TOPOLOGICAL HOCHSCHILD HOMOLOGY OF THOM SPECTRA AND THE FREE LOOP SPACE

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Abstract. We describe the topological Hochschild homology (THH) of A_∞-ring spectra that arise as the Thom spectra of A_∞-maps $f : X \to BG$, for suitable groups and monoids G (e.g. the infinite orthogonal group). To do this, we construct symmetric monoidal models of the category of spaces in which A_∞-spaces correspond to monoids and E_∞-spaces to commutative monoids, along with strongly symmetric monoidal versions of the Thom spectrum functor from the corresponding categories of "spaces over BG" to a symmetric monoidal category of spectra. Our main result identifies $\mathrm{THH}(\mathrm{T}(f))$ as the Thom spectrum of a certain stable bundle over the free loop space $L(BX)$. In the presence of additional commutativity, we obtain a splitting $\mathrm{THH}(\mathrm{T}(f)) \simeq (\mathrm{T}(f)) \wedge BX_+$. This splitting leads to immediate calculations of the THH of cobordism spectra such as MO, MSO, $MSpin$, and MU. We also discuss the calculation of the THH of the Eilenberg-Mac Lane spectra $H\mathbb{Z}/2$, $H\mathbb{Z}/p$, and $H\mathbb{Z}$ from this perspective.

1. Introduction

Many interesting ring spectra arise naturally as Thom spectra. It is well-known that one may associate a Thom spectrum $\mathrm{T}(f)$ to any map $f\colon\ X \to BF$, where BF denotes the classifying space for stable spherical fibrations. The stable homotopy type of $\mathrm{T}(f)$ only depends on the homotopy class of f [17, 21]. When the classifying map f is suitably multiplicative, $\mathrm{T}(f)$ inherits the structure of a ring spectrum. In particular, if f is an A_∞ map, then it follows from a result of Lewis [17] that $\mathrm{T}(f)$ is an A_∞ ring spectrum. More generally, Lewis' theorem shows that if f is an E_n map, then $\mathrm{T}(f)$ is an E_n ring spectrum, where E_n denotes an operad equivalent to the little n-cubes operad. As a consequence, for such multiplicative f the topological Hochschild homology spectrum $\mathrm{THH}(T(f))$ is defined. For example, all of the Thom spectra MG representing the classical cobordism theories (where G denotes one of the stabilized Lie groups O, SO, $Spin$, U, or Sp) arise from canonical E_∞ maps $BG \to BF$.

In this paper, we will provide an explicit description of the topological Hochschild homology of such ring spectra in terms of the Thom spectrum of a certain stable bundle over a free loop space. In order to state our main result we begin by recalling some elementary results about free loop space $L(B)$ of a connected space B. Fixing a base point in B, we have the usual fibration sequence

$$\Omega(B) \longrightarrow L(B) \longrightarrow B$$

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obtained by evaluating a loop at the base point of S^1. This sequence is split by the inclusion $B \to L(B)$ as the constant loops. When B has the structure of a homotopy associative and commutative H-space, $L(B)$ inherits that structure and the composition

$$\Omega(B) \times B \longrightarrow L(B) \times L(B) \longrightarrow L(B) \tag{1}$$

is an equivalence of H-spaces. If we assume that B has the homotopy type of a CW complex, the same holds for $L(B)$, and inverting the above equivalence specifies a well-defined homotopy class $L(B) \xrightarrow{\sim} \Omega(B) \times B$.

Applying this to the delooping B^2F of the infinite loop space BF, we obtain a splitting

$$L(B^2F) \simeq \Omega(B^2F) \times B^2F \simeq BF \times B^2F.$$

Let $\eta : S^3 \to S^2$ denote the unstable Hopf map and also in mild abuse of notation the map obtained by precomposing as follows

$$\eta : B^2F \simeq \mathrm{Map}_*(S^2, B^4F) \xrightarrow{\eta^*} \mathrm{Map}_*(S^3, B^4F) \simeq BF.$$

The following result is the main theorem of the paper.

Theorem 1.1. *Let $f\colon X \to BF$ be an A_∞ map such that X is grouplike ($\pi_0(X)$ is a group) and let $\mathrm{T}(f)$ be the associated Thom ring spectrum. Then there is a natural stable equivalence*

$$\mathrm{THH}(\mathrm{T}(f)) \simeq \mathrm{T}(L^\eta(Bf)),$$

where $L^\eta(Bf)$ denotes the composite

$$L^\eta(Bf)\colon\ L(BX) \xrightarrow{L(Bf)} L(B^2F) \simeq BF \times B^2F \xrightarrow{id\times\eta} BF \times BF \longrightarrow BF.$$

Here $Bf\colon\ BX \to B^2F$ is a delooping of f (for instance, as defined in [25]*) and the last arrow represents multiplication in the H-space BF.*

For clarity, in the statement of the preceding theorem we suppressed the specification of the operad structuring the A_∞ map; tacitly, the A_∞ map in question is compatible with the canonical A_∞ structure on BF associated to the standard E_∞ structure. In the body of the paper we will work with certain specific choices of operad for technical felicity. However, by the usual arguments one can convert maps structured by other operads into the form we require: we discuss "change of operad" techniques in Section 11.

When f is the constant map, $\mathrm{T}(f)$ is equivalent to the spherical group ring $\Sigma^\infty\Omega(BX)_+$, where $+$ indicates a disjoint base point. In this case we recover the stable equivalence of Bökstedt and Waldhausen,

$$\mathrm{THH}(\Sigma^\infty\Omega(BX)_+) \simeq \Sigma^\infty L(BX)_+.$$

The real force of Theorem 1.1 comes from the fact that the Thom spectrum $T(L^\eta Bf)$ can be analyzed effectively in many cases. We will say that f is an n-fold loop map if there exists an $(n-1)$-connected based space B^nX and a homotopy commutative diagram of the form

$$\begin{array}{ccc}
\Omega^n(B^nX) & \xrightarrow{\Omega^n(B^nf)} & \Omega^n(B^{n+1}F) \\
\uparrow\sim & & \uparrow\sim \\
X & \xrightarrow{f} & BF,
\end{array}$$

where the vertical maps are equivalences as indicated. It follows from [25] that a map satisfying the condition in the theorem is a loop map in this sense. Conversely, as discussed in Section 11, such a loop map can always be represented by an A_∞ map of the form required for Theorem 1.1. When X is a 2-fold loop space, the product decomposition in Equation 1 can be applied to $L(BX)$. In this setting, we prove the following theorem:

Theorem 1.2. *If f is a 2-fold loop map, then there is a stable equivalence*

$$\mathrm{THH}(\mathrm{T}(f)) \simeq \mathrm{T}(f) \wedge \mathrm{T}(\eta \circ Bf),$$

where $T(\eta \circ Bf)$ denotes the Thom spectrum of $BX \xrightarrow{Bf} B^2F \xrightarrow{\eta} BF$.

In the presence of slightly more commutativity, we can describe $THH(T(f))$ without referring to η.

Theorem 1.3. *If f is a 3-fold loop map, then there is a stable equivalence*

$$\mathrm{THH}(\mathrm{T}(f)) \simeq \mathrm{T}(f) \wedge BX_+.$$

This splitting also applies when f factors through BG such that $\eta\colon BBG \to BG$ is null-homotopic; for instance, when $G = SO$.

When f is an E_∞ ring spectrum, the equivalence in Theorem 1.3 is an equivalence of E_∞-ring spectra.

Theorem 1.4. *If f is an E_∞ map, then there is a stable equivalence of commutative S-algebras*

$$\mathrm{THH}(\mathrm{T}(f)) \simeq \mathrm{T}(f) \wedge BX_+.$$

In fact, we expect a more general version of this statement is true. The operadically sophisticated reader will note that our terminology of "n-fold loop maps" is a device-independent way of describing maps f which are structured by E_n operads. We have chosen this elementary description in order to make clear the connection with many examples of multiplicative Thom spectra that arise in the classical theory [21]. It is known [9, 24] that $\mathrm{THH}(R)$ for an E_n ring spectrum R is an E_{n-1} ring spectrum. The equivalence of Theorem 1.3 should be an equivalence of E_{n-1} ring spectra when X is an E_n space and f is a map of E_n spaces.

Computations and applications. These results, particularly the splitting of Theorem 1.3, allow us to easily calculate the THH of many interesting ring spectra. Let G be one of the stabilized Lie groups O, SO, $Spin$, U, or Sp. Then the Thom spectrum MG arises from an infinite loop map $BG \to BF$ and so Theorem 1.3 applies to give a stable equivalence:

$$\mathrm{THH}(MG) \simeq MG \wedge BBG_+.$$

Spelling this out using the Bott periodicity theorem, we get the following corollary.

Corollary 1.5. *There are stable equivalences of spectra*

$$\begin{aligned}
\mathrm{THH}(MO) &\simeq MO \wedge (U/O)\langle 1\rangle_+ \\
\mathrm{THH}(MSO) &\simeq MSO \wedge (U/O)\langle 2\rangle_+ \\
\mathrm{THH}(MSpin) &\simeq MSpin \wedge (U/O)\langle 3\rangle_+ \\
\mathrm{THH}(MU) &\simeq MU \wedge SU_+ \\
\mathrm{THH}(MSp) &\simeq MSp \wedge (U/Sp)\langle 1\rangle_+,
\end{aligned}$$

where here $(U/O)\langle n\rangle$ and $(U/Sp)\langle n\rangle$ denote the n-connected covers of U/O and U/Sp respectively.

These results also admit a cobordism interpretation.

Corollary 1.6. *Let G be one of the stabilized Lie groups considered above, and let Ω^G_* denote the corresponding G-bordism theory. Then there is an isomorphism*

$$\pi_* \operatorname{THH}(MG) \simeq \Omega^G_*(BBG)$$

There are many other examples of cobordism spectra for which Theorem 1.3 applies, see [35]. In the case of the identity map $BF \to BF$ we get the spectrum MF, and we again have an equivalence

$$\operatorname{THH}(MF) \simeq MF \wedge BBF_+.$$

However, in this case Corollary 1.6 does not hold [35, §IV.25].

As a further application we show how the methods of this paper can be used to recover the calculations of the topological Hochschild homology of the Eilenberg-Mac Lane spectra $H\mathbb{Z}/p$ and $H\mathbb{Z}$. These calculations are originally due to Bökstedt [6], via rather different methods.

By an old observation of Mahowald, we can realize certain Eilenberg-MacLane spectra as Thom spectra [21, 10]. Furthermore, these Thom spectrum models turn out to have sufficient multiplicative structure to compute THH. We exploit these constructions to recover the computations of THH of $H\mathbb{Z}/2$, $H\mathbb{Z}/p$ (for p odd), and $H\mathbb{Z}$.

Specifically, recall that $H\mathbb{Z}/2$ is the Thom spectrum associated to a certain map $\alpha : \Omega^2 S^3 \to BO$. We can regard this map as obtained by looping down the generator $S^3 \to BO$ in $\pi_3(BO)$. Similarly, one can realize $H\mathbb{Z}/p$ as a Thom spectrum. Any Thom spectrum over BF will necessarily have π_0 either $\mathbb{Z}$ or $\mathbb{Z}/2$ (depending on the existence of an orientation), and so we must work at the prime p. There is a space $BF_{(p)}$ which classifies p-local stable spherical fibrations by replacing self-equivalences of spheres with self-equivalences of p-local spheres [28], and it is an insight of Hopkins that $H\mathbb{Z}/p$ is the Thom spectrum associated to a certain double-loop map $\alpha_p \colon \Omega^2 S^3 \to BF_{(p)}$ [22].

The case of $H\mathbb{Z}$ is slightly more complicated. $H\mathbb{Z}$ can be modeled as a Thom spectrum over $\Omega^2 S^3\langle 3\rangle$ [10]. However, the specific construction of $H\mathbb{Z}$ as a Thom spectrum studied therein does not have sufficient multiplicative structure; only an H-space structure is obtained. Nonetheless, in work of the first author [4] a new construction realizing $H\mathbb{Z}$ as an E_2-ring spectrum is given, and this suffices for us to perform calculations of THH.

Although none of these Thom spectra are E_3 ring spectra (triple loop maps), it turns out that the splitting phenomenon of Theorem 1.3 occurs in these examples, and so we obtain the following descriptions of THH.

Theorem 1.7. *There is an equivalence of spectra*

$$\operatorname{THH}(H\mathbb{Z}/p) \simeq H\mathbb{Z}/p \wedge \Omega(S^3)_+$$

for each prime p.

On the level of homotopy groups, we get the isomorphims

$$\pi_* \operatorname{THH}(H\mathbb{Z}/p) = H_*(\Omega(S^3), \mathbb{Z}/p) = \mathbb{Z}/p[x],$$

where x has degree 2, see e.g., [37, 1.18]. In the case of $H\mathbb{Z}$, we prove the following.

Theorem 1.8. *There is an equivalence*

$$\mathrm{THH}(H\mathbb{Z}) \simeq H\mathbb{Z} \wedge \Omega(S^3\langle 3\rangle)_+,$$

where $S^3\langle 3\rangle$ *denotes the 3-fold connected cover of* S^3.

This gives the isomorphism

$$\pi_i \,\mathrm{THH}(H\mathbb{Z}) = H_i(\Omega(S^3\langle 3\rangle), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z}/(\frac{i+1}{2}), & \text{for } i > 0 \text{ odd}, \\ 0, & \text{for } i > 0 \text{ even}. \end{cases}$$

The last isomorphism easily follows by applying the Serre spectral sequence to the fibration sequence

$$S^1 \longrightarrow \Omega(S^3\langle 3\rangle) \longrightarrow \Omega(S^3).$$

Foundations. We now begin to explain the ideas and constructions going into the proof of the main results. The central object of interest is the cyclic bar construction. Let $(\mathcal{A}, \boxtimes, 1_{\mathcal{A}})$ be a symmetric monoidal category. Recall that if A is a monoid in $\mathcal{A}$, then the cyclic bar construction is the cyclic object

$$B^{\mathrm{cy}}_\bullet(A)\colon\ [k] \mapsto \underbrace{A \boxtimes \cdots \boxtimes A}_{k+1}$$

with face operators

$$d_i(a_0 \boxtimes \cdots \boxtimes a_k) = \begin{cases} a_0 \boxtimes \cdots \boxtimes a_i a_{i+1} \boxtimes \cdots \boxtimes a_k, & i = 0, \dots k-1, \\ a_k a_0 \boxtimes \cdots \boxtimes a_{k-1}, & i = k, \end{cases}$$

degeneracy operators

$$s_i(a_0 \boxtimes \cdots \boxtimes a_k) = a_0 \boxtimes \dots a_i \boxtimes 1_{\mathcal{C}} \boxtimes a_{i+1} \cdots \boxtimes a_k, \quad i = 0, \dots, k,$$

and cyclic operators

$$t_i(a_0 \boxtimes \cdots \boxtimes a_k) = a_k \boxtimes a_0 \boxtimes \cdots \boxtimes a_{k-1}.$$

Here the notation is supposed to be self-explanatory. We denote the geometric realization of this object (when this notion makes sense) as $B^{\mathrm{cy}}A$.

When $\mathcal{A}$ is a symmetric monoidal category of spectra, then for cofibrant S-algebras R the cyclic bar construction $B^{\mathrm{cy}}R$ is a model of $\mathrm{THH}(R)$. The cyclic bar construction was originally introduced by Waldhausen in the context of the category of spaces, and the intuitive picture underlying our results is the notion that the Thom spectrum should take the cyclic bar construction in spaces (over BG) to the cyclic bar construction in spectra.

More specifically, let M be a topological monoid and temporarily imagine that BF could be realized as a commutative topological monoid. Then associated to a monoid map $f\colon M \to BF$ we could define the simplicial map

$$B^{\mathrm{cy}}_\bullet(f)\colon\ B^{\mathrm{cy}}_\bullet(X) \longrightarrow B^{\mathrm{cy}}_\bullet(BF) \longrightarrow BF_\bullet,$$

where the last map is given by levelwise multiplication and $BF_\bullet$ denotes the constant simplicial object on BF. Since a basic property of the Thom spectrum functor is that it takes products to smash products, the Thom spectrum of the map $B^{\mathrm{cy}}_\bullet(f)$ should be the cyclic bar construction $B^{\mathrm{cy}}_\bullet\,\mathrm{T}(f)$; ignoring issues of cofibrancy, this is precisely $\mathrm{THH}(\mathrm{T}(f))$.

This picture makes contact with the free loop space in the following fashion: When A is a topological monoid, then the topological realization $B^{\mathrm{cy}}(A)$ inherits an action of the circle group $\mathbb{T}$ from the cyclic structure. This gives rise to the composite map

$$\mathbb{T} \times B^{\mathrm{cy}}(A) \longrightarrow B^{\mathrm{cy}}(A) \longrightarrow B(A),$$

where $B(A)$ is the realization of the usual bar construction and the second map is the realization of the simplicial map

$$(a_0, \dots, a_k) \mapsto (a_1, \dots, a_k).$$

The adjoint map $B^{\mathrm{cy}}(A) \to L(B(A))$ fits in a commutative diagram of spaces

$$\begin{array}{ccccc} A & \longrightarrow & B^{\mathrm{cy}}(A) & \longrightarrow & B(A) \\ \downarrow & & \downarrow & & \downarrow {=} \\ \Omega(B(A)) & \longrightarrow & L(B(A)) & \longrightarrow & B(A), \end{array} \tag{2}$$

where the vertical map on the left is the usual group-completion. Standard results on realizations of simplicial quasifibrations imply that if A is a group-like topological monoid (i.e. $\pi_0 A$ is a group) which is well-based (the inclusion of the unit is a Hurewicz cofibration), then the upper sequence is a fibration sequence up to homotopy and the vertical maps are weak homotopy equivalences. This suggests that we should be able to connect the description in terms of the cyclic bar construction to the free loop space.

However, there are formidable technical impediments to making this sketch precise. For one thing, BF cannot be realized as a commutative topological monoid. Furthermore, it is not straightforward to give a suitable description of the cyclic bar construction of an A_∞ or E_∞ space. Moreover, the classical comparisons of the Thom spectrum of a cartesian product to the smash product of the Thom spectra are insufficiently rigid; one obtains a simplicial object in the homotopy category, and this is not sufficient for computing THH.

The bulk of this paper is concerned with developing suitable technical foundations to carry out the program above. Our approach is as follows: We introduce a symmetric monoidal category $(\mathcal{A}, \boxtimes, 1_{\mathcal{A}})$ which is a model of the category of spaces in the sense that

- The homotopy category of $\mathcal{A}$ is equivalent to the homotopy category of spaces, and
- The derived bifunctor of $\boxtimes$ is the cartesian product $\times$.

