}$ and $g_2K_2g_2^{-1}$. Since $K_i \subset g_i^{-1}Lg_i$, the maps $\alpha_i\colon G/H \longrightarrow G/K_i$ factor through the maps $\beta_i\colon G/K_i \longrightarrow G/L$ specified by $\beta_i(eK_i) = g_iL$ and there result factorizations of the τ_i as

$$\Delta_n \longrightarrow \tilde{X}(g_i^{-1}Lg_i,y_i) \xrightarrow{(q,[c_z])^*} \tilde{X}(K_i,y_i),$$

where q denotes either quotient map $G/K_i \longrightarrow G/g_i^{-1}Lg_i$. By our choice of the generators τ, this can only happen if $g_i^{-1}Lg_i = K_i$, giving $g_1K_1g_1^{-1} = g_2K_2g_2^{-1}$.

In terms of g_1 and g_2, we see that our equation $(\alpha_1, [c_z])^*\tau_1 = (\alpha_2, [c_z])^*\tau_2$ says that $g_1\tau_1 = g_2\tau_2$, that is, $\tau_2 = g_2^{-1}g_1\tau_1$. Since $g_2^{-1}g_1$ defines an isomorphism $G/K_2 \longrightarrow G/K_1$, we again see by our choice of the generators τ that $\tau_1 = \tau_2$ and that $g_2^{-1}g_1 \in K_1 = K_2$. This in turn implies that the maps $\alpha_i : G/H \longrightarrow G/K_i$ defined by the g_i are identical. The conclusion is that $(\alpha_1, [\gamma_1])^*\tau_1 = (\alpha_2, [\gamma_2])^*\tau_2$ implies $\tau_1 = \tau_2$ and $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$, as desired.

It only remains to show that we have accounted for all elements of $C_n(\tilde{X})(H, x)$. For any map $\sigma : \Delta_n \longrightarrow \tilde{X}(H, x)$, $\sigma(v)$ is a homotopy class of paths from (H, x) to (H, x') in X^H. Call this class $[\gamma]$. Then $(\mathrm{id}, [\gamma])$ is an isomorphism with inverse $(\mathrm{id}, [\gamma^{-1}])$, and $\sigma' = (\mathrm{id}, [\gamma^{-1}])^*\sigma$ takes v to the homotopy class of the constant path at (H, x'); it follows that $\sigma = (\mathrm{id}, [\gamma])^*\sigma'$. Similarly, if σ' factors through $\tilde{X}(K, y)$ for some K properly containing a conjugate of H, then by definition $\sigma = (\alpha, [\gamma])^*\tau$ for some $\tau : \Delta_n \longrightarrow \tilde{X}(K, y)$. We can choose a τ that does not itself factor and is in our chosen set of generators. □

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