The importance of the category $\mathcal{A}$ is that monoids and commutative monoids in $\mathcal{A}$ will correspond to A_∞ and E_∞ spaces. In particular, BF will admit a model as a commutative $\boxtimes$-monoid $BF_{\mathcal{A}}$. Next, we show that the Thom spectrum functor can be refined to a strong symmetric monoidal functor

$$\mathrm{T}_{\mathcal{A}} \colon \mathcal{A}/BF_{\mathcal{A}} \longrightarrow \mathcal{S}.$$

Here $\mathcal{A}/BF_{\mathcal{A}}$ denotes the category of objects in $\mathcal{A}$ over $BF_{\mathcal{A}}$. $\mathcal{A}/BF_{\mathcal{A}}$ inherits a symmetric monoidal structure from $\mathcal{A}$: given two such objects $f \colon X \to BF_{\mathcal{A}}$ and $g \colon Y \to BF_{\mathcal{A}}$, the monoidal product is defined by

$$f \boxtimes g \colon X \boxtimes Y \longrightarrow BF_{\mathcal{A}} \boxtimes BF_{\mathcal{A}} \longrightarrow BF_{\mathcal{A}}.$$

This is symmetric monoidal precisely because $BF_{\mathcal{A}}$ is commutative. Recall that T is strict monoidal if there is a natural isomorphism

$$\mathrm{T}(f) \wedge \mathrm{T}(g) \xrightarrow{\cong} \mathrm{T}(f \boxtimes g).$$

Now, in this setting a monoid map $X \to BF_{\mathcal{A}}$ gives rise to a well-defined map,

$$B^{\mathrm{cy}}_{\boxtimes} f \colon\ B^{\mathrm{cy}}_{\boxtimes} X \longrightarrow B^{\mathrm{cy}}_{\boxtimes}(BF_{\mathcal{A}}) \longrightarrow BF_{\mathcal{A}}$$

and applying the strong symmetric monoidal Thom spectrum functor yields an isomorphism

$$\mathrm{T}(B^{\mathrm{cy}}_{\boxtimes} f) \cong B^{\mathrm{cy}}_{\wedge} \mathrm{T}(f).$$

Therefore we can directly implement the intuitive strategy discussed above. Of course, there is significant technical work necessary to retain homotopical control over the quantities involved in the formula above when computing THH, but the basic approach does become as simple as indicated.

We construct two different possible realizations of the category $\mathcal{A}$. In a precise sense, our constructions herein are space-level analogues of the constructions of the modern symmetric monoidal categories of spectra. Just as there is an operadic approach to a symmetric monoidal category of spectra given by EKMM and a "diagrammatic" approach given by (for example) symmetric spectra, we have operadic and diagrammatic approaches to producing $\mathcal{A}$. Put another way, we develop foundations for infinite loop space theory akin to the modern foundations for categories of spectra.

Since there are several good choices for the category $\mathcal{A}$, we shall in fact give an axiomatic description of the properties needed to prove Theorem 1.1. The point is that even though these settings are equivalent, the natural input for the respective Thom spectrum functors is very different. Working in an axiomatic setting gives a flexible framework for adapting the constructions to fit the input provided in particular cases.

$\mathcal{L}(1)$-spaces and S-modules. Our first construction is intimately related to the S-modules of [12]. Let $\mathcal{L}(n)$ denote the space of linear isometries $\mathcal{L}(U^n, U)$, for a fixed countably infinite-dimensional real inner product space U. The object $\mathcal{L}(1)$ is a monoid under composition, and we consider the category of $\mathcal{L}(1)$-spaces. Following the approach of [15], we construct an "operadic smash product" on this category of spaces defined as the coequalizer

$$X \boxtimes Y = \mathcal{L}(2) \times_{\mathcal{L}(1)\times\mathcal{L}(1)} (X \times Y).$$

This product has the property that an A_∞-space is a monoid and an E_∞-space is a commutative monoid. Therefore BG is a commutative monoid with respect to the $\boxtimes$ product, and so we can adapt the Lewis-May Thom spectrum functor to construct a Thom spectrum functor from the category of $\mathcal{L}(1)$-spaces over BG to S-modules which is strong symmetric monoidal. The observation that one could carry out the program of [12] in the setting of spaces is due to Mike Mandell, and was worked out in the thesis of the second author [3].

$\mathcal{I}$-spaces and symmetric spectra. Our second construction is intimately related to the symmetric spectra of [19]. Let $\mathcal{I}$ denote Bökstedt's indexing category, with objects the finite sets $\mathbf{n} = \{1, \ldots, n\}$ and morphisms are the injective maps. The empty set $\mathbf{0}$ is an initial object. The usual concatenation $\mathbf{m} \sqcup \mathbf{n}$ of finite ordered sets makes this a symmetric monoidal category with symmetric structure maps $\mathbf{m} \sqcup \mathbf{n} \to$

$\mathbf{n} \sqcup \mathbf{m}$ given by the obvious shuffles. By definition, an $\mathcal{I}$-*space* is a continuous functor $X\colon \mathcal{I} \to \mathcal{U}$, and we write $\mathcal{I}\mathcal{U}$ for the category of such functors. Just as in the setting of diagram spectra, this category inherits a symmetric monoidal structure from $\mathcal{I}$ via left Kan extension:

$$(X \boxtimes Y)(n) \cong \operatorname{colim}_{\ell \sqcup m \to n} X(\ell) \wedge X(m).$$

Given an $\mathcal{I}$-space X, there is a "suspension spectrum" functor which takes values in symmetric spectra: we may define a symmetric spectrum $\Sigma^\infty(X)$ whose nth space is $S^n \wedge X(n)_+$ with Σ_n acting diagonally on S^n and $X(n)$. The construction of symmetric Thom spectra from $\mathcal{I}$-spaces generalizes this construction. This point of view on Thom spectra is related to early constructions of May, Quinn, and Ray [27] and has been worked out in detail by the third author [30].

Organization of the paper. We begin in Section 2 by reviewing the construction of the Lewis-May operadic Thom spectrum functor. In Section 3, we describe an axiomatic framework which specifies the properties of $\mathcal{A}$ needed to prove Theorem 1.1. Following this, in Section 4 we then show how our main theorems can be deduced from these axioms. The rest of the paper is devoted to discussing the implementations. We collect the relevant background material for the S-module approach in Section 5 and verify the axioms in this setting in Section 6. In Section 7 we discuss modifications needed to work in the context of universal quasifibrations. We then switch gears and consider the setting of $\mathcal{I}$-spaces. In Section 8 we collect and formulate some background material on symmetric spectra and we verify the axioms in this setting in Section 9. The commutative case is discussed in Section 10. Finally, in Section 11 we discuss the technical details of our application to computing the THH of Eilenberg-Mac Lane spectra, including the issues surrounding the use of "change of operad" techniques.

2. The Lewis-May operadic Thom spectrum functor

In this section, we review the details of the operadic Thom spectrum functor studied in Lewis' thesis [17]. Our rigid Thom spectrum functors are built on this foundation, and this construction provides the "reference" homotopy type we expect from a Thom spectrum functor.

The Lewis-May Thom spectrum functor takes values in the category $\mathscr{S}$ of coordinate-free spectra, which is a full subcategory of the category $\mathscr{P}$ of coordinate-free prespectra [17]. Recall that a coordinate-free prespectrum consists of a sequence of based spaces $X(V)$ and a transitive system of based structure maps $S^{W-V} \wedge X(V) \to X(W)$, where V ranges over the finite-dimensional subspaces of a fixed countably infinite dimensional real inner product space U. A spectrum is a prespectrum X such that the adjoint structure maps $X(V) \to \Omega^{W-V} X(W)$ are homeomorphisms; denote the category of spectra by $\mathscr{S}$. The forgetful functor from $\mathscr{S}$ to $\mathscr{P}$ has a left adjoint, the spectrification functor L.

Underlying both of these categories is the category of coordinatized prespectra; denote this category by *Sp*. Explicitly, a prespectrum in *Sp* is given as a sequence of based spaces E_n for $n \geq 0$, equipped with a transitive system of structure maps $S^1 \wedge E_n \to E_{n+1}$. Clearly, there is a forgetful functor from $\mathscr{P}$ to *Sp*. Furthermore, *Sp* determines the homotopy theory on $\mathscr{P}$ and $\mathscr{S}$: the homotopy groups of a coordinate-free prespectrum or spectrum are defined as the stable homotopy groups of the underlying coordinatized prespectrum. Each of the categories $\mathscr{S}$, $\mathscr{P}$, and

Sp is equipped with a stable model structure in which the weak equivalences are defined as the maps which induce isomorphisms on homotopy groups; it follows that the forgetful functors $\mathbb{U}\colon \mathscr{S} \to \mathscr{P}$ and $\mathbb{U}\colon \mathscr{P} \to Sp$ induce Quillen equivalences. Note however that none of these categories are symmetric monoidal prior to passage to the homotopy category.

We now begin to set up the context in which to define the Lewis-May Thom spectrum functor. Let $\mathscr{I}_c$ denote the category of finite-dimensional or countably-infinite real inner product spaces and linear isometries.

Definition 2.1. An $\mathscr{I}_c$-space is a continuous functor X from $\mathscr{I}_c$ to the category $\mathcal{T}$ of based topological spaces. An $\mathscr{I}_c$-FCP (functor with cartesian product) is a $\mathscr{I}_c$-space equipped with a unital and associative "Whitney sum" natural transformation $\omega\colon X \times X \to X \circ \oplus$. A commutative $\mathscr{I}_c$-FCP is an $\mathscr{I}_c$-FCP for which the natural transformation $\tau\colon X \times X \to X \circ \oplus$ is commutative.

We will restrict attention to $\mathscr{I}_c$-spaces that satisfy the following properties:

- The map $X(V) \to X(W)$ induced by an isometry $V \to W$ is a homeomorphism onto a closed subset, and
- $X(V)$ is the colimit over the inclusions $W \subset V$ of $X(W)$, for the finite-dimensional subspaces $W \subset V$.

These requirements implies that it is sufficient to consider the restriction of X to the full subcategory of $\mathscr{I}_c$ consisting of the finite-dimensional real inner product spaces [27, 1.1.8-1.1.9]. The idea of using $\mathscr{I}_c$ to capture structure about infinite loop spaces and operad actions dates back to Boardman and Vogt's original treatment [5]. However, we have adopted the terminology of [26, §23] when discussing the multiplicative structure on this category.

Lemma 2.2. [27, 1.1.6] *An $\mathscr{I}_c$-FCP X yields a non-Σ $\mathcal{L}$-space structure on $X(\mathbb{R}^\infty)$. A commutative $\mathscr{I}_c$-FCP X yields an $\mathcal{L}$-space structure on $X(\mathbb{R}^\infty)$.*

There is an obvious product structure on the category of $\mathscr{I}_c$-spaces specified by the levelwise cartesian product. An $\mathscr{I}_c$-FCP with values in monoids is a monoid-valued $\mathscr{I}_c$-FCP such that the levelwise monoid product specifies a morphism of $\mathscr{I}_c$-spaces. Our central example is the commutative $\mathscr{I}_c$-FCP F given by taking $F(V)$ to be the space of based homotopy equivalences of S^V. Analogously, we will consider $\mathscr{I}_c$-FCP's with values in groups; the commutative $\mathscr{I}_c$-FCP O specified by taking $O(V)$ to be the space of orthogonal transformations of V is a familiar example.

Given a commutative $\mathscr{I}_c$-FCP with values in monoids X augmented over F, there is an associated universal quasifibration. Namely, define $EX(V) = B(*, X(V), S^V)$ and $BX(V) = B(*, X(V), *)$, where $B(-,-,-)$ denotes the two-sided bar construction with respect to the cartesian product in the category of spaces and X acts on S^V via the map to F. These assemble to give $\mathscr{I}_c$-FCP's EX and BX. There is a projection map which is a map of $\mathscr{I}_c$-FCP's $\pi : EX \to BX$ and a section defined by the basepoint inclusion $* \hookrightarrow S^V$. This section is a Hurewicz cofibration, π is a quasifibration, and π has fiber S^V [17, 7.2]. When X takes values in groups, π is a bundle. We are following Lewis in letting $EG(V)$ denote the total space of the universal spherical quasifibration rather than the associated principal quasifibration.

Next, we specify the Thom spectrum construction. Assume we are given a map $f\colon Y \to BG$, by which we mean a map from Y to $BG(\mathbb{R}^\infty)$ where BG is the classifying space of a commutative $\mathscr{I}_c$-FCP G with values in monoids equipped with a map to F. The filtration of BG by inner-product spaces V induces a filtration of Y by pullback, where we define $Y(V)$ to be $f^{-1}(BG(V))$.

Definition 2.3. Define the Thom prespectrum associated to $f : Y \to BG$ as follows. Set $\mathrm{T}_{\mathscr{P}} f(V)$ to be the Thom space of the pullback of the diagram

$$\begin{array}{ccc} Z(V) & \longrightarrow & EG(V) \\ \downarrow & & \downarrow \\ Y(V) & \longrightarrow & BG(V). \end{array}$$

Specifically, the map $Z(V) \to Y(V)$ has a section i, and so we define $T_{\mathscr{P}} f(V)$ by quotienting out by the image of the section.

To see that $\mathrm{T}_{\mathscr{P}} f$ is actually a prespectrum (in $\mathscr{P}$), we must describe the suspension maps. Associated to the inclusion $V \subset W$ is an inclusion $Y(V) \subset Y(W)$. Therefore, there is a map of sphere spaces between the pullbacks

$$\begin{array}{ccc} Z_W & \longrightarrow & EG(W) \\ \downarrow & & \downarrow \\ Y(W) & \longrightarrow & BG(W) \end{array} \qquad \begin{array}{ccccc} Q_V & \longrightarrow & EG(V) & \longrightarrow & EG(W) \\ \downarrow & & \downarrow & & \downarrow \\ Y(V) & \longrightarrow & BG(V) & \longrightarrow & BG(W) \end{array}$$

However we can identify Q_V as $\Sigma^{W-V} Z_V$ [17, 7.2.2], and so the map in question is a suspension map. One checks that the map is appropriately coherent [17, 7.2.1].

Thus, we have specified a functor

$$\mathrm{T}_{\mathscr{P}} \colon \mathcal{U}/BG \longrightarrow \mathscr{P}.$$

By applying the spectrification functor L, we obtain a functor

$$\mathrm{T}_{\mathscr{S}} \colon \mathcal{U}/BG \longrightarrow \mathscr{S}.$$

This is the classical definition of the Lewis-May Thom spectrum functor. We can also obtain a functor

$$\mathrm{T}_{Sp} \colon \mathcal{U}/BG \longrightarrow Sp$$

by applying the forgetful functor to $\mathrm{T}_{\mathscr{P}}$ or equivalently restricting to indexing spaces $\mathbb{R}^n$. Note that other filtrations can also be used in this construction, but it can be shown that the choice of filtration does not matter up to isomorphism of spectra [17, 7.4.4].

The definition above requires a commutative $\mathscr{I}_c$-FCP's with values in monoids augmented with a distinguished map to F, which gives the action of X on S^V. However, we can develop such a theory for commutative $\mathscr{I}_c$-FCP's augmented over other monoids provided we specify an action on the fiber of the universal quasifibration. In particular, in Section 11 when considering models of Eilenberg-Maclane spectra as Thom spectra we will use a "localized" version of F and universal quasifibrations with fibers $S^V_{(p)}$.

Due to the fact that quasifibrations and cofibrations are not in general preserved under pullback, this is not a homotopy functor unless G takes values in groups. This is the main added difficulty when working with BF instead of BO. The standard remedy is to functorially replace f by a Hurewicz fibration $\Gamma(f)\colon \Gamma_f(X) \to BF$ in the usual way. It then follows from [17, §IX.4.9] that the composite functor

$$T_{\mathscr{S}}\Gamma\colon \mathcal{U}/BF \longrightarrow \mathscr{S}$$

is a homotopy functor in the sense that it takes weak homotopy equivalences over BF to stable equivalences. Given a map $f\colon X \to BF$, there is a natural homotopy equivalence $X \to \Gamma_f(X)$, which we may view as a natural transformation from the identity functor on $\mathcal{U}/BF$ to Γ. We think of $\mathrm{T}(f)$ as representing the "correct" homotopy type if the induced map $\mathrm{T}(f) \to \mathrm{T}(\Gamma(f))$ is a stable equivalence, and in this case we say that f is *T-good*. It follows from the above discussion that the restriction of T to the full subcategory of T-good objects is a homotopy functor. Thus, for the statement in Theorem 1.1 to be homotopically meaningful we have must have chosen T-good representatives before applying the Thom spectrum functor.

For simplicity we only formulate the axioms needed to prove Theorem 1.1 in the case where G is group valued. One may also formulate such axioms in the general case by repeated use of the functor Γ, but we feel that this added technicality would obscure the presentation. In the implementation of the axioms in Section 6 and Section 9, we discuss how to modify the constructions so as to obtain Theorem 1.1 in general.

3. Rigid Thom spectrum functors

In this section, we formulate the axiomatic setup and prove the main theorems from the axioms.

3.1. **Preliminary setup and assumptions.** Let $\mathcal{S}$ denote a symmetric monoidal category of "spectra", for example symmetric spectra. Formally, we require that $\mathcal{S}$ be a symmetric monoidal topological category and that there is a continuous functor $U\colon \mathcal{S} \to Sp$, which we think of as a forgetful functor. A morphism in $\mathcal{S}$ is said to be a *weak equivalence* if the image under U is a stable equivalence of spectra. We further require that $\mathcal{S}$ be cocomplete and tensored over unbased spaces. As we recall in Section 4.1, this implies that there is an internal notion of geometric realization for simplicial objects in $\mathcal{S}$. We also assume that the category of monoids in $\mathcal{S}$ comes equipped with a full subcategory whose objects we call *flat monoids*. Given a flat monoid in $\mathcal{S}$, we define its topological Hochschild homology to be the geometric realization of the cyclic bar construction. (In the implementations, the flat objects are sufficiently "cofibrant" for this to represent the correct homotopy type. We recall that for an ordinary discrete ring, flatness is sufficient for its Hochschild homology to be represented by the cyclic bar construction).

We write $\mathcal{A}$ for our refined category of spaces. Formally, we require that $\mathcal{A}$ be a closed symmetric monoidal topological category with monoidal product $\boxtimes$ and unit $1_{\mathcal{A}}$, and we assume that there is a continuous functor $U\colon \mathcal{A} \to \mathcal{U}$ which we again think of as a forgetful functor. A morphism in $\mathcal{A}$ is said to be a *weak equivalence* if the image under U is a weak homotopy equivalence of spaces. We also require that $\mathcal{A}$ be cocomplete and tensored over unbased spaces and that U preserves colimits and tensors.

3.2. **Rigid Thom spectrum functors.** We assume the presence of the additional structure listed in **A1**–**A6**.

A1: There exists a commutative monoid $BG_{\mathcal{A}}$ in $\mathcal{A}$ and a weak homotopy equivalence $\zeta\colon BG_U \xrightarrow{\sim} BG$, where BG_U denotes the image of $BG_{\mathcal{A}}$ under the functor U. We further assume that $BG_{\mathcal{A}}$ is augmented in the sense that there is a map of monoids $BG_{\mathcal{A}} \to 1_{\mathcal{A}}$.

A2: There exists a strong symmetric monoidal "Thom spectrum" functor

$$\mathrm{T}_{\mathcal{A}}\colon \mathcal{A}/BG_{\mathcal{A}} \longrightarrow \mathcal{S}$$

that preserves weak equivalences, and commutes with colimits and tensors with unbased spaces. We require that $\mathrm{T}_{\mathcal{A}}$ be a lift of T_{Sp} in the sense that the two compositions in the diagram

$$\begin{array}{ccc} \mathcal{A}/BG_{\mathcal{A}} & \xrightarrow{\mathrm{T}_{\mathcal{A}}} & \mathcal{S} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathcal{U}/BG_U & \xrightarrow[\mathrm{T}_{Sp}]{} & Sp \end{array}$$

are related by a chain of natural stable equivalences. Here T denotes the composite functor

$$\mathcal{U}/BG_U \xrightarrow{\zeta_*} \mathcal{U}/BG \xrightarrow{\mathrm{T}_{Sp}} Sp.$$

These two axioms already guarantee that we can carry out the argument sketched in the introduction. Let $\alpha\colon A \to BG_{\mathcal{A}}$ be a monoid morphism, and let $B^{\mathrm{cy}}(\alpha)$ be the realization of the simplicial map

$$B^{\mathrm{cy}}_\bullet(\alpha)\colon B^{\mathrm{cy}}_\bullet(A) \longrightarrow B^{\mathrm{cy}}_\bullet(BG_{\mathcal{A}}) \longrightarrow BG_{\mathcal{A}},$$

where we view $BG_{\mathcal{A}}$ as a constant simplicial object.

Theorem 3.1. *Let $\alpha\colon A \to BG_{\mathcal{A}}$ be a monoid morphism in $\mathcal{A}$. Then $\mathrm{T}_{\mathcal{A}}(\alpha)$ is a monoid in $\mathcal{S}$ and there is an isomorphism*

$$B^{cy}(\mathrm{T}_{\mathcal{A}}(\alpha)) \cong \mathrm{T}_{\mathcal{A}}(B^{cy}(\alpha)).$$

Furthermore, there is a stable equivalence

$$U\,\mathrm{T}_{\mathcal{A}}(B^{cy}(\alpha)) \xrightarrow{\sim} \mathrm{T}(UB^{cy}(\alpha)).$$

The simplicity of this result, once we have set up the framework of the two axioms, is very satisfying. However, since we are really interested in topological Hochschild homology, we must be able to represent our Thom spectra as flat monoids in $\mathcal{S}$ and for this reason we introduce the functor C below. This should be thought of as a kind of cofibrant replacement functor and for the application of Theorem 3.1 it is essential that this "replacement" takes place in the category $\mathcal{A}$ before applying T. Adapting the usual convention for topological monoids to our setting, we say that a monoid in $\mathcal{A}$ is *well-based* if the unit $1_{\mathcal{A}} \to A$ has the homotopy extension property, see Section 4.1 for details. We write $\mathcal{A}[\mathbb{T}]$ for the category of monoids in $\mathcal{A}$.

A3: There exists a functor

$$C\colon \mathcal{A}[\mathbb{T}] \longrightarrow \mathcal{A}[\mathbb{T}], \quad A \mapsto A^c,$$

and a natural weak equivalence $A^c \to A$ in $\mathcal{A}[\mathbb{T}]$. We require that the monoid A^c be well-based and that the composite functor

$$\mathcal{A}[\mathbb{T}]/BG_{\mathcal{A}} \xrightarrow{C} \mathcal{A}[\mathbb{T}]/BG_{\mathcal{A}} \xrightarrow{\mathrm{T}_{\mathcal{A}}} \mathcal{S}, \quad \alpha \mapsto \mathrm{T}_{\mathcal{A}}(A^c \longrightarrow A \xrightarrow{\alpha} BG_{\mathcal{A}})$$

takes values in the full subcategory of flat monoids in $\mathcal{S}$.

As explained earlier, we think of the symmetric monoidal category $\mathcal{A}$ as a refined model of the category of spaces in which we can represent E_∞ monoids by strictly commutative monoids. Whereas the functor U should be thought of as a forgetful functor, the functor Q introduced below encodes the relationship between the monoidal product $\boxtimes$ and the cartesian product of spaces.

A4: There exists a strong symmetric monoidal functor $Q\colon \mathcal{A} \to \mathcal{U}$ that preserves colimits and tensors with unbased spaces. We further assume that there is a natural transformation $U \to Q$ that induces a weak homotopy equivalence

$$U(A_1^c \boxtimes \cdots \boxtimes A_k^c) \longrightarrow Q(A_1^c \boxtimes \cdots \boxtimes A_k^c)$$

for all $k \geq 0$ and all k-tuples of monoids $A_1, \dots, A_k$.

For $k = 0$, the last requirement amounts to the condition that $U(1_{\mathcal{A}}) \to *$ be a weak homotopy equivalence. Until now we have not made any assumptions on the homotopical behavior of $BG_{\mathcal{A}}$ with respect to the monoidal structure. The next axiom ensures that we may replace $BG_{\mathcal{A}}$ by a commutative monoid which is in a certain sense well-behaved.

A5: There exists a well-based commutative monoid $BG'_{\mathcal{A}}$ in $\mathcal{A}$ and a weak equivalence of monoids $BG'_{\mathcal{A}} \to BG_{\mathcal{A}}$. We assume that the canonical map (induced by the augmentation)

$$U\big(\underbrace{BG'_{\mathcal{A}} \boxtimes \cdots \boxtimes BG'_{\mathcal{A}}}_{k}\big) \longrightarrow \underbrace{UBG'_{\mathcal{A}} \times \cdots \times UBG'_{\mathcal{A}}}_{k}$$

is an equivalence for all k.

Notice, that since Q is monoidal, it takes monoids in $\mathcal{A}$ to ordinary topological monoids. In particular, if A is a monoid in $\mathcal{A}$, then QA^c is a topological monoid and we say that A is *grouplike* if QA^c is grouplike in the usual sense. Let now $\alpha\colon A \to BG_{\mathcal{A}}$ be a monoid morphism and write $X = UA$ and $f = U\alpha$. We define BX to be the realization of $UB_\bullet(A^c)$ and B^2G_U to be the realization of $UB_\bullet(BG^c_{\mathcal{A}})$. Let $\alpha^c\colon A^c \to BG_{\mathcal{A}}$ be the composition of α with the natural map from **A3** and let Bf be the realization of $UB_\bullet(\alpha^c)$, that is,

$$Bf\colon BX \longrightarrow B^2G_U. \tag{3}$$

We shall see in Section 4.2 that **A3** and **A4** imply that this is a delooping of f if A is grouplike.

Theorem 3.2. *Suppose that* **A1**–**A5** *hold and that A is grouplike. Then there is a stable equivalence*

$$T(UB^{cy}(\alpha^c)) \simeq T(L^\eta(Bf)),$$

where $L^\eta(Bf)$ is the map

$$L(BX) \xrightarrow{L(Bf)} L(B^2G_U) \simeq BG_U \times B^2G_U \xrightarrow{id\times\eta} BG_U \times BG_U \longrightarrow BG_U,$$

defined as in Theorem 1.1.

Combining this result with Theorem 3.1, we get a stable equivalence

$$UB^{\mathrm{cy}}(\mathrm{T}_{\mathcal{A}}(\alpha^c)) \simeq \mathrm{T}(L^\eta(Bf))$$

and since $\mathrm{T}_{\mathcal{A}}(\alpha^c)$ is a flat replacement of $\mathrm{T}_{\mathcal{A}}(\alpha)$ by assumption, this gives an abstract version of Theorem 1.1. In order to obtain the latter, we must be able to lift space level data to $\mathcal{A}$. This is the purpose of our final axiom. Here $\mathcal{C}_{\mathcal{A}}$ denotes an A_∞ operad and $\mathcal{U}[\mathcal{C}_{\mathcal{A}}]$ is the category of spaces with $\mathcal{C}_{\mathcal{A}}$-action.

A6: There exists an A_∞ operad $\mathcal{C}_{\mathcal{A}}$ that acts on BG_U and a functor

$$R\colon \mathcal{U}[\mathcal{C}_{\mathcal{A}}]/BG_U \longrightarrow \mathcal{A}[\mathbb{T}]/BG_{\mathcal{A}}, \quad (X \xrightarrow{f} BG_U) \mapsto (R_f(X) \xrightarrow{R(f)} BG_{\mathcal{A}}),$$

such that $R(\mathrm{id})\colon R_{\mathrm{id}}(BG_U) \to BG_{\mathcal{A}}$ is a weak aquivalence and the composite functor

$$\mathcal{U}[\mathcal{C}_{\mathcal{A}}]/BG_U \xrightarrow{R} \mathcal{A}[\mathbb{T}]/BG_{\mathcal{A}} \longrightarrow \mathcal{A}[\mathbb{T}] \xrightarrow{C} \mathcal{A}[\mathbb{T}] \xrightarrow{Q} \mathcal{U}[\mathbb{T}] \longrightarrow \mathcal{U}[\mathcal{C}_{\mathcal{A}}]$$

is related to the forgetful functor by a chain of natural weak homotopy equivalences in $\mathcal{U}[\mathcal{C}_{\mathcal{A}}]$.

The second arrow represents the forgetful functor and the last arrow represents the functor induced by the augmentation from the A_∞ operad $\mathcal{C}_{\mathcal{A}}$ to the associativity operad, see [25]. It follows from **A3**, **A4** and **A6** that there is a chain of weak homotopy equivalences relating X to $UR_f(X)$. Thus, in this sense R is a partial right homotopy inverse of U. We shall later see that if X is grouplike, then the conditions in **A6** ensure that the delooping of f implied by the operad action is homotopic to the map defined in 3.

4. Proofs of the main results from the axioms

We first recall some background material on tensored categories and geometric realization.

4.1. **Simplicial objects and geometric realization.** Let $\mathcal{A}$ be a cocomplete topological category. Thus, we assume that $\mathcal{A}$ is enriched over $\mathcal{U}$ in the sense that the morphism sets $\mathcal{A}(A,B)$ are topologized and the composition maps continuous. The category $\mathcal{A}$ is tensored over unbased spaces if there exists a continuous functor $\otimes\colon \mathcal{A}\times\mathcal{U}\to\mathcal{A}$, together with a natural homeomorphism

$$\mathrm{Map}(X,\mathcal{A}(A,B)) \cong \mathcal{A}(A\otimes X, B),$$

where A and B are objects in $\mathcal{A}$ and X is a space. For the category $\mathcal{U}$ itself, the tensor is given by the cartesian product, and in *Sp* the tensor of a spectrum A with an (unbased) space X is the level-wise smash product $A\wedge X_+$. Assuming that $\mathcal{A}$ is tensored, there is an internal notion of geometric realization of simplicial objects. Let $[p]\mapsto\Delta^p$ be the usual cosimplicial space used to define the geometric realization. Given a simplicial object $A_\bullet$ in $\mathcal{A}$, we define the realization $|A_\bullet|$ to be the coequalizer of the diagram

$$\coprod_{[p]\to[q]} A_q\otimes\Delta^p \rightrightarrows \coprod_{[r]} A_r\otimes\Delta^r,$$

where the first coproduct is over the morphisms in the simplicial category and the two arrows are defined as for the realization of a simplicial space. Notice, that if we view an object A as a constant simplicial object, then its realization is isomorphic to A. In the case where $\mathcal{A}$ is the category $\mathcal{U}$, the above construction gives the usual geometric realization of a simplicial space and if $A_\bullet$ is a simplicial spectrum, then $|A_\bullet|$ is the usual level-wise realization. The following lemma is an immediate consequence of the definitions.

Lemma 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be cocomplete topological categories that are tensored over unbased spaces and let $\Psi\colon \mathcal{A} \to \mathcal{B}$ be a continuous functor that preserves colimits and tensors. Then Ψ also preserves realization of simplicial objects in the sense that there is a natural isomorphism $|\Psi A_\bullet| \cong \Psi|A_\bullet|$.*

Following [23], we say that a morphism $U \to V$ in $\mathcal{A}$ is an *h-cofibration* if the induced morphism from the mapping cylinder

$$V \cup_U U \otimes I \longrightarrow V \otimes I$$

admits a retraction in $\mathcal{A}$. This generalizes the usual notion of a cofibration in $\mathcal{U}$, that is, of a map having the homotopy extension property. Applying the usual terminology from the space level setting as in [33], Appendix A, we say that a simplicial object in $\mathcal{A}$ is *good* if the degeneracy operators are h-cofibrations. We observe that a functor that preserves colimits and tensors as in Lemma 4.1 also preserves h-cofibrations. It therefore also preserves the goodness condition for simplicial objects. If $\mathcal{A}$ has a monoidal structure, then we say that a monoid A is *well-based* if the unit $1_{\mathcal{A}} \to A$ is an h-cofibration.

Lemma 4.2. *Let $\mathcal{A}$ be a closed symmetric monoidal topological category that is cocomplete and tensored over unbased spaces. If A is a well-based monoid in $\mathcal{A}$, then the simplicial objects $B_\bullet(A)$ and $B^{cy}_\bullet(A)$ are good.*

Proof. We claim that if $U \to V$ is an h-cofibration in $\mathcal{A}$, then the induced morphism $U \boxtimes W \to V \boxtimes W$ is again an h-cofibration for any object W. In order to show this we use that $\mathcal{A}$ is closed to establish a canonical isomorphism

$$(V \cup_U U \otimes I) \boxtimes W \cong V \boxtimes W \cup_{V \boxtimes W} U \boxtimes W \otimes I.$$

Similarly, we may identify $(V \otimes I) \boxtimes W$ with $(V \boxtimes W) \otimes I$ and the claim follows. Since A is well-based, this implies the statement of the lemma. □

4.2. **Consequences of the axioms.** Let now $\mathcal{A}$ and $\mathcal{S}$ be as in Section 3.1, and assume that the axioms **A1**–**A6** hold. We shall then prove the consequences of the axioms stated in Section 3.2.

Proof of Theorem 3.1. If $\alpha\colon A \to BF_{\mathcal{A}}$ is a monoid morphism in $\mathcal{A}$, then the assumption that $T_{\mathcal{A}}$ be strong symmetric monoidal implies that we have an isomorphism of cyclic objects $B^{\mathrm{cy}}_\bullet(T_{\mathcal{A}}(\alpha)) \cong T_{\mathcal{A}}(B^{\mathrm{cy}}_\bullet(\alpha))$. The first statement then follows from Lemma 4.1 since we have assumed that $T_{\mathcal{A}}$ preserves colimits and tensors. The second statement follows from the assumption that the diagram in **A2** commutes up to stable equivalence.

The following lemma is an immediate consequence of Lemma 4.2 and the assumption that U and Q preserve colimits and tensors.

Lemma 4.3. *If A is a well-based monoid in $\mathcal{A}$, then the simplicial objects $B_\bullet(A)$ and $B^{cy}_\bullet(A)$ are good and so are the simplicial spaces obtained by applying U and Q.*

If $Z_\bullet$ is a simplicial object in $\mathcal{A}$ with internal realization Z, then it follows from Lemma 4.1 that UZ is homeomorphic to the realization of the simplicial space $UZ_\bullet$ obtained by applying U degree-wise. If $Z_\bullet$ is a cyclic object, then $UZ_\bullet$ is a cyclic space and UZ inherits an action of the circle group $\mathbb{T}$. Recall that a monoid A in $\mathcal{A}$ is said to be grouplike if the topological monoid QA^c is grouplike in the usual sense.

Proposition 4.4. *If A is grouplike, then $UB(A^c)$ is a delooping of UA.*

Proof. The natural transformation in **A4** gives rise to a map of simplicial spaces $UB_\bullet(A^c) \to QB_\bullet(A^c)$ which is a level-wise weak homotopy equivalence by assumption. Since by Lemma 4.3 these are good simplicial spaces, it follows that the topological realization is also a weak homotopy equivalence. Furthermore, since Q is strong symmetric monoidal, $QB(A^c)$ is isomorphic to the classifying space of the grouplike topological monoid QA^c, hence is a delooping of the latter. Thus, we have a chain of weak homotoppy equivalences

$$\Omega(UB(A^c)) \simeq \Omega(QB(A^c)) \simeq \Omega(B(QA^c)) \simeq QA^c \simeq UA^c \simeq UA,$$

where the two last equivalences are implied by **A3** and **A4** . □

Suppose now that A and therefore also A^c is augmented over the unit $1_{\mathcal{A}}$. We then have the following analogue of (2),

$$\mathbb{T} \times UB^{\mathrm{cy}}(A^c) \longrightarrow UB^{\mathrm{cy}}(A^c) \longrightarrow UB(A^c),$$

where the last arrow is defined using the augmentation.

Proposition 4.5. *If A is grouplike, then the adjoint map*

$$UB^{cy}(A^c) \longrightarrow L(UB(A^c))$$

is a weak homotopy equivalence.

Proof. It follows from the proof of Proposition 4.4 that there is a weak homotopy equivalence $UB(A^c) \to QB(A^c)$ and by a similar argument we get a weak homotopy equivalence $UB^{\mathrm{cy}}(A^c) \to QB^{\mathrm{cy}}(A^c)$. These maps are related by a commutative diagram

$$\begin{array}{ccc} UB^{\mathrm{cy}}(A^c) & \longrightarrow & L(UB(A^c)) \\ \downarrow\sim & & \downarrow\sim \\ Q(B^{\mathrm{cy}}(A^c)) & \longrightarrow & L(QB(A^c)), \end{array}$$

where, replacing U by Q, the bottom map is defined as the map in the proposition. Using that Q is strong symmetric monoidal, we can write the latter map in the form

$$B^{\mathrm{cy}}(QA^c) \longrightarrow L(B(QA^c)),$$

and as discussed in the introduction, this map is a weak homotopy equivalence. This implies the result. □

Let now $\alpha\colon A \to BG_{\mathcal{A}}$ be a monoid morphism in $\mathcal{A}$ of the form considered in Theorem 3.2. We wish to analyze the map obtained by applying U to the composite morphism

$$B^{\mathrm{cy}}(\alpha^c)\colon\ B^{\mathrm{cy}}(A^c) \longrightarrow B^{\mathrm{cy}}(A) \longrightarrow B^{\mathrm{cy}}(BG_{\mathcal{A}}) \longrightarrow BG_{\mathcal{A}}.$$

Notice first, that we have a commutative diagram

$$\begin{array}{ccc} UB^{\mathrm{cy}}(A^c) & \longrightarrow & UB^{\mathrm{cy}}(BG^c_{\mathcal{A}}) \\ \downarrow\simeq & & \downarrow\simeq \\ L(UB(A^c)) & \longrightarrow & L(UB(BG^c_{\mathcal{A}})), \end{array}$$

where the vertical maps are weak homotopy equivalences by Proposition 4.5. Writing B^2G_U for the delooping $UB(BG^c_{\mathcal{A}})$ as usual, we must identify the homotopy class represented by the diagram

$$L(B^2G_U) \xleftarrow{\sim} UB^{\mathrm{cy}}(BG^c_{\mathcal{A}}) \longrightarrow UB^{\mathrm{cy}}(BG_{\mathcal{A}}) \longrightarrow BG_U. \tag{4}$$

We shall do this by applying the results of [31] and for this we need to recall some general facts aboout Γ-spaces. Consider in general a commutative well-based monoid Z in $\mathcal{A}$ that is augmented over the unit $1_{\mathcal{A}}$. Such a monoid gives rise to a Γ-object in $\mathcal{A}$, that is, to a functor $Z\colon \Gamma^o \to \mathcal{A}$, where Γ^o denotes the category of finite based sets. It suffices to define Z on the skeleton subcategory specified by the objects $\mathbf{n}_+ = \{*, 1, \dots, n\}$, where we let $Z(\mathbf{0}_+) = 1_{\mathcal{A}}$ and

$$Z(\mathbf{n}_+) = \underbrace{Z \boxtimes \cdots \boxtimes Z}_{n}$$

with an implicit choice of placement of the parenthesis in the iterated monoidal product. The Γ-structure is defined using the symmetric monoidal structure of $\mathcal{A}$, together with the multiplication and augmentation of Z. From this point of view the diagram of simplicial objects

$$Z \longrightarrow B^{\mathrm{cy}}_\bullet(Z) \longrightarrow B_\bullet(Z)$$

may be identified with that obtained by evaluating Z degree-wise on the cofibration sequence of simplicial sets $S^0 \to S^1_{\bullet+} \to S^1_\bullet$, see [31], Section 5.2. Composing with the functor U we get the Γ-space UZ and the assumption that the monoid Z be well-based assures that this construction is homotopically well-behaved. Notice also that UZ is degree-wise equivalent to the reduced Γ-space $\tilde{U}Z$ defined by the quotient spaces

$$\tilde{U}Z(\mathbf{n}_+) = UZ(\mathbf{n}_+)/U(1_{\mathcal{A}}).$$

Following Bousfield and Friedlander [7], we say that UZ is a *special* Γ*-space* if the canonical maps

$$U(\underbrace{Z \boxtimes \cdots \boxtimes Z}_{n}) \longrightarrow \underbrace{UZ \times \cdots \times UZ}_{n}$$

are weak homotopy equivalences for all n. In this case the underlying space UZ inherits a weak H-space structure, and we say that the Γ-space is *very special* if the monoid of components is a group. This is equivalent to our previous condition for a monoid in $\mathcal{A}$ to be grouplike. Consider now the composition

$$\mathbb{T} \times UZ(S^1_+) \longrightarrow UZ(S^1_+) \longrightarrow UZ(S^1)$$

defined in analogy with (2). The following result is an immediate consequence of [31], Proposition 7.3.

Proposition 4.6 ([31])**.** *If Z is a well-based commutative monoid in $\mathcal{A}$ such that the Γ-space UZ is very special, then the adjoint map*

$$UZ(S^1_+) \xrightarrow{\sim} L(UZ(S^1))$$

is a weak homotopy equivalence and the diagram

$$UZ(S^1) \longrightarrow L(UZ(S^1)) \xleftarrow{\sim} UZ(S^1_+) \longrightarrow UZ(S^0)$$

represents multiplication by η.

The last map in the above diagram is induced by the retraction $S^1_+ \to S^0$.

Proof of Theorem 3.2. It remains to identify the homotopy class represented by (4) and as explained in the introduction we have a splitting

$$L(B^2G_U) \simeq BG_U \times B^2G_U.$$

We must prove that the homotopy class specified by the diagram

$$B^2G_U \longrightarrow L(B^2G_U) \xleftarrow{\sim} UB^{\mathrm{cy}}(BG^c_{\mathcal{A}}) \longrightarrow BG_U \tag{5}$$

is multiplication by η in the sense of the theorem. Let $BG'_{\mathcal{A}} \to BG_{\mathcal{A}}$ be as in **A6** and let the spaces BG'_U and $B^2G'_U$ be defined as $UBG'_{\mathcal{A}}$ and $UB(BG'^c_{\mathcal{A}})$, respectively. We then obtain a diagram

$$B^2G'_U \longrightarrow L(B^2G'_U) \xleftarrow{\sim} UB^{\mathrm{cy}}(BG'^c_{\mathcal{A}}) \longrightarrow BG'_U \tag{6}$$

which is term-wise weakly homotopy equivalent to (5). Writing Z for the commutative monoid $BG'_{\mathcal{A}}$, the assumptions in **A5** ensure that Z gives rise to a very special Γ-space UZ. We claim that the diagram (6) is term-wise weakly equivalent to the diagram in Proposition 4.6. Indeed, it follows from **A5** that the natural weak equivalence in **A3** gives rise to degree-wise weak homotopy equivalences

$$UB_\bullet(BG'_{\mathcal{A}}) \longrightarrow UZ(S^1_\bullet), \quad \text{and} \quad UB^{\mathrm{cy}}_\bullet(BG'_{\mathcal{A}}) \longrightarrow UZ(S^1_{\bullet+}).$$

Since these are good simplicial spaces by Lemma 4.3, the induced maps of realizations are then also weak homotopy equivalences. The statement of the theorem now follows immediately from Proposition 4.6.

4.3. **Proofs of the main theorems.** We first recall some general facts about deloopings of A_∞ maps from [25]. Thus, let $\mathcal{C}$ be an A_∞ operad with augmentation $\mathcal{C} \to \mathcal{M}$ where $\mathcal{M}$ denotes the associative operad. Let C and M be the associated monads and consider for a $\mathcal{C}$-space X the diagram of weak homotopy equivalences of $\mathcal{C}$-spaces

$$X \xleftarrow{\sim} B(C, C, X) \xrightarrow{\sim} B(M, C, X)$$

defined as in [25], Theorem 13.5. The $\mathcal{C}$-space $B(M, C, X)$ is in fact a topological monoid and we define $B'X$ to be the its classifying space, defined by the usual bar construction. If X is grouplike, then $B'X$ is a delooping in the sense that $\Omega(B'X)$ is related to X by a chain of weak homotopy equivalences. This construction is clearly functorial in X: given a map of $\mathcal{C}$-spaces $X \to Y$, we have an induced map $B'X \to B'Y$.

Let now $f\colon X \to BG_U$ be a $\mathcal{C}$-map and let $\alpha\colon A \to BG_{\mathcal{A}}$ be the object in $\mathcal{A}[\mathbb{T}]/BG_{\mathcal{A}}$ obtained by applying the functor R. We let $Bf\colon BX \to B^2G_U$ be defined as in (3) where we recall that BX and B^2G_U denote the spaces $UB(A^c)$ and $UB(BG^c_{\mathcal{A}})$, respectively. The first step in the proof of Theorem 1.1 is to compare the maps Bf and $B'f$.

Lemma 4.7. *There is a commutative diagram*

$$\begin{array}{ccc} B'X & \xrightarrow{\sim} & BX \\ \Big\downarrow{\scriptstyle B'f} & & \Big\downarrow{\scriptstyle Bf} \\ B'BG_U & \xrightarrow{\sim} & B^2G_U \end{array}$$

in which the vertical arrows represent chains of compatible weak homotopy equivalences.

Proof. By definition, A is the monoid $R_f(X)$ and it follows from **A6** that there is a chain of weak homotopy equivalences of $\mathcal{C}$-spaces relating X to $QR_f(X)^c$. Applying the bar construction from [25] we obtain a chain of weak homotopy equivalences of topological monoids

$$B(M,C,X) \simeq B(M,C,QR_f(X)^c) \simeq QR_f(X)^c.$$

The last equivalence comes from the fact that $QR_f(X)^c$ is itself a topological monoid, see [25, 13.5]. This chain in turn gives a chain of equivalences of the classifying spaces and composing with the equivalence induced by the natural transformation $U \to Q$ we get the upper row in the diagram. In particular, applied to the identity on BG_U, this construction gives a chain of weak homotopy equivalences $B'(BG_U) \simeq B(BG_U)$. Furthermore, the weak equivalence $R_{\mathrm{id}}(BG_U) \to BG_{\mathcal{A}}$ from **A6** gives rise to a weak homotopy equivalence $B(BG_U) \to B^2G_U$ and composing with this we get the bottom row in the diagram. It is clear from the construction that the horizontal rows are compatible as claimed. □

Proof of Theorem 1.1. We first reformulate the theorem in a more precise form that is consistent with the notation introduced above. Thus, we consider a map of $\mathcal{C}$-spaces $f\colon X \to BG_U$ and we again write $\alpha\colon A \to BG_{\mathcal{A}}$ for the monoid morphism in $\mathcal{A}$ obtained by applying the functor R. Then $UB^{\mathrm{cy}}(T_{\mathcal{A}}(\alpha^c))$ represents the topological Hochschild homology spectrum $\mathrm{THH}(T(f))$ and it follows from Theorem 3.1 and Theorem 3.2 that there is a stable equivalence

$$UB^{\mathrm{cy}}(T_{\mathcal{A}}(\alpha^c)) \simeq T(L^{\eta}(Bf))$$

where Bf is defined as in (3). Using the homotopy invariance of the Thom spectrum functor, the result then follows from Lemma 4.7.

In preparation for the proof of Theorem 1.2 we recall that the Thom spectrum functor T is multiplicative in the following sense: given maps $f\colon X \to BG_U$ and $g\colon Y \to BG_U$, there is a stable equivalence

$$T(f \times g) \simeq T(f) \wedge T(g),$$

where $f \times g$ denotes the map

$$f \times g\colon\ X \times Y \xrightarrow{f\times g} BG_U \times BG_U U \longrightarrow BG_U$$

defined using the H-space structure of BG_U. We refer to [17] and [30] for different accounts of this basic fact. Of course, one of the main points of this paper is to "rigidify" this stable equivalence.

Proof of Theorem 1.2. As explained in the introduction, the loop space structure on BX gives rise to a weak homotopy equivalence

$$X \times BX \xrightarrow{\sim} L(BX).$$

This fits in a homotopy commutative diagram

$$\begin{array}{ccccc}
X \times BX & \xrightarrow{f\times Bf} & BG_U \times B^2G_U & \xrightarrow{\mathrm{id}\times\eta} & BG_U \times BG_U \\
\big\downarrow{\scriptstyle\sim} & & \big\downarrow{\scriptstyle\sim} & & \big\downarrow \\
L(BX) & \xrightarrow{L(Bf)} & L(B^2G_U) & \longrightarrow & BG_U,
\end{array}$$

where the composition in the bottom row is the map $L^\eta(Bf)$. By homotopy invariance of the Thom spectrum functor we get from this the stable equivalences

$$T(L^\eta(Bf)) \simeq T(f \times (\eta \circ Bf)) \simeq T(f) \wedge T(\eta \circ Bf)$$

and the result follows from Theorem 1.1.

Proof of Theorem 1.3. Notice first, that if X is a 3-fold loop map, then the unstable Hopf map η gives rise to a map

$$\eta\colon\ BX \simeq \mathrm{Map}_*(S^2, B^3X) \xrightarrow{\eta^*} \mathrm{Map}_*(S^3, B^3X) \simeq X.$$

Let Φ be the self homotopy equivalence of $X \times BX$ defined by

$$\Phi = \begin{bmatrix} \mathrm{id} & \eta \\ 0 & \mathrm{id} \end{bmatrix}\colon\ X \times BX \xrightarrow{\sim} X \times BX.$$

Given a 3-fold loop map $f\colon\ X \to BG_U$, we then have a homotopy commutative diagram

$$\begin{array}{ccccccc} X \times BX & \xrightarrow{f\times Bf} & BG_U \times B^2G_U & \xrightarrow{\mathrm{id}\times\eta} & BG_U \times BG_U & \longrightarrow & BG_U \\ \sim\downarrow\Phi & & \sim\downarrow\Phi & & & & \| \\ X \times BX & \xrightarrow{f\times Bf} & BG_U \times B^2G_U & \xrightarrow{\mathrm{id}\times *} & BG_U \times BG_U & \longrightarrow & BG_U, \end{array}$$

where $*$ denotes the trivial map. It follows from the proof of Theorem 1.2 that the composition of the maps in the upper row is weakly homotopy equivalent to $L^\eta(Bf)$. Thus, by homotopy invariance of the Thom spectrum functor we get the stable equivalence

$$T(L^\eta(Bf)) \simeq T(f \times *) \simeq T(f) \wedge BX_+$$

and the result again follows from Theorem 1.1.

5. Operadic products in the category of spaces

In this section, we adapt the construction of the "operadic" smash product of spectra from [12] to the context of topological spaces. The conceptual foundation of the approach to structured ring spectra undertaken in [12] is the observation that by exploiting special properties of the linear isometries operad, it is possible to define a weakly symmetric monoidal product on a certain category of spectra such that the (commutative) monoids are precisely the (E_∞) A_∞-ring spectra. Since many of the good properties of this product are a consequence of the nature of the operad, such a construction can be carried out in other categories — for instance, [15] studied an algebraic version of this definition. Following an observation of Mandell, we introduce a version of this construction in the category of spaces. Much of this work appeared in the first author's University of Chicago thesis [3].

5.1. **The weakly symmetric monoidal category of $\mathcal{L}(1)$-spaces.** Fix an countably infinite-dimensional real inner product space U, and let $\mathcal{L}(n)$ denote the nth space of the linear isometries operad associated to U; recall that this is the space of linear isometries $\mathcal{L}(U^n, U)$. In particular, the space $\mathcal{L}(1) = \mathcal{L}(U, U)$ is a topological monoid. We begin by considering the category of $\mathcal{L}(1)$-spaces: unbased spaces equipped with a map $\mathcal{L}(1) \times X \to X$ which is associative and unital. We can equivalently regard this category as the category $\mathcal{U}[\mathbb{L}]$ of algebras over the monad $\mathbb{L}$ on the category $\mathcal{U}$ of unbased spaces which takes X to $\mathcal{L}(1) \times X$.

The category $\mathcal{U}[\mathbb{L}]$ admits a product $X \boxtimes_{\mathcal{L}} Y$ defined in analogy with the product $\wedge_{\mathcal{L}}$ on the category of $\mathbb{L}$-spectra [12, §I.5.1]. Specifically, there is an obvious action of $\mathcal{L}(1) \times \mathcal{L}(1)$ on $\mathcal{L}(2)$ via the inclusion of $\mathcal{L}(1) \times \mathcal{L}(1)$ in $\mathcal{L}(2)$. In addition, there is a natural action of $\mathcal{L}(1) \times \mathcal{L}(1)$ on $X \times Y$ given by the isomorphism

$$(\mathcal{L}(1) \times \mathcal{L}(1)) \times (X \times Y) \cong (\mathcal{L}(1) \times X) \times (\mathcal{L}(1) \times Y).$$

We define $\boxtimes_{\mathcal{L}}$ as the balanced product of these two actions.

Definition 5.1. The product $X \boxtimes_{\mathcal{L}} Y$ is the coequalizer of the diagram

$$\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \rightrightarrows \mathcal{L}(2) \times X \times Y.$$

The coequalizer is itself an $\mathcal{L}(1)$-space via the action of $\mathcal{L}(1)$ on $\mathcal{L}(2)$.

The arguments of [12, §I.5] now yield the following proposition.

Proposition 5.2. *The product $\boxtimes_{\mathcal{L}}$ is associative and commutative.*

A useful consequence of the proof that $\boxtimes_{\mathcal{L}}$ is associative is the following analogue of [12, §I.5.6].

Proposition 5.3. *For any j-tuple $M_1, M_2, \ldots, M_j$ of $\mathcal{L}(1)$-spaces, there is a canonical isomorphism of $\mathcal{L}(1)$-spaces*

$$M_1 \boxtimes_{\mathcal{L}} M_2 \boxtimes_{\mathcal{L}} \ldots \boxtimes_{\mathcal{L}} M_j \cong \mathcal{L}(j) \times_{\mathcal{L}(1)^j} (M_1 \times M_2 \times \ldots \times M_j),$$

where the iterated product on the left is associated in any fashion.

There is a corresponding mapping space $F_{\boxtimes_{\mathcal{L}}}(X, Y)$ which satisfies the usual adjunction; in fact, the definition is forced by the adjunctions.

Definition 5.4. The mapping space $F_{\boxtimes_{\mathcal{L}}}(X, Y)$ is the equalizer of the diagram

$$\mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(2) \times X, Y) \rightrightarrows \mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times X, Y).$$

Here one map is given by the action of $\mathcal{L}(1) \times \mathcal{L}(1)$ on $\mathcal{L}(2)$ and the other via the adjunction

$$\mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times X, Y) \cong \mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(2) \times \mathcal{L}(1) \times X, \mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(1), Y))$$

along with the action $\mathcal{L}(1) \times X \to X$ and coaction

$$Y \longrightarrow \mathrm{Map}_{\mathcal{L}(1)}(\mathcal{L}(1), Y).$$

A diagram chase verifies the following proposition.

Proposition 5.5. *Let X, Y, and Z be $\mathcal{L}(1)$-spaces. Then there is an adjunction homeomorphism*

$$\mathrm{Map}_{\mathcal{L}(1)}(X \boxtimes_{\mathcal{L}} Y, Z) \cong \mathrm{Map}_{\mathcal{L}(1)}(X, F^{\boxtimes}(Y, Z)).$$

The natural choice for the unit of the product $\boxtimes_{\mathcal{L}}$ is the point $*$, endowed with the trivial $\mathcal{L}(1)$-action. As in [12, §1.8.3], there is a unit map $* \boxtimes_{\mathcal{L}} X \to X$ which is compatible with the associativity and commutativity properties of $\boxtimes_{\mathcal{L}}$.

Proposition 5.6. *Let X and Y be $\mathcal{L}(1)$-spaces. There is a natural map of $\mathcal{L}(1)$-spaces $\lambda \colon * \boxtimes_{\mathcal{L}} X \to X$. The symmetrically defined map $X \boxtimes_{\mathcal{L}} * \to X$ coincides with the composite $\lambda\tau$. Under the associativity isomorphism $\lambda\tau \boxtimes_{\mathcal{L}} \mathrm{id} \cong \mathrm{id} \boxtimes_{\mathcal{L}} \lambda$, and, under the commutativity isomorphism, these maps also agree with $* \boxtimes_{\mathcal{L}} (X \boxtimes_{\mathcal{L}} Y) \to X \boxtimes_{\mathcal{L}} Y$.*

However, just as is the case in the category of $\mathbb{L}$-spectra, $*$ is not a strict unit for $\boxtimes_{\mathcal{L}}$ — the unit map $* \boxtimes_{\mathcal{L}} X \to X$ is not necessarily an isomorphism. However, it is always a weak equivalence. We do not reprise the proof of this, as it is technical and very similar to the proof of the analogous fact for $\mathbb{L}$-spectra [12, 1.8.5]. We remark only that it is a consequence of point-set properties of the linear isometries operad and the isomorphism $* \boxtimes_{\mathcal{L}} * \to *$.

Proposition 5.7. *For any $\mathcal{L}(1)$-space X, the unit map $\lambda\colon * \boxtimes_{\mathcal{L}} X \to X$ is a weak equivalence of $\mathcal{L}(1)$-spaces.*

In summary, we have shown that the category $\mathcal{U}[\mathbb{L}]$ is a closed weak symmetric monoidal category, with product $\boxtimes_{\mathcal{L}}$ and weak unit $*$. Recall that this means that the $\mathcal{U}[\mathbb{L}]$, $\boxtimes_{\mathcal{L}}$, $F^{\boxtimes}(-,-)$, and $*$ satisfy all of the axioms of a closed symmetric monoidal category except that the unit map is not required to be an isomorphism [12, §II.7.1].

5.2. **Monoids and commutative monoids for $\boxtimes_{\mathcal{L}}$.** In this section, we study $\boxtimes_{\mathcal{L}}$-monoids and commutative $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{U}[\mathbb{L}]$; these are defined as algebras over certain monads, following [12, §2.7]. We show that $\boxtimes_{\mathcal{L}}$-monoids are A_∞-spaces and commutative $\boxtimes_{\mathcal{L}}$-monoids are E_∞-spaces. One can prove this directly, as is done in the algebraic setting in [15, §V.3.1], but we prefer to follow the categorical approach given for $\mathbb{L}$-spectra under $\wedge_{\mathcal{L}}$ in [12, §II.4].

In any weakly symmetric monoidal category, monoids and commutative monoids can be regarded as algebras over the monads $\mathbb{T}$ and $\mathbb{P}$ defined as follows. Let $X^{\boxtimes j}$ denote the j-fold power with respect to $\boxtimes_{\mathcal{L}}$, where $X^0 = *$. Then we define the monads on the category of $\mathcal{L}(1)$-spaces as

$$\mathbb{T}X = \bigvee_{j\geq 0} X^{\boxtimes j} \qquad\qquad \mathbb{P}X = \bigvee_{j\geq 0} X^{\boxtimes j}/\Sigma_j \ ,$$

where the unit is given by the inclusion of X into the wedge and the product is induced by the obvious identifications (and the unit map, if any indices are 0). We regard A_∞ and E_∞ spaces as algebras over the monads $\mathbb{B}$ and $\mathbb{C}$ on based spaces. Recall that these monads are defined as

$$\mathbb{B}Y = \bigvee_{j\geq 0} \mathcal{L}(j) \times X^j \qquad\qquad \mathbb{C}Y = \bigvee_{j\geq 0} \mathcal{L}(j) \times_{\Sigma_j} X^j,$$

subject to an equivalence relation which quotients out the basepoint (where here X^n indicates the iterated cartesian product). If we ignore the quotient, we obtain corresponding monads on unbased spaces which we will denote $\mathbb{B}'$ and $\mathbb{C}'$. The main tool for comparing these various categories of algebras is the lemma [12, §II.6.1], which we write out below for clarity.

Lemma 5.8. *Let $\mathbb{S}$ be a monad in a category $\mathcal{C}$ and let $\mathbb{R}$ be a monad in the category $\mathcal{C}[\mathbb{S}]$ of $\mathbb{S}$-algebras. Then the category $\mathcal{C}[\mathbb{S}][\mathbb{R}]$ of $\mathbb{R}$-algebras in $\mathcal{C}[\mathbb{S}]$ is isomorphic to the category $\mathcal{C}[\mathbb{R}\mathbb{S}]$ of algebras over the composite monad $\mathbb{R}\mathbb{S}$ in $\mathcal{C}$. Moreover, the unit of $\mathbb{R}$ defines a map $\mathbb{S} \to \mathbb{R}\mathbb{S}$ of monads in $\mathcal{C}$. An analogous assertion holds for comonads.*

Recall that the category of based spaces can be viewed as the category of algebras over the monad $\mathbb{U}$ in unbased spaces which adjoins a disjoint basepoint. In mild abuse of notation, we will also refer to the monad on $\mathcal{L}(1)$-spaces which adjoins a disjoint basepoint with trivial $\mathcal{L}(1)$-action as $\mathbb{U}$.

Proposition 5.9. *An $\mathcal{L}(1)$-space X which is a $\boxtimes_{\mathcal{L}}$-monoid is an A_∞-space over the non-Σ linear isometries operad, and the data of a map of non-Σ $\mathcal{L}$-spaces is equivalent to the data of a map of $\boxtimes_{\mathcal{L}}$-monoids. An $\mathcal{L}(1)$-space X which is a commutative $\boxtimes_{\mathcal{L}}$-monoid is an E_∞-space over the linear isometries operad, and the data of a map of $\mathcal{L}$-spaces is equivalent to the data of a map of commutative $\boxtimes_{\mathcal{L}}$-monoids.*

Proof. A straightforward verification shows that $\mathbb{C}' \cong \mathbb{C}\mathbb{U}$ as monads on unbased spaces (and similarly $\mathbb{B}' \cong \mathbb{B}\mathbb{U}$). Then Lemma 5.8 shows that $\mathbb{C}'$-algebras in unbased spaces are equivalent to $\mathbb{C}$-algebras in based spaces. Next, there is an identification of monads on unbased spaces $\mathbb{C}' \cong \mathbb{T}\mathbb{L}$; Proposition 5.3 implies an isomorphism of objects, and the comparison of monad structures is immediate. Lemma 5.8 then implies that $\mathbb{C}'$-algebras in unbased spaces are equivalent to $\mathbb{T}$-algebras in $\mathcal{L}(1)$-spaces, and combined with the initial observation this implies the desired result. □

5.3. **The symmetric monoidal category of $*$-modules.** In this section we define a subcategory of $\mathcal{L}(1)$-spaces which forms a closed symmetric monoidal category with respect to $\boxtimes_{\mathcal{L}}$. This is necessary for our application to topological Hochschild homology — in order to define the cyclic bar construction as a strict simplicial object, we need a unital product to define the degeneracies. In fact, there are two possible approaches to constructing a symmetric monoidal category: These parallel the approaches developed in [15] and [12]. If we restrict attention to the category $\mathcal{T}[\mathbb{U}]$ of based $\mathcal{L}(1)$-spaces (where the $\mathcal{L}(1)$ action is trivial on the basepoint), there is a unital product $\star_{\mathcal{L}}$ formed as the pushout

$$\begin{array}{ccc} (X \boxtimes_{\mathcal{L}} *) \vee (* \boxtimes_{\mathcal{L}} Y) & \longrightarrow & X \vee Y \\ \downarrow & & \downarrow \\ X \boxtimes_{\mathcal{L}} Y & \longrightarrow & X \star_{\mathcal{L}} Y. \end{array}$$

In the algebraic setting of [15], this kind of construction is our only option. However, since there is an isomorphism $* \boxtimes_{\mathcal{L}} * \cong *$ we can also pursue the strategy of considering a subcategory of $\mathcal{L}(1)$-spaces analogous to the category of S-modules [12]. Specifically, observe that the $\mathcal{L}(1)$-space $* \boxtimes_{\mathcal{L}} X$ is unital in the sense that the unit map $* \boxtimes_{\mathcal{L}} (* \boxtimes_{\mathcal{L}} X) \to (* \boxtimes_{\mathcal{L}} X)$ is an isomorphism.

Definition 5.10. The category $\mathcal{M}_*$ of $*$-modules is the subcategory of $\mathcal{L}(1)$-spaces such that the unit map $\lambda\colon * \boxtimes_{\mathcal{L}} X \to X$ is an isomorphism. For $*$-modules X and Y, define $X \boxtimes Y$ as $X \boxtimes_{\mathcal{L}} Y$ and $F_{\boxtimes}(X, Y)$ as $* \boxtimes_{\mathcal{L}} F_{\boxtimes_{\mathcal{L}}}(X, Y)$.

The work of the previous section implies that $\mathcal{M}_*$ is a closed symmetric monoidal category. For use in establishing a model structure on $\mathcal{M}_*$ in Theorem 5.18, we review a more obscure aspect of this category, following the analogous treatment for the category of S-modules [12, §II.2]. The functor $* \boxtimes_{\mathcal{L}} -$ is not a monad in $\mathcal{L}(1)$-spaces. However, the category of $*$-modules has a "mirror image" category to which it is naturally equivalent, and this equivalence facilitates formal analysis of the category of $*$-modules.

Let $\mathcal{M}^*$ be the full subcategory of $\mathcal{L}(1)$-spaces Z such that the counit map $Z \to F^{\boxtimes}(*, Z)$ is an isomorphism. Following the notation of [12, §II.2], let f denote the functor $F^{\boxtimes}(*, -)$ and s denote the functor $* \boxtimes_{\mathcal{L}} (-)$. Let r be the inclusion of the counital $\mathcal{L}(1)$-spaces into the category of $\mathcal{L}(1)$-spaces, and ℓ the inclusion of the unital $\mathcal{L}(1)$-spaces ($*$-modules) into $\mathcal{L}(1)$-spaces. We have the following easy lemma about these functors.

Lemma 5.11. *The functor f is right adjoint to the functor s and left adjoint to the inclusion r.*

Now, we obtain a pair of mirrored adjunctions

$$\mathcal{U}[\mathbb{L}] \underset{rf\ell}{\overset{s}{\rightleftarrows}} \mathcal{M}_* \underset{s}{\overset{\ell}{\rightleftarrows}} \mathcal{U}[\mathbb{L}] \qquad\qquad \mathcal{U}[\mathbb{L}] \underset{r}{\overset{f}{\rightleftarrows}} \mathcal{M}^* \underset{f}{\overset{\ell sr}{\rightleftarrows}} \mathcal{U}[\mathbb{L}].$$

The composite of the first two left adjoints is $* \boxtimes_{\mathcal{L}} (-)$ and the composite of the second two right adjoints is $F^{\boxtimes}(*, -)$. These are themselves adjoints, and now by the uniqueness of adjoints we have the following consequence.

Lemma 5.12. *For an $\mathcal{L}(1)$-space X, the maps*

$$* \boxtimes_{\mathcal{L}} X \longrightarrow * \boxtimes_{\mathcal{L}} F^{\boxtimes}(*, X)$$

and

$$F^{\boxtimes}(*, * \boxtimes_{\mathcal{L}} X) \longrightarrow F^{\boxtimes}(*, X)$$

are natural isomorphisms.

An immediate consequence of this is that the category $\mathcal{M}_*$ and $\mathcal{M}^*$ are equivalent, and in particular we see that the category of $*$-modules is equivalent to the category of algebras over the monad rf determined by the adjunction (see the proof of [12, §II.2.7] for details).

5.4. **Monoids and commutative monoids in $\mathcal{M}_*$.** The monads $\mathbb{T}$ and $\mathbb{P}$ on $\mathcal{L}(1)$-spaces restrict to define monads on $\mathcal{M}_*$. The algebras over these monads are monoids and commutative monoids for $\boxtimes$, respectively. Thus, a $\boxtimes$-monoid in $\mathcal{M}_*$ is a $\boxtimes_{\mathcal{L}}$-monoid in $\mathcal{L}(1)$ which is also a $*$-module (and similarly for commutative $\boxtimes$-monoids). The functor $*\boxtimes_{\mathcal{L}}(-)$ gives us a means to functorially replace $\boxtimes_{\mathcal{L}}$-monoids and commutative $\boxtimes_{\mathcal{L}}$-monoids with $\boxtimes$-monoids and commutative $\boxtimes$-monoids.

Proposition 5.13. *Given a $\boxtimes_{\mathcal{L}}$-monoid X, the object $* \boxtimes_{\mathcal{L}} X$ is a $\boxtimes$-monoid and the map $\lambda\colon *\boxtimes_{\mathcal{L}}X \to X$ is a weak equivalence of $\boxtimes_{\mathcal{L}}$-monoids, and similarly in the commutative case.*

Furthermore, note that the standard formal arguments imply that $\boxtimes$ is the coproduct in the category of commutative monoids.

5.5. **Functors to spaces.** In this section, we discuss two functors which allow us to compare the categories $\mathcal{U}[\mathbb{L}]$ and $\mathcal{M}_*$ to spaces. There is a continuous forgetful functor $U\colon \mathcal{U}[\mathbb{L}] \to \mathcal{U}$ which preserves weak equivalences, and takes the unit $*$ to $*$. This functor restricts to a continuous forgetful functor $U\colon \mathcal{M}_* \to U$. In addition, we have another continuous functor from $\mathcal{L}(1)$-spaces to $\mathcal{U}$. There is a map of topological monoids $\theta\colon \mathcal{L}(1) \to *$. Associated to any map of monoids is an adjoint pair (θ_*, θ^*). The right adjoint $\theta^*\colon \mathcal{U} \to \mathcal{U}[\mathbb{L}]$ is the functor which assigns a trivial action to a space, and the left adjoint is described in the next definition.

Definition 5.14. The monoid map $\mathcal{L}(1) \to *$ induces a functor Q from $\mathcal{L}(1)$-spaces to spaces which takes an $\mathcal{L}(1)$-space X to $* \times_{\mathcal{L}(1)} X$; Q is the left adjoint to the functor which gives a space Y the trivial $\mathcal{L}(1)$-action. Q restricts to a functor from $\mathcal{M}_*$ to $\mathcal{U}$.

The interest of this second functor Q is that it is strong symmetric monoidal.

Lemma 5.15. *The functor $* \times_{\mathcal{L}(1)} X$ from $\mathcal{M}_*$ to spaces is strong symmetric monoidal with respect to the symmetric monoidal structures induced by $\boxtimes_{\mathcal{L}}$ and $\times$ respectively.*

Proof. Let X and Y be $\mathcal{L}(1)$-spaces. We need to compare $* \times_{\mathcal{L}(1)} (X \boxtimes_{\mathcal{L}} Y)$ and $(* \times_{\mathcal{L}(1)} X) \times (* \times_{\mathcal{L}(1)} Y)$. Observe that $\mathcal{L}(2)$ is homeomorphic to $\mathcal{L}(1)$ as a left $\mathcal{L}(1)$-space, by composing with an isomorphism $U^2 \to U$. Therefore we have isomorphisms

$$\begin{aligned} * \times_{\mathcal{L}(1)} (X \boxtimes_{\mathcal{L}} Y) &= * \times_{\mathcal{L}(1)} \mathcal{L}(2) \times_{\mathcal{L}(1)\times\mathcal{L}(1)} (X \times Y) \\ &\cong (* \times_{\mathcal{L}(1)} X) \times (* \times_{\mathcal{L}(1)} Y) \end{aligned}$$

One checks that the required coherence diagrams commute. The result now follows, as $* \times_{\mathcal{L}(1)} * \cong *$. □

The preceding result implies that $* \times_{\mathcal{L}(1)} (-)$ takes $\boxtimes_{\mathcal{L}}$-monoids to topological monoids. In Section 5.7, we will investigate conditions under which the natural map $UX \to QX$ is a weak equivalence.

5.6. **Model category structures.** In this section, we describe the homotopy theory of the categories described in the previous sections. We begin by establishing model category structures on the various categories and identifying the cofibrant objects. We rely on the following standard lifting result (e.g. [32, A.3]).

Theorem 5.16. *Let $\mathcal{C}$ be a cofibrantly generated model category where all objects are fibrant, with generating cofibrations I and acyclic generating cofibrations J. Assume that the domains of I and J are small relative to the classes of transfinite pushouts of maps in I and J respectively. Let $\mathbb{A}$ be a continuous monad on $\mathcal{C}$ which commutes with filtered direct limits and such that all $\mathbb{A}$-algebras have a path object. Then the category $\mathcal{C}[\mathbb{A}]$ has a cofibrantly generated model structure in which the weak equivalences and fibrations are created by the forgetful functor to $\mathcal{C}$. The generating cofibrations and acyclic cofibrations are the sets $\mathbb{A}I$ and $\mathbb{A}J$ respectively.*

Furthermore, when $\mathcal{C}$ is topological, provided that the monad $\mathbb{A}$ preserves reflexive coequalizers, the category $\mathcal{C}[\mathbb{A}]$ will also be topological [12, §VII.2.10]. All of the monads that arise in this paper preserve reflexive coequalizers [12, §II.7.2]. Moving on, we begin to record the model structures on the categories we study.

Theorem 5.17. *The category of $\mathcal{L}(1)$-spaces admits a cofibrantly generated topological model structure in which the fibrations and weak equivalences are detected by the forgetful functor to spaces. Limits and colimits are constructed in the underlying category of spaces.*

The functor $*\boxtimes_{\mathcal{L}}(-)$ from $\mathcal{L}(1)$-spaces to $*$-modules is not a monad. Nonetheless, we can use the same kind of argument to deduce the existence of a topological model structure on $*$-modules; we employ the technique used in the proof of [12, §VII.4.6]. The point is that counital $\mathcal{L}(1)$-spaces are algebras over a monad, and moreover there is an equivalence of categories between counital $\mathcal{L}(1)$-spaces and unital $\mathcal{L}(1)$-spaces; recall the discussion of the "mirror image" categories above. The following theorem then follows once again from Theorem 5.16.

Theorem 5.18. *The category $\mathcal{M}_*$ admits a cofibrantly generated topological model structure in which the weak equivalences are detected by the forgetful functor to $\mathcal{L}(1)$-spaces. A map $f\colon X \to Y$ of $*$-modules is a fibration if the induced map $F_{\boxtimes}(*,X) \to F_{\boxtimes}(*,Y)$ is a fibration of spaces. Colimits are created in the category of $\mathcal{L}(1)$-spaces, and limits are created by applying $*\boxtimes(-)$ to the limit in the category of $\mathcal{L}(1)$-spaces.*

Notice that although the fibrations have changed (since the functor to spaces which we're lifting over has changed), nonetheless this category still has the useful property that all objects are fibrant.

Lemma 5.19. *All objects in the category of $*$-modules are fibrant.*

Proof. This is an immediate consequence of the isomorphism $F^{\boxtimes}(*,*) \cong *$ and the fact that all spaces are fibrant. □

As a consequence, we obtain the following summary theorem about model structures on monoids and commutative monoids.

Theorem 5.20. *The categories $(\mathcal{U}[\mathbb{L}])[\mathbb{T}]$ and $(\mathcal{U}[\mathbb{L}])[\mathbb{P}]$ of $\boxtimes_{\mathcal{L}}$-monoids and commutative $\boxtimes_{\mathcal{L}}$-monoids in $\mathcal{L}(1)$-spaces admit cofibrantly generated topological model structures in which the weak equivalences and fibrations are maps which are weak equivalences and fibrations of $\mathcal{L}(1)$-spaces. Similarly, the categories $\mathcal{M}_*[\mathbb{T}]$ and $\mathcal{M}_*[\mathbb{P}]$ of $\boxtimes$-monoids and commutative $\boxtimes$-monoids in $\mathcal{M}_*$ admit cofibrantly generated topological model structures in which the weak equivalences and fibrations are the maps which are are weak equivalences and fibrations in $\mathcal{M}_*$. Limits are created in the underlying category and colimits are created as a certain coequalizer* [12, §II.7.4].

Next, in order to work with left derived functors associated to functors with domain one of these categories, we describe convenient characterizations of the cofibrant objects. Let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations $\{A_i \to B_i\}$ and let $\mathbb{A}$ be a continuous monad. A cellular object in the category $\mathcal{C}[\mathbb{A}]$ is a colimit $\operatorname{colim}_i X_i$, with $X_0 = *$ and X_{i+1} defined as the pushout

$$\begin{array}{ccc} \bigvee_\alpha \mathbb{A}A_i & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ \bigvee_\alpha \mathbb{A}B_i & \longrightarrow & X_{i+1}, \end{array}$$

where here α is some indexing subset of the indexing set of the generating cofibrations. The following result is then a formal consequence of the proof of Theorem 5.16.

Proposition 5.21. *In the model structures of Theorem 5.17, Theorem 5.18, and Theorem 5.20, the cofibrant objects are retracts of cellular objects.*

5.7. **Homotopical analysis of $\boxtimes_{\mathcal{L}}$.** In this section, we discuss the homotopical behavior of $\boxtimes_{\mathcal{L}}$. We will show that the left derived functor of $\boxtimes_{\mathcal{L}}$ is the cartesian product. The analysis begins with an essential proposition based on a useful property of $\mathcal{L}(2)$.

Proposition 5.22. *For spaces X and Y, there are isomorphisms of $\mathcal{L}(1)$-spaces*

$$(\mathcal{L}(1)\times X)\boxtimes_{\mathcal{L}}(\mathcal{L}(1)\times Y) \cong \mathcal{L}(2)\times(X\times Y) \cong \mathcal{L}(1)\times(X\times Y).$$

As a consequence, if M and N are cell $\mathcal{L}(1)$-spaces then so is $M \boxtimes_{\mathcal{L}} N$. If M and N are cell $*$-modules, then so is $M \boxtimes N$.

Proof. The first isomorphism is immediate from the definitions. The second is a consequence of the fact that $\mathcal{L}(2)$ is isomorphic to $\mathcal{L}(1)$ as an $\mathcal{L}(1)$-space via choice of a linear isometric isomorphism $f\colon U^2 \to U$; there is then a homeomorphism $\gamma\colon \mathcal{L}(1) \times \{f\} \to \mathcal{L}(2)$. The last two statement now follow from the analogous result for the cartesian product of cell spaces. □

With this in mind, we can prove the following result regarding the homotopical behavior of $\boxtimes$.

Theorem 5.23. *The category $\mathcal{M}_*$ is a monoidal model category with respect to the symmetric monoidal product $\boxtimes$ and unit $*$.*

Proof. We need to verify that $\mathcal{M}_*$ satisfies the pushout-product axiom [32, 2.1]. Thus, for cofibrations $A \to B$ and $X \to Y$, we must show that the map

$$(A \boxtimes Y) \coprod_{A \boxtimes X} (B \boxtimes X) \longrightarrow B \boxtimes Y$$

is a cofibration, and a weak equivalence if either $A \to B$ or $X \to Y$ was. Since $\mathcal{M}_*$ is cofibrantly generated, it suffices to check this on generating (acyclic) cofibrations. Therefore, we can reduce to considering a pushout-product of the form

$$P \longrightarrow (* \boxtimes (\mathcal{L}(1) \times B')) \boxtimes (* \boxtimes (\mathcal{L}(1) \times Y')),$$

where P is the pushout

$$\begin{array}{ccc}
* \boxtimes (\mathcal{L}(1) \times A') \boxtimes * \boxtimes (\mathcal{L}(1) \times X') & \longrightarrow & * \boxtimes (\mathcal{L}(1) \times B') \boxtimes * \boxtimes (\mathcal{L}(1) \times X') \\
\downarrow & & \downarrow \\
* \boxtimes (\mathcal{L}(1) \times A') \boxtimes * \boxtimes (\mathcal{L}(1) \times Y') & \longrightarrow & P,
\end{array}$$

for $A' \to B'$ and $X' \to Y'$ generating cofibrations in $\mathcal{U}$. Using the fact that $(* \boxtimes M) \boxtimes (* \boxtimes N) \cong * \boxtimes (M \boxtimes N)$ and the fact that $* \boxtimes (-)$ is a left adjoint, we can bring the $* \boxtimes (-)$ outside. Similarly, using Proposition 5.22 we can bring $\mathcal{L}(1) \times (-)$ outside and rewrite as

$$* \boxtimes (\mathcal{L}(1) \times \left(A' \times Y' \coprod_{A' \times X'} B' \times X' \longrightarrow B' \times Y' \right)).$$

Finally, the pushout-product axiom for $\mathcal{U}$ implies that it suffices to show that $* \boxtimes (\mathcal{L}(1) \times -)$ preserves cofibrations and weak equivalences. By construction of the model structures, it follows that $\mathcal{L}(1) \times (-)$ and $* \boxtimes (-)$ preserve cofibrations. Furthermore, $\mathcal{L}(1) \times (-)$ evidently preserves weak equivalences, and the analogous result for $* \boxtimes (-)$ follows from the fact that the unit map λ is a weak equivalence. □

The previous theorem implies that for a cofibrant $*$-module X, the functor $X \boxtimes (-)$ is a Quillen left adjoint. In particular, we can compute the derived $\boxtimes$ product by working with cofibrant objects in $\mathcal{M}_*$. Having established the existence of the derived $\boxtimes$ product, we now will compare it to the cartesian product of spaces.

The analogues in this setting of [12, §I.4.6] and [12, §II.1.9] yield the following helpful lemma.

Lemma 5.24. *If X is a cofibrant $\mathcal{L}(1)$-space, then X is homotopy equivalent as an $\mathcal{L}(1)$-space to a free $\mathcal{L}(1)$-space $\mathcal{L}(1) \times X'$, where X' is a cofibrant space. If Z is a cofibrant $*$-module, then Z is homotopy equivalent as a $*$-module to a free $*$-module $* \boxtimes Z'$, where Z' is a cofibrant $\mathcal{L}(1)$-space.*

We now use this to analyze the behavior of the forgetful functor on $\boxtimes$. Choosing a linear isometric isomorphism $f\colon U^2 \to U$, for $\mathcal{L}(1)$-spaces X and Y there is a natural map $\alpha\colon UX \times UY \to \mathcal{L}(2) \times (X \times Y) \to X \boxtimes_{\mathcal{L}} Y$.

Proposition 5.25. *Let X and Y be cofibrant $\mathcal{L}(1)$-spaces. Then the natural map $\alpha\colon UX \times UY \to U(X \boxtimes_{\mathcal{L}} Y)$ is a weak equivalence of spaces. Let X and Y be cofibrant $*$-modules. Then the natural map $\alpha\colon UX \times UY \to U(X \boxtimes_{\mathcal{L}} Y)$ is a weak equivalence of spaces.*

Proof. Lemma 5.24 allows us to reduce to the case of free $\mathcal{L}(1)$-spaces and free $*$-modules, and the result then follows from Proposition 5.22. □

There is also a map $U(X \boxtimes_{\mathcal{L}} Y) \to UX \times UY$ induced by the universal property of the product; it follows from Proposition 5.25 that this map is a weak equivalence as well under the hypotheses of the proposition.

The analogous result for commutative $\boxtimes$-monoids follows from a result of [2, 6.8]; they prove that for cofibrant $\mathcal{L}$-spaces X and Y, the natural map $X \vee Y \to X \times Y$ is a weak equivalence. To handle the case of $\boxtimes$-monoids, we exploit the following analysis of the underlying $\mathcal{L}(1)$-space of a cell $\boxtimes$-monoid, following [12, §VII.6.2]. The proof of this result is somewhat technical.

Proposition 5.26. *The underlying $*$-module associated to a cell associative monoid is cell.*

To prove this, we need to briefly recall some facts about simplicial objects in $\mathcal{L}(1)$-spaces and then prove Proposition 5.26, following the outline of [12, §VII.6]. Recall that for the categories we consider there are internal and external notions of geometric realization. We need the following compatibility result.

Lemma 5.27. *Let X be a simplicial object in any of the categories $\mathcal{U}[\mathbb{L}]$, $\mathcal{M}_*$, $(\mathcal{U}[\mathbb{L}])[\mathbb{T}])$, $\mathcal{M}_*[\mathbb{T}]$, $(\mathcal{U}[\mathbb{L}][\mathbb{P}])$, or $\mathcal{M}_*[\mathbb{P}]$. Then there is an isomorphism between the internal realization $|X.|$ and the external realization $|UX.|$.*

Proof. First, assume that X is a simplicial object in $\mathcal{U}[\mathbb{L}]$, $(\mathcal{U}[\mathbb{L}])[\mathbb{T}]$, or $(\mathcal{U}[\mathbb{L}])[\mathbb{P}]$. In all cases, the argument is the same; we discuss the argument for $\mathcal{U}[\mathbb{L}]$. The essential point is that for any simplicial space $Z.$, there is an isomorphism of spaces $\mathbb{L}|Z| \cong |\mathbb{L}Z|$. Now, since $* \boxtimes_{\mathcal{L}} |X.| \cong |* \boxtimes_{\mathcal{L}} X.|$, the realization of a $*$-module is a $*$-module. The remaining parts of the lemma follow. □

We now begin the proof of Proposition 5.26.

Proof. First, observe that we have the following analogue of [12, §VII.6.1], which holds by essentially the same proof: for A and B $\boxtimes$-monoids which are cell $*$-modules, the underlying $*$-module of the coproduct in the category of $\boxtimes$-monoids is a cell $*$-module.

Next, assume that X is a cell $*$-module. Then $X \boxtimes X \boxtimes \ldots \boxtimes X$ is a cell $*$-module. Since $X \to CX$ is cellular, the induced map $\mathbb{T}X \to \mathbb{T}CX$ is the inclusion of a subcomplex. Let Y_n be a cell $*$-module and consider the pushout of $\boxtimes$-monoids

$$\begin{array}{ccc} \mathbb{T}X & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \mathbb{T}CX & \longrightarrow & Y_{n+1}. \end{array}$$

By passage to colimits, it suffices to show that Y_{n+1} is a cell $*$-module. We rely on a description of the pushout of $\boxtimes$-monoids as the realization of a simplicial $*$-module. By the argument of [12, §XII.2.3], we know that $\mathbb{T}$ preserves Hurewicz cofibrations of $*$-modules. Moreover, $\mathbb{T}$ preserves tensors and pushouts and thus $\mathbb{T}CX \cong * \coprod_{\mathbb{T}X}(\mathbb{T}X \otimes I)$. By the argument of [12, §VII.3.8], we can describe this pushout as the double mapping cylinder $M(\mathbb{T}CX, \mathbb{T}X, *)$, and the argument of [12, §VII.3.7] establishes that this double mapping cylinder is isomorphic to the realization of the two-sided bar construction with k-simplices

$$[k] \mapsto \mathbb{T}CX \coprod \underbrace{\mathbb{T}X \coprod \mathbb{T}X \coprod \ldots \coprod \mathbb{T}X}_{k} \coprod *.$$

Since the k-simplices of this bar construction are cell $*$-modules and the face and degeneracy maps are cellular, the realization is itself a cell $*$-module. □

Finally, we study the functor Q. Since we wish to use Q to provide a functorial rectification of associative monoids, we need to determine conditions under which the natural map $UX \to QX$ is a weak equivalence. Note that we do not expect this map to be a weak equivalence for commutative monoids, as that would provide a functorial rectification of E_∞-spaces to commutative topological monoids.

Proposition 5.28. *Let X be a cell $\mathcal{L}(1)$-space. Then the map $UX \to QX$ is a weak equivalence of spaces. Let X be a cell $*$-module. Then the map $UX \to QX$ is a weak equivalence of spaces. Finally, let X be a $\boxtimes$ monoid in $\mathcal{M}_*$. Then the map $UX \to QX$ is a weak equivalence of spaces.*

Proof. First, assume that X is a cell $\mathcal{L}(1)$-space. Since colimits in $\mathcal{L}(1)$-spaces are created in $\mathcal{U}$, U commutes with colimits. Moreover, since Q is a left adjoint, it commutes with colimits and so by naturality we can reduce to considering the attachment of a single cell. Applying Q to the diagram

$$\begin{array}{ccc} \mathcal{L}(1) \times A & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \mathcal{L}(1) \times B & \longrightarrow & X_{n+1}, \end{array}$$

since $Q(\mathcal{L}(1) \times A) \cong A$ we obtain the pushout

$$\begin{array}{ccc} A & \longrightarrow & QX_n \\ \downarrow & & \downarrow \\ B & \longrightarrow & QX_{n+1} \end{array}$$

and so we can induct. Next, assume that X is a cell $*$-module. Once again, we can reduce to the consideration of the free cells. Since Q is strong symmetric monoidal, $Q(* \boxtimes_{\mathcal{L}} (\mathcal{L}(1) \times Z)) \cong * \times Q(\mathcal{L}(1) \times Z)$. Since the unit map λ is always a weak equivalence, once again we can induct. The case for $\boxtimes$-monoids now follows from Proposition 5.26. $\square$

Therefore, for cell monoids in $*$-modules Q provides a functorial rectification to topological monoids.

6. Implementing the axioms for $\mathcal{L}(1)$-spaces

In this section, we will show that the Lewis-May Thom spectrum functor T restricted to the category of $*$-modules over $* \boxtimes BG$ satisfies our axioms. We begin by reviewing the essential properties of the Lewis-May functor.

6.1. **Review of the properites of** $\mathrm{T}_{\mathscr{S}}$**.** As discussed in Section 2, the Lewis-May construction of the Thom spectrum yields a functor $\mathrm{T}_{\mathscr{S}}\colon \mathcal{U}/BG \to \mathscr{S}$. If we work in the based setting $\mathcal{T}/BG$, we obtain a functor to $\mathcal{S}\backslash S$, spectra under S, where for $f\colon X \to BG$ the unit $S \to \mathrm{T}(f)$ is induced by the inclusion $* \to X$ over BG. In this section, we review various properties of $\mathrm{T}_{\mathscr{S}}$ which we will need in verifying the axioms for the version of the Thom spectrum functor we will construct in the context of $\mathcal{L}(1)$-spaces.

Theorem 6.1.

(i) *Let $f\colon * \to BG$ be the basepoint inclusion. Then $\mathrm{T}(f) \cong S$.*
(ii) *The functor $\mathrm{T}\colon \mathcal{U}/BG \to \mathcal{S}$ preserves colimits* [17, 7.4.3].
(iii) *The functor $\mathrm{T}\colon \mathcal{T}/BG \to \mathcal{S}\backslash S$ preserves colimits.*
(iv) *Let $f\colon X \to BG$ be a map and A a space. Let g be the composite*

$$X \times A \xrightarrow{\pi} X \xrightarrow{f} BG ,$$

where π is the projection away from A. Then

$$\mathrm{T}(g) \cong A_+ \wedge \mathrm{T}(f)$$

[17, 7.4.6].
(v) *If $f : X \to BG$ and $g : X' \to BG$ are T-good maps such that there is a weak equivalence $h : X \simeq X'$ over BG, then there is a stable equivalence $Mf \simeq Mg$ given by the map of Thom spectra induced by h* [17, 7.4.9].
(vi) *If $f : X \to BG$ and $g : X \to BG$ are T-good maps which are homotopic, then there is a stable equivalence $Mf \simeq Mg$. However, the stable equivalence depends on the homotopy* [17, 7.4.10].

Notice that taking $A = I$, item (4) of the preceding theorem implies that functor T converts fiberwise homotopy equivalences into homotopy equivalences in $\mathcal{S}\backslash S$. Similarly, item (3) and (4) imply that T preserves Hurewicz cofibrations. The requirement that the maps $X \to BG$ be T-good that appears in the homotopy invariance results suggest that when dealing with spaces over BF, a better functor to consider might be the composite $\mathrm{T}\,\Gamma$. Unfortunately, the interaction of Γ with some of the constructions we are interested in (notably extended powers) is complicated; see Section 7 for further discussion.

Next, we review the multiplicative properties of this version of the Thom spectrum functor. Based spaces with actions by the linear isometries operad $\mathcal{L}$ can be regarded as algebras with respect to an associated monad $\mathbb{C}$ and spectra in $\mathcal{S}\backslash S$ which are E_∞-ring spectra structured by the linear isometries operad can be regarded as algebras with respect to the monad $\mathbb{C}$. In order to understand the interaction of the Thom spectrum functor with these monads, we need to describe the role of the twisted half-smash product. Given a map $\chi\colon X \to \mathcal{L}(U^j, U)$ and maps $f_i\colon Y_i \to BG$, $1 \le i \le j$, we define a map $\chi \times \Pi_i f_i$ as the composite

$$\Pi_i X_i \xrightarrow{\Pi_i f_i} \Pi_i BG \xrightarrow{\chi_*} BG$$

where here χ_* denotes the map induced on BG by χ. Given a subgroup $\pi \subset \Sigma_n$ such that X is a π-space and χ is a π-map, then $\chi \times \Pi_i f_i$ is a π-map (letting π act trivially on BG) and $\mathrm{T}(\chi \times \Pi_i f_i)$ is a spectrum with π-action.

The following theorem is [17, 7.6.1].

Theorem 6.2.

(i) *In the situation above,* $\mathrm{T}(\chi \times \Pi_i f_i) \cong X \ltimes (S\bar{m}a_i\, \mathrm{T}(f_i))$.

(ii) *Passing to orbits, there is an isomorphism*

$$\mathrm{T}(\chi \times_{\Sigma_n} \Pi_i f_i) \cong X \ltimes_{\Sigma_n} (\bar{\bigwedge_i} \mathrm{T}(f_i)).$$

Here $\bar{\wedge}$ *denotes the external smash product.*

The following corollary is an immediate consequence.

Corollary 6.3. *For* $f\colon X \to BG$ *a map of spaces, let* $\mathbb{L}f$ *denote the composite*

$$\mathcal{L}(1) \times X \longrightarrow \mathcal{L}(1) \times BG \longrightarrow BG,$$

where the last map is given by the $\mathcal{L}$*-space structure of* BG*. Then* $\mathrm{T}(\mathbb{L}f) \cong \mathbb{L}\,\mathrm{T}(f)$*.*

This result is the foundation of the essential technical result that describes the behavior of this Thom spectrum functor in the presence of operadic multiplications. Given a map $f\colon X \to BG$, there is an induced monad $\mathbb{C}_{BG}$ on $\mathcal{T}/BG$ specified by defining $\mathbb{C}_{BG} f$ as the composite $\mathbb{C}X \to \mathbb{C}BG \to BG$. Lewis proves the following [17, 7.7.1].

Theorem 6.4. *Given a map* $f : X \to BF$*, there is an isomorphism* $\mathbb{C}\,\mathrm{T}(f) \cong \mathrm{T}(\mathbb{C}_{BG} f)$*, and this isomorphism is coherently compatible with the unit and multiplication maps for these monads.*

This result has the following essential corollary; the first part of this is one of the central conclusions of Lewis' thesis, and the second part is a consequence explored at some length in the first author's thesis and forthcoming paper [3, 4].

Theorem 6.5.

(i) *The functor* T *restricts to a functor* $\mathrm{T}\colon (\mathcal{T}/BG)[\mathbb{C}_{BG}] \to (\mathcal{S}\backslash S)[\mathbb{C}]$.

(ii) *The functor* $\mathrm{T}\colon (\mathcal{T}/BG)[\mathbb{C}_{BG}] \to \mathcal{S}[\mathbb{C}]$ *preserves colimits and tensors.*

6.2. **Verification of the axioms.** We begin by studying the behavior of T in the context of $\mathcal{L}(1)$-spaces. The E_∞-space BG constructed as the colimit over the inclusions of the $\mathscr{I}_c$-space $V \mapsto BG(V)$ is a commutative monoid for $\boxtimes$. Therefore, the category $\mathcal{U}[\mathbb{L}]/BG$ has a symmetric monoidal product where given $f\colon X \to BG$ and $g\colon Y \to BG$, the product $f \boxtimes_{\mathcal{L}} g$ is defined as $f \boxtimes_{\mathcal{L}} g\colon X \boxtimes_{\mathcal{L}} Y \to BG \boxtimes_{\mathcal{L}} BG \to BG$. The unit is given by the trivial map $* \to BG$. We define $\mathrm{T}_{\mathcal{L}}\colon \mathcal{U}[\mathbb{L}]/BG \to \mathcal{S}[\mathbb{L}]$ as the Lewis-May Thom spectrum functor T restricted to $\mathcal{U}[\mathbb{L}]/BG$: Corollary 6.3 implies that T takes values in $\mathbb{L}$-spectra on $\mathcal{U}[\mathbb{L}]/BG$. Moreover, $\mathrm{T}_{\mathcal{L}}$ is strong symmetric monoidal (up to unit).

Proposition 6.6. *Given $f\colon X \to BG$ and $g\colon Y \to BG$, there is a coherently associative isomorphism $\mathrm{T}_{\mathcal{L}}(f \boxtimes g) \cong \mathrm{T}_{\mathcal{L}}(f) \wedge_{\mathcal{L}} \mathrm{T}_{\mathcal{L}}(g)$.*

Proof. We can describe $f \boxtimes g\colon X \boxtimes Y \to BG$ as the natural map to BG associated to the coequalizer describing $X \boxtimes Y$. Theorem 6.1 implies that $\mathrm{T}_{\mathcal{L}}$ commutes with this coequalizer and Theorem 6.2 implies that

$$\mathrm{T}_{\mathcal{L}}(\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1) \times (X \times Y)) \cong (\mathcal{L}(2) \times \mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (X \bar{\wedge} Y)$$

and

$$\mathrm{T}_{\mathcal{L}}(\mathcal{L}(2) \times (X \times Y)) \cong \mathcal{L}(2) \ltimes (X \bar{\wedge} Y).$$

Inspection of the maps then verifies that the resulting coequalier is precisely the the coequalizer defining $\mathrm{T}_{\mathcal{L}}(f) \wedge_{\mathcal{L}} \mathrm{T}_{\mathcal{L}}(g)$. □

Now we restrict to the subcategory $\mathcal{M}_*$. For a $\boxtimes$-monoid in $\mathcal{M}_*$ with multiplication μ, unitality implies that there is a commutative diagram

$$\begin{array}{ccc} X \boxtimes * & \longrightarrow & X \boxtimes X \\ & \underset{\lambda}{\searrow} & \downarrow \mu \\ & & X. \end{array}$$

In conjunction with Proposition 6.6, this diagram implies that given a map $f\colon X \to BG$, there is an isomorphism

$$\mathrm{T}_{\mathcal{L}}(* \boxtimes_{\mathcal{L}} X \longrightarrow * \boxtimes_{\mathcal{L}} BG \longrightarrow BG) \cong S \wedge_{\mathcal{L}} \mathrm{T}_{\mathcal{L}}(f),$$

which implies that the following definition is sensible.

Definition 6.7. Define $BG_{\mathcal{A}}$ to be $* \boxtimes_{\mathcal{L}} BG$; the category $\mathcal{A}$ is then the overcategory $\mathcal{M}_*/BG_{\mathcal{A}}$. This category is symmetric monoidal: Given $f_1\colon X_1 \to BG_{\mathcal{A}}$ and $f_2\colon X_2 \to BG_{\mathcal{A}}$, the product is the composite $f_1 \boxtimes f_2\colon X_1 \boxtimes X_2 \to BG_{\mathcal{A}} \boxtimes BG_{\mathcal{A}} \to BG_{\mathcal{A}}$. We define a Thom spectrum functor $\mathrm{T}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{M}_S$ given $f\colon X \to BG_{\mathcal{A}}$ by applying $\mathrm{T}_{\mathcal{L}}$ to the composite

$$X \xrightarrow{f} BG_{\mathcal{A}} \xrightarrow{\lambda} BG.$$

It follows immediately from Proposition 6.6 and the observation preceding the definition that $\mathrm{T}_{\mathcal{A}}$ is a strong symmetric monoidal functor from $\mathcal{M}_*/BG_{\mathcal{A}}$ to $\mathcal{M}_S$. Since Theorem 6.5 implies that $\mathrm{T}_{\mathcal{A}}$ commutes with colimits and tensors, $\mathrm{T}_{\mathcal{A}}$ commutes with geometric realization and therefore the pair $\mathcal{A}$, $\mathrm{T}_{\mathcal{A}}$ satisfies axioms **A1** and **A2**.

Next, we use Theorem 5.20 to choose a cofibrant replacement functor c on the category $\mathcal{M}_*[\mathbb{T}]$ such that for any object X in $\mathcal{A}$, cX is a cell monoid [13]. Note that cell monoid are well-based. Then given an object $f\colon X \to BG_{\mathcal{A}}$, we define Cf to be the composite

$$cX \xrightarrow[\simeq]{} X \longrightarrow BG_{\mathcal{A}}\ .$$

We will work the notion of flatness encapsulated in the definition of the class of S-modules $\bar{\mathcal{F}}_S$ [1, 9.6] (see also [12, VII.6.4]). Let $\mathcal{F}_S$ denote the collection of modules of the form $S \wedge_{\mathcal{L}} \mathcal{L}(j) \ltimes_G K$ where K is a G-spectrum (for $G \subset \Sigma_i$) which has the homotopy type of a G-CW spectrum. Then $\bar{\mathcal{F}}_S$ is the closure of $\mathcal{F}_S$ under finite $\wedge$, wedges, pushouts along cofibrations, colimits of countable sequences of cofibrations, homotopy equivalences, and "stabilization" (in which if ΣM is in $\bar{\mathcal{F}}_S$ then so is M). The point of this definition is that for S-modules M and N in the class $\bar{\mathcal{F}}_S$, the point-set smash product $M \wedge N$ represents the derived smash product [12, VII.6.7], [1][9.5].

Lemma 6.8. *Let $f\colon X \to BG_{\mathcal{A}}$ be an object of $\mathcal{M}_*[\mathbb{T}]/BG_{\mathcal{A}}$ such that X is a cell monoid. Then the underlying S-module of the S-algebra $\mathrm{T}(f)$ is in the class $\bar{\mathcal{F}}_S$.*

Proof. We proceed by induction. Since $\mathrm{T}_{\mathcal{A}}$ commutes with colimits, we can reduce to consideration of a pushout square of the form

$$\begin{array}{ccc} \mathrm{T}_{\mathcal{A}}(\mathbb{T}(* \boxtimes (\mathcal{L}(1) \times A))) & \longrightarrow & \mathrm{T}_{\mathcal{A}}(X_n) \\ \downarrow & & \downarrow \\ \mathrm{T}_{\mathcal{A}}(\mathbb{T}(* \boxtimes (\mathcal{L}(1) \times B))) & \longrightarrow & \mathrm{T}_{\mathcal{A}}(X_{n+1}) \end{array}$$

where X_n can be assumed to be in the class $\bar{\mathcal{F}}_S$ and A and B are CW-complexes. Here we are suppressing the maps from A, B, X_n, and X_{n+1} in our notation. Next, since both $\mathrm{T}_{\mathcal{A}}$ and $\mathbb{T}$ preserves cofibrations, it suffices to show that for any CW-complex Z, $\mathrm{T}_{\mathcal{A}}(\mathbb{T}(* \boxtimes (\mathcal{L}(1) \times Z)))$ is in the class $\bar{\mathcal{F}}_S$. Since $\mathrm{T}(Z)$ has the homotopy type of a CW-complex (see the proof of [17, 7.5.6]), the result follows from Proposition 5.3 and Theorem 6.2. □

For **A4**, we use the functor Q of Definition 5.14; we have already verified that it has the desired properties in Proposition 5.28. For **A5**, we choose a cofibrant replacement functor on the category of commutative monoids provided by the model structure of Theorem 5.20 and use this to define $BG'_{\mathcal{A}}$.

Finally, rectification is very straightforward in this context; since Proposition 5.9 tells us that an A_∞-map $X \to BG_{\mathcal{A}}$ over the linear isometries operad specifies the data of a monoid map in $(\mathcal{U}[\mathbb{L}]/BG_{\mathcal{A}})[\mathbb{T}]$, by applying $* \boxtimes (-)$ we obtain a monoid map in $*$-modules. To complete the verification of **A6**, we use Proposition 5.28. Given a topological monoid M, we can regard this as an A_∞-space over the non-Σ linear isometries operad by pulling back along the augmentation to the associative operad. Equivalently, M regarded as an $\mathcal{L}(1)$-space with trivial action is a $\boxtimes_{\mathcal{L}}$-monoid with multiplication induced from the monoid multiplication $M \times M \to M$ and the fact that $M \boxtimes_{\mathcal{L}} M \cong M \times M$ by the argument of Lemma 5.15.

Let X be an A_∞-space for the linear isometries operad. Since Q is left adjoint to the functor which assigns the trivial $\mathcal{L}(1)$-action, the unit of the adjunction induces a map of $\mathcal{L}(1)$-spaces $X \boxtimes_{\mathcal{L}} X \to Q(X \boxtimes_{\mathcal{L}} X) \cong QX \times QX$. To show that axiom **A6** holds, it suffices to show that this is a map of $\boxtimes_{\mathcal{L}}$-monoids; this map constructs the homotopy commutative diagram of the axiom. But since $QX \times QX \cong QX \boxtimes_{\mathcal{L}} QX$ this map is a map of $\boxtimes_{\mathcal{L}}$-monoids by the definition of the multiplication on QX.

Finally, we discuss the proof of Theorem 1.4. Given a map of E_∞-spaces $X \to BG$, Proposition 5.9 yields a map of commutative $\boxtimes$-monoids $* \boxtimes X \to * \boxtimes BG$, and the Thom spectrum functor of Definition 6.7 clearly restricts to a functor $\mathcal{M}_*[\mathbb{P}]/BG_{\mathcal{A}} \to \mathcal{M}_S[\mathbb{P}]$. Using the cofibrant replacement functor in the category $\mathcal{M}_*[\mathbb{P}]$, the argument of Lemma 6.8 applies to show that the underlying S-module of the resulting Thom spectrum is in the class $\bar{\mathcal{F}}_S$.

With this setup, the proof of Theorem 1.3 is straightforward; there is a sequence of commutative $\boxtimes$-monoids

$$X \longrightarrow B^{\mathrm{cy}}_{\boxtimes} X \longrightarrow B_{\boxtimes} X,$$

and the first map is split by the levelwise multiplication map $B^{\mathrm{cy}}_{\boxtimes} X \to X$. Therefore, given a map $f \colon X \to BG_{\mathcal{A}}$, we obtain a commutative diagram

$$\begin{array}{ccc} B^{\mathrm{cy}}_{\boxtimes} X & \longrightarrow & X \boxtimes B_{\boxtimes} X \\ & \searrow & \downarrow{\scriptstyle f\circ\pi} \\ & & BG_{\mathcal{A}}, \end{array}$$

where π is induced by the projection away from $B_{\boxtimes} X$. Since the top vertical map is a weak equivalence when X is cofibrant [31, 4], passing to Thom spectra yields the desired result.

7. Modifications when working over BF

In this section, we discuss the situation when working over monoids (such as BF), which involves some complication with the homotopical aspects of the axiomatic framework. We focus on the associative situation, working with the category $\mathcal{M}_*[\mathbb{T}]$; the commutative case is substantially more intricate, and so we refer the reader either to the Bokstedt-style solution discussed in this paper for $\mathscr{I}$-spaces or the arguments of [4] when working with commutative objects.

7.1. **A review of the properties of Γ.** When working with BF, there are technical complications which arise from the fact that the projection

$$\pi_n : B(*, F(n), S^n) \to B(*, F(n), *)$$

is a universal quasifibration, with section a Hurewicz cofibration. Quasifibrations are not preserved under pullback, and in general the pullback of the section will not be a Hurewicz cofibration. If the section of π_n could be shown to be a fiberwise cofibration, pullback along any map would provide a section which was a fiberwise cofibration. However, it is not known whether the section is in fact a fiberwise cofibration or not (the proof that it is a Hurewicz cofibration is indirect insofar as it exploits the fact that the spaces in question are LEC and retractions between LEC spaces are cofibrations).

The standard solution to these issues (pioneered by Lewis) is to use an explicit functor Γ which replaces a map by a Hurewicz fibration. Since various properties of Γ play an essential role in our work, we will review relevant details here.

Definition 7.1. Given a map $f : X \to B$, define

$$\Pi B = \{(\theta, r) \in B^{[0,\infty]} \times [0, \infty] \,|\, \theta(t) = \theta(r), t > r\}.$$

The end-point projection $\nu : \Pi B \to B$ is defined as $(\theta, r) \mapsto \theta(r)$. There is also the evaluation map $e_0 : \Pi B \to B$ defined as $(\theta, r) \mapsto \theta(0)$. Define ΓX to be the pullback

$$\begin{array}{ccc} \Gamma X & \longrightarrow & \Pi B \\ \downarrow & & \downarrow{\scriptstyle e_0} \\ X & \xrightarrow{f} & B. \end{array}$$

Let Γf denote the the induced map

$$\Gamma X \longrightarrow \Pi B \xrightarrow{\nu} BF.$$

There is a map $\delta : X \to \Gamma X$ specified by taking $x \in X$ to the pair (x, ζ_x) where ζ_x is the path of length zero at x.

The map Γf is a Hurewicz fibration.

Lemma 7.2. *The maps δ specify a natural transformation* $\mathrm{id} \to \Gamma$*, and $\delta : X \to \Gamma X$ is a homotopy equivalence (although not a homotopy equivalence over B).*

Moreover, Γ has very useful properties in terms of interaction with the naive model structures on $\mathcal{U}/B$ and $\mathcal{T}/B$.

Proposition 7.3. *The functor Γ on spaces over B takes cofibrations to fiberwise cofibrations and homotopy equivalences over B to fiberwise homotopy equivalences. As a functor on ex-spaces, it takes ex-spaces with sections which are cofibrations to ex-spaces with sections which are fiberwise cofibrations.*

There is a useful related lemma.

Lemma 7.4. *If $X \to X'$ is a weak equivalence over BF, $\Gamma X \to \Gamma X'$ is a weak equivalence.*

There are two possible ways we might use Γ to resolve the problems with BF; we could replace π_n with $\Gamma\pi_n$, which will be a Hurewicz fibration and will have section a fiberwise cofibration, or we could replace a given map $f : X \to BF$ with a Hurewicz fibration via Γ. The latter approach will yield a homotopically well-behaved Thom spectrum construction, since the pullback of a quasifibration along a Hurewicz fibration is a quasifibration and the pullback of a section which is a cofibration will be a cofibration. Moreover, Lewis shows that the two approaches yield stably equivalent Thom spectra. Since the first approach is much more felicitous for the study of multiplicative structures, we will employ it exclusively.

Γ behaves well with respect to colimits and unbased tensors.

Proposition 7.5. *As a functor on $\mathcal{U}/BF$ and $\mathcal{T}/BF$, Γ commutes with colimits.*

There is a related result for tensors with unbased spaces (although note however that this is false for based tensors).

Proposition 7.6. *As a functor on $\mathcal{U}/BF$ and $\mathcal{T}/BF$, Γ commutes with the tensor with an unbased space A.*

Finally, we recall some salient facts about the interaction of Γ with operadic multiplications. May ([25, 1.8]) shows that Γ lifts to a functor on $\mathcal{T}[\mathbb{C}]$, for an operad $\mathcal{C}$ augmented over the linear isometries operad, and Lewis observes that in fact Γ extends to a functor on $\mathcal{T}[\mathbb{C}]/BF$. We are particularly interested in this in the cases of $\mathbb{T}$ and $\mathbb{P}$, of course. An essential aspect of these results is that all of the various maps associated with Γ (notably δ) are maps of $\mathbb{C}$-algebras, and so in particular the map δ yields a homotopy equivalence in the category $\mathcal{T}[\mathbb{C}]$.

7.2. **Γ and Cofibrant replacement.** In order to verify axiom **A3** in this setting, we must amend the cofibrant replacement process. Given a map $f : X \to BF$, regarded as a map in $(\mathcal{U}[\mathbb{L}])[\mathbb{T}]$, we consider the map $\Gamma(c\Gamma f) : \Gamma(c\Gamma X) \to BF$ obtained by applying Γ, then the cofibrant replacement functor in $(\mathcal{U}[\mathbb{L}])[\mathbb{T}]$, then

Γ again. We have the following commutative diagram:

$$\begin{array}{ccccc} c\Gamma X & \longrightarrow & \Gamma X & \longrightarrow & BF \\ \downarrow{\scriptstyle \delta} & & \simeq \downarrow{\scriptstyle \delta} & & \downarrow \\ \Gamma c\Gamma X & \xrightarrow{\simeq} & \Gamma\Gamma X & \longrightarrow & BF. \end{array}$$

Since the labelled weak equivalences in the preceding diagram connect objects of $(\mathcal{U}[\mathbb{L}])[\mathbb{T}]/BF$ which are T-good, there is a zig-zag of stable equivalences connecting $\mathrm{T}_{\mathcal{L}}(\Gamma f)$ to $\mathrm{T}_{\mathcal{L}}(\Gamma(c(\Gamma f)))$. If f itself was T-good, then there is a further stable equivalence to $\mathrm{T}_{\mathcal{L}}\, f$. Therefore this process does not change the homotopy type of the Thom spectrum. Finally, we apply $* \boxtimes (-)$ to ensure that we land in $\mathcal{M}_*[\mathbb{T}]/BG_{\mathcal{A}}$. Denote this composite functor by γ.

Next, we must verify that this process produces something which allows us to compute derived functors with respect to $\boxtimes$ and $\wedge$, in $\mathcal{M}_*[\mathbb{T}]$ and $\mathcal{M}_S[\mathbb{T}]$ respectively. Although $*\boxtimes\Gamma(c\Gamma X)$ is not a cofibrant object in $\mathcal{M}_*[\mathbb{T}]$, it has the homotopy type of a cofibrant object and this suffices to ensure that it can be used to compute the derived $\boxtimes$ product. This observation also implies that the functor Q satisfies axiom **A4** in this setting as well.

Moving on, we now need to show that $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(c\Gamma f))$ can be used to compute the derived smash product in $\mathcal{M}_S$.

Lemma 7.7. *Let $f : X \to BG_{\mathcal{A}}$ be a map in $\mathcal{M}_*[\mathbb{T}]$. Let U denote the forgetful functors $\mathcal{M}_*[\mathbb{T}] \to \mathcal{M}_*$ and $\mathcal{M}_S[\mathbb{T}] \to \mathcal{M}_S$ respectively. Then there is an isomorphism $\mathrm{T}_{\mathcal{A}}(Uf) \cong U\,\mathrm{T}_{\mathcal{A}}(f)$.*

By Lemma 7.7, it will suffice to show that the underlying S-module of $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(c\Gamma f))$ is in the class $\bar{\mathcal{F}}_S$. Since a slight modification of the argument of Proposition 5.26 shows that the underlying $\mathcal{L}(1)$-space of a cell $\boxtimes_{\mathcal{L}}$-monoid is a cell $\mathcal{L}(1)$-space, in fact Lemma 7.7 implies that it will suffice to show that given

$$f : X \to BF_{\mathcal{A}}$$

such that X is a cell $\mathcal{L}(1)$-space, $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma f)$ is in the class $\bar{\mathcal{F}}_S$. We will make an inductive argument. Using the fact that Γ commutes with colimits in $\mathcal{L}(1)$-spaces and preserves cofibrations, it suffices to show that the Thom spectra of $* \boxtimes \Gamma\mathbb{T}(CE_n)$ and $* \boxtimes \Gamma\mathbb{T}(E_n)$ are in the class $\bar{\mathcal{F}}_S$, where E_n is a wedge of cells D^n. We can further reduce to the case where we are considering a single cell. Abusing notation by suppressing the maps to BG, we will refer to the relevant Thom spectra as $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(\mathcal{L}(1) \times S^n))$ and $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(\mathcal{L}(1) \times D^n))$. Finally, since S^n can be constructed as the pushout $D^n \cup_{S^{n-1}} D^n$, it suffices to consider $\mathrm{T}_{\mathcal{A}}(\Gamma(* \boxtimes \mathcal{L}(1) \times *))$ and $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(\mathcal{L}(1) \times D^n))$. Recall that by Proposition 6.6, these spectra are isomorphic as S-modules to $S \wedge_{\mathcal{L}} \mathrm{T}_{\mathcal{L}}(\Gamma(\mathcal{L}(1) \times *))$ and $S \wedge_{\mathcal{L}} \mathrm{T}_{\mathcal{L}}(\Gamma(\mathcal{L}(1) \times D^n))$ respectively.

Lemma 7.8. *The S-modules $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(\mathcal{L}(1) \times *))$ and $\mathrm{T}_{\mathcal{A}}(* \boxtimes \Gamma(\mathcal{L}(1) \times *))$ are in the class $\bar{\mathcal{F}}_S$.*

Proof. Given a map $D^n \to BF$, by choosing a point in BF in the image we obtain a map $* \to D^n$ over BF. This induces a map $\mathcal{L}(1) \times * \to \mathcal{L}(1) \times D^n$ which is a weak equivalence of $\mathcal{L}(1)$-spaces over BF. Since these are cofibrant $\mathcal{L}(1)$-spaces, this is a homotopy equivalence over BF. Applying Γ turns this into a fiberwise homotopy equivalence. Since T takes fiberwise homotopy equivalences to homotopy equivalences of spectra, the resulting spectra are homotopy equivalent.

Therefore, we are reduced to considering the Thom spectra associated to $\Gamma(\mathcal{L}(1)\times *)$ associated to the various choices of a target for point. But since BF is path-connected, an argument analogous to the one in the preceding paragraph allows us to show that all of these spectra are homotopy equivalent. Thus, it suffices to consider the trivial map $* \to BF$. But then $\Gamma(\mathcal{L}(1) \times *))$ is homeomorphic as a space over BF to $\pi_2 : \mathcal{L}(1) \times \Gamma(*)$, where π_2 is the projection away from $\mathcal{L}(1)$. Applying T yields the Thom spectrum $\mathcal{L}(1)_+ \wedge \mathrm{T}(\Gamma(*))$. Finally, $S \wedge_{\mathcal{L}} (\mathcal{L}(1)_+ \wedge \mathrm{T}(\Gamma(*))$ is in the class $\bar{\mathcal{F}}_S$; this follows from the proof of part (ii) of [17, 7.3.7], in which the homotopy type of the Thom spectrum $\mathrm{T}(\Gamma(*))$ is explicitly described. □

In the previous proof, we are implicitly exploiting the "untwisting" proposition I.2.1 from EKMM which provides an isomorphism of spectra $A \ltimes \Sigma^\infty X \cong \Sigma^\infty(A_+ \wedge X)$.

Finally, we need to show that we can understand the homotopy type of the Thom spectrum associated to $\Gamma f \boxtimes \Gamma f$.

Proposition 7.9. *Let $f\colon X \to BF$ be a map of $\mathcal{L}(1)$-spaces, and assume that X is a cell $\mathcal{L}(1)$-space. Then there is a weak equivalence between $\mathrm{T}_{\mathcal{L}}\Gamma(f \boxtimes_{\mathcal{L}} f)$ and $\mathrm{T}_{\mathcal{L}}(\Gamma f \boxtimes_{\mathcal{L}} \Gamma f)$.*

Proof. Recall from Proposition 5.22 that given a choice of a linear isometric isomorphism $g\colon U^2 \to U$, there is a chain of weak equivalences

$$X \times X \to \mathcal{L}(2) \times (X \times X) \to X \boxtimes_{\mathcal{L}} X.$$

Moreover, these equivalences are given by maps over BG.

Lewis shows that there is a map $\Gamma f \times \Gamma f \to \Gamma(f \times f)$ given by multiplication of paths which is a weak equivalence [17, 7.5.5]. In addition, he shows that $\Gamma f \times \Gamma f$ is a good map. This is the heart of our comparison. Applying Γ to the chain of equivalences above, we have a composite

$$\Gamma(f \times f) \to \Gamma(\mathcal{L}(2) \times (f \times f)) \to \Gamma(f \boxtimes_{\mathcal{L}} f)$$

which induces weak equivalences of Thom spectra $\mathrm{T}(\Gamma(f \times f)) \to \mathrm{T}(\Gamma(f \boxtimes_{\mathcal{L}} f))$. On the other hand, there is also the composite

$$\Gamma f \times \Gamma f \to \mathcal{L}(2) \times (\Gamma f \times \Gamma f) \to \Gamma f \boxtimes_{\mathcal{L}} \Gamma f.$$

Although these are not all good maps, the induced maps of Thom spectra are

$$g_*(\mathrm{T}(\Gamma f) \bar{\wedge}\, \mathrm{T}(\Gamma f) \to \mathcal{L}(2) \ltimes (\mathrm{T}(\Gamma f) \bar{\wedge}\, \mathrm{T}(\Gamma f) \to \mathrm{T}(\Gamma f) \boxtimes_{\mathcal{L}} \mathrm{T}(\Gamma f),$$

and these are weak equivalences. Since $\Gamma f \times \Gamma f \to \Gamma(f \times f)$ induces a weak equivalence of Thom spectra, the result follows. □

This has the the following consequence.

Corollary 7.10. *Let $f : X \to BG_{\mathcal{A}}$ be a map of $\boxtimes$-monoids. Then there is a weak equivalence of spectra $\mathrm{T}_{\mathcal{A}}(N^{cy}_{\boxtimes}(\gamma f)$ and $\mathrm{T}_{\mathcal{A}}(\gamma N^{cy}_{\boxtimes} f)$.*

8. Preliminaries on symmetric spectra

Let Sp^{Σ} be the category of topological symmetric spectra as defined in [23]. Thus, a symmetric spectrum T is a spectrum in which the spaces $T(n)$ come equipped with base point preserving Σ_n-actions such that the iterated structure maps $S^m \wedge T(n) \to T(m+n)$ are Σ_{m+n}-equivariant.

8.1. **The detection functor.** We recall that as defined in [19] and [23], the stable model equivalences (that is, the weak equivalences in the stable model structure) of symmetric spectra need not be stable equivalences of the underlying prespectra. In order to characterize the stable model equivalences in terms of ordinary stable equivalences, Shipley [34] has defined an explicit "detection" functor $D\colon Sp^{\Sigma} \to Sp^{\Sigma}$. This functor takes a symmetric spectrum T to the symmetric spectrum DT with nth space

$$DT(n) = \operatorname{hocolim}_{\mathbf{m}\in\mathcal{I}} \Omega^m(T(m) \wedge S^n).$$

Thus, DT is the 0'th space in Bökstedt's model of the topological Hochschild complex, see [20]. In order to avoid trouble with degenerate base points we implicitly replace the spaces in the definition of DT by spaces which are well-based, for example the realization of their singular simplicial complexes. It then follows as in [34], Theorem 3.1.2, that a map of symmetric spectra is a stable model equivalence if and only if applying D gives an ordinary stable equivalence of the underlying spectra. Furthermore, by [34], Theorem 3.1.6, the functor D is related to the identity functor on Sp^{Σ} by a chain of natural stable model equivalences of symmetric spectra.

8.2. **The flatness condition for symmetric spectra.** It is proved in [34], Section 4, that if T is a cofibrant symmetric ring spectrum, then the cyclic bar construction $B^{\mathrm{cy}}(T)$ represents the topological Hochschild homology of T. However, in the study of Thom spectra we find it useful to introduce the notion of a *flat symmetric spectrum*, which is a more general type of symmetric spectrum for which the smash product is homotopically well-behaved. We first consider flat symmetric spectra in general and then define what we mean by a flat symmetric ring spectrum. For this we need to recall some convenient notation from [30]. In the following T denotes a symmetric spectrum and $\mathcal{I}$ is the category of finite sets and injective maps defined in Section 1. Given a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$, we write $\mathbf{n} - \alpha$ for the set $\mathbf{n} - \alpha(\mathbf{m})$ and let $S^{n-\alpha}$ denote the one-point compactification of $\mathbb{R}^{\mathbf{n}-\alpha}$. We define $S^{n-\alpha} \wedge T(m) \to T(n)$ to be the composite map

$$S^{n-\alpha} \wedge T(m) \longrightarrow S^{n-m} \wedge T(m) \longrightarrow T(n) \xrightarrow{\bar{\alpha}} T(n),$$

where the first map is the homeomorphism induced by the ordering of $\mathbf{n}-\alpha$ inherited from $\mathbf{n}$, the second map is the structure map of the symmetric spectrum, and $\bar{\alpha}$ is the extension of α to a permutation which is order preserving on the complement of $\mathbf{m}$. The advantage of this notation is that it will make some of our constructions self explanatory. Consider for each object $\mathbf{n}$ the $\mathcal{I}/\mathbf{n}$-diagram of based spaces that to an object $\alpha\colon \mathbf{m} \to \mathbf{n}$ associates $S^{n-\alpha} \wedge T(m)$. If $\beta\colon (\mathbf{m}, \alpha) \to (\mathbf{m}', \alpha')$ is a morphism in $\mathcal{I}/\mathbf{n}$, then $\alpha = \alpha' \circ \beta$ by definition, and the induced map is defined by

$$S^{n-\alpha} \wedge T(m) \xrightarrow{\sim} S^{n-\alpha'} \wedge S^{m'-\beta} \wedge T(m) \longrightarrow S^{n-\alpha'} \wedge T(m').$$

Applying this functor to a commutative diagram in $\mathcal{I}$ of the form

$$\begin{array}{ccc} \mathbf{m} & \xrightarrow{\alpha_1} & \mathbf{n}_1 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} \\ \mathbf{n}_2 & \xrightarrow{\beta_2} & \mathbf{n}, \end{array} \tag{7}$$

we get a commutative diagram of based spaces

$$\begin{array}{ccc} S^{n-\gamma} \wedge T(m) & \longrightarrow & S^{n-\beta_1} \wedge T(n_1) \\ \downarrow & & \downarrow \\ S^{n-\beta_2} \wedge T(n_2) & \longrightarrow & T(n), \end{array} \tag{8}$$

where γ denotes the composite $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$. We say that T is *flat* if each of the spaces $T(n)$ is well-based, and if for each diagram (7), such that the intersection of the images of β_1 and β_2 equals the image of γ, the induced map

$$S^{n-\beta_1} \wedge T(n_1) \cup_{S^{n-\gamma} \wedge T(m)} S^{n-\beta_2} \wedge T(n_2) \longrightarrow T(n)$$

is a cofibration. Here we use the term cofibration in its traditional sense, that is, for a map having the homotopy extension property. By Lillig's union theorem for cofibrations [18], a level-wise well-based symmetric spectrum T is flat if and only if (i) any morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces a cofibration $S^{n-\alpha} \wedge T(m) \to T(n)$, and (ii) for any diagram of the form (7), satisfying the above condition, the intersection of the images of $S^{n-\beta_1} \wedge T(n_1)$ and $S^{n-\beta_2} \wedge T(n_2)$ equals the image of $S^{n-\gamma} \wedge T(m)$.

Example. The symmetric Thom spectra MO, MSO, MF and MSF are flat.

We shall now prove that the smash product of flat symmetric spectra is homotopically well-behaved. Recall that the smash product of a family of symmetric spectra $T_1, \dots, T_k$ is the symmetric spectrum with nth space

$$T_1 \wedge \cdots \wedge T_k(n) = \operatorname{colim}_{\alpha\colon \mathbf{n}_1 \sqcup \cdots \sqcup \mathbf{n}_k \to \mathbf{n}} S^{n-\alpha} \wedge T(n_1) \wedge \cdots \wedge T(n_k).$$

Here the colimit is over the comma category $\sqcup^k/\mathbf{n}$, where $\sqcup^k\colon \mathcal{I}^k \to \mathcal{I}$ denotes the iterated monoidal product. We introduce a homotopy invariant version by the analogous based homotopy colimit construction,

$$T_i \wedge^h \cdots \wedge^h T_k(n) = \operatorname{hocolim}_{\alpha\colon \mathbf{n}_1 \sqcup \cdots \sqcup \mathbf{n}_k \to \mathbf{n}} S^{n-\alpha} \wedge T(n_1) \wedge \cdots \wedge T(n_k).$$

As we shall see in Proposition 8.2, the latter construction always represents the "correct" homotopy type of the smash product for symmetric spectra that are level-wise well-based.

Proposition 8.1. *If the symmetric spectra $T_1, \dots, T_k$ are flat, then the canonical projection*

$$T_1 \wedge^h \cdots \wedge^h T_k \longrightarrow T_1 \wedge \cdots \wedge T_k$$

is a level-wise equivalence.

Proof. For notational reasons we only carry out the proof for a pair of flat symmetric spectra T_1 and T_2. The proof in the general case is completely analogous. Let $\mathcal{A}(n)$ be the full subcategory of $\sqcup/\mathbf{n}$ whose objects $\alpha\colon \mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}$ are such that the restrictions to $\mathbf{n}_1$ and $\mathbf{n}_2$ are order preserving. Since this is a skeleton subcategory, it suffices to show that the canonical map

$$\operatorname{hocolim}_{\mathcal{A}(n)} S^{n-\alpha} \wedge T_1(n_1) \wedge T_2(n_2) \longrightarrow \operatorname{colim}_{\mathcal{A}(n)} S^{n-\alpha} \wedge T_1(n_1) \wedge T_2(n_2)$$

is a weak homotopy equivalence for each n. Notice that $\mathcal{A}(n)$ may be identified with the partially ordered set of pairs (U_1, U_2) of disjoint subsets of $\mathbf{n}$, so that we may write the diagram in the form

$$Z(U_1, U_2) = S^{\mathbf{n}-U_1\cup U_2} \wedge T_1(U_1) \wedge T_2(U_2).$$

Notice also, that since the base points are non-degenerate and the categories $\mathcal{A}(n)$ are contractible, it suffices to consider the unbased homotopy colimit instead of the based homotopy colimit. We now use that the categories $\mathcal{A}(n)$ are very small in the sense of [11, §10.13]. By general model theoretical arguments using the Strøm model category structure [36] on $\mathcal{U}$, we are therefore left with showing that the canonical map

$$\operatorname{colim}_{(U_1,U_2)\subsetneq(U_1^0,U_2^0)} Z(U_1, U_2) \longrightarrow Z(U_1^0, U_2^0)$$

is an cofibration for each fixed object (U_1^0, U_2^0). Since the structure maps in the $\mathcal{A}(n)$-diagram Z are cofibrations and since cofibrations are closed inclusions, we may view each of the spaces $Z(U_1, U_2)$ as a subspace of $Z(U_1^0, U_2^0)$. By the flatness assumptions on T_1 and T_2 we then have the equality

$$Z(U_1, U_2) \cap Z(V_1, V_2) = Z(U_1 \cap V_1, U_2 \cap V_2)$$

for each pair of objects (U_1, U_2) and (V_1, V_2). Thus, it follows from the pasting lemma for maps defined on a union of closed subspaces that the colimit in question may be identified with the union of the subspaces $Z(U_1, U_2)$. The conclusion now follows from an inductive argument using Lillig's union theorem for cofibrations [18]. □

Proposition 8.2. *If $T_1, \dots, T_k$ are level-wise well-based symmetric spectra, then there is a chain of level-wise equivalences*

$$T_1 \wedge^h \cdots \wedge^h T_k \xleftarrow{\sim} T_1' \wedge^h \cdots \wedge^h T_k' \xrightarrow{\sim} T_1' \wedge \cdots \wedge T_k',$$

where $T_i' \to T_i$ are cofibrant replacements in the stable model structure on symmetric spectra.

Proof. It follows from [23], Proposition 9.9, that we may choose cofibrant replacements $T_i' \to T_i$, such that the symmetric spectra T_i' are cofibrant and the maps are level-wise acyclic fibrations. It follows that the left hand map is a level-wise equivalence since homotopy colimits preserve equivalences of well-based diagrams. That the right hand map is an equivalence follows from Proposition 8.1 since cofibrant symmetric spectra are retracts of relative cell-complexes, see [23], hence in particular flat. □

Combining these propositions we get the following corollary which states that smash products of flat symmetric spectra represent the "derived" smash products.

Corollary 8.3. *If $T_1, \dots, T_k$ are flat symmetric spectra, then there is a level-wise equivalence*

$$T_1' \wedge \cdots \wedge T_k' \xrightarrow{\sim} T_1 \wedge \cdots \wedge T_k,$$

where $T_i' \to T_i$ are cofibrant replacements in the stable model category on symmetric spectra.

In the following definition we use the notion of an h-cofibration introduced in Section 4.1.

Definition 8.4. A flat symmetric ring spectrum T is a symmetric ring spectrum whose underlying symmetric spectrum is flat and whose unit $S \to T$ is an h-cofibration.

Proposition 8.5. *If T is a flat symmetric ring spectrum, then $B^{cy}(T)$ represents the topological Hochschild homology of T.*

Proof. Let $T' \to T$ be a cofibrant replacement of T as a symmetric ring spectrum. Then it follows from [34], Section 4, that $B^{\mathrm{cy}}(T')$ represents the topological Hochschild of T. Using Corollary 8.3, we see that the induced map of simplicial symmetric spectra $B^{\mathrm{cy}}_\bullet(T') \to B^{\mathrm{cy}}_\bullet(T)$ is a level-wise equivalence in each simplicial degree. Furthermore, the assumption that the unit be an h-cofibration implies by Lemma 4.2 that these are good simplicial spaces. Therefore the topological realizations are also level-wise equivalent. □

8.3. **Flat replacement of symmetric spectra.** We define an endofunctor on the category of symmetric spectra by associating to a symmetric spectrum T the symmetric spectrum $\bar{T}$ defined by

$$\bar{T}(n) = \mathrm{hocolim}_{\alpha\colon\, \mathbf{m}\to\mathbf{n}}\, S^{n-\alpha} \wedge T(m),$$

where the (based) homotopy colimit is over the category $\mathcal{I}/\mathbf{n}$. This is not quite a flat replacement functor since $\bar{T}$ need not be level-wise well-based. However, we do have the following.

Proposition 8.6. *If T is a symmetric spectrum that is level-wise well-based, then $\bar{T}$ is flat and the canonical projection $\bar{T} \to T$ is a level-wise weak homotopy equivalence.*

Proof. We show that $\bar{T}$ is flat by verifying the conditions (i) and (ii) above. Thus, given a morphism $\alpha\colon\, \mathbf{m} \to \mathbf{n}$, we must show that the structure map $S^{n-\alpha}\wedge\bar{T}(m) \to \bar{T}(n)$ is a cofibration. Let $\alpha_*\colon\, \mathcal{I}/\mathbf{m} \to \mathcal{I}/\mathbf{n}$ be the functor induced by α. Using that based homotopy colimits commute with smash products, the map in question may be viewed as a map of homotopy colimits,

$$\begin{aligned}\mathrm{hocolim}_{\mathcal{I}/\mathbf{m}}\, S^{n-\alpha} \wedge S^{m-\bullet} \wedge T(\bullet) \longrightarrow \mathrm{hocolim}_{\mathcal{I}/\mathbf{m}}\, S^{n-\bullet} \wedge T(\bullet) \circ \alpha_* \\ \longrightarrow \mathrm{hocolim}_{\mathcal{I}/\mathbf{n}}\, S^{n-\bullet} \wedge T(\bullet),\end{aligned}$$

where the first map is induced by the natural isomorphism of $\mathcal{I}/\mathbf{m}$-diagrams

$$S^{n-\alpha} \wedge S^{m_1-\beta} \wedge T(m_1) \cong S^{n-\alpha\beta} \wedge T(m_1), \qquad \beta\colon\, \mathbf{m}_1 \longrightarrow \mathbf{m},$$

and the second map is induced by α_*. Notice that the latter induces an isomorphism of $\mathcal{I}/\mathbf{m}$ onto a full subcategory of $\mathcal{I}/\mathbf{n}$. The conclusion therefore follows from the general fact that the map of homotopy colimits obtained by restricting a diagram to a full subcategory is a cofibration, see [12], X, Lemma 3.5. In order to verify (ii) one first checks the condition in each simplicial degree of the simplicial replacement defining the homotopy colimit, see [8]. Then result then follows from the fact that topological realization preserves pullback diagrams. The map $\bar{T} \to T$ is defined by the canonical projection from the homotopy colimit to the colimit

$$\bar{T}(n) = \mathrm{hocolim}_{\mathcal{I}/\mathbf{n}}\, T \longrightarrow \mathrm{colim}_{\mathcal{I}/\mathbf{n}}\, T = T(n),$$

where the identification of the colimit comes from the fact that $\mathcal{I}/\mathbf{n}$ has a terminal object. The existence of a terminal object implies that this is a weak homotopy equivalence. □

The relationship between the replacement functor and the $\wedge^h$-product is recorded in the following proposition whose proof we leave with the reader.

Proposition 8.7. *There is a natural isomorphism of symmetric spectra*

$$\bar{T}_1 \wedge \cdots \wedge \bar{T}_k \cong T_1 \wedge^h \cdots \wedge^h T_k.$$

Recall the notion of a monoidal functor from [16, §XI.2]. This is what is sometimes called *lax monoidal*.

Proposition 8.8. *The replacement functor $T \mapsto \bar{T}$ is monoidal and the canonical map $\bar{T} \to T$ is a monoidal natural transformation.*

Proof. The replacement of the sphere spectrum has 0th space $\bar{S}(0) = S^0$ and we let $S \to \bar{S}$ be the unique map of symmetric spectra that is the identity in degree 0. We must define an associative and unital natural transformation $\bar{T}_1 \wedge \bar{T}_2 \to \overline{T_1 \wedge T_2}$, which by the universal property of the smash product amounts to an associative and unital natural transformation of $\mathcal{I}_S \times \mathcal{I}_S$-diagrams

$$\bar{T}_1(\mathbf{m}) \wedge \bar{T}_2(\mathbf{n}) \longrightarrow \overline{T_1 \wedge T_2}(\mathbf{m} \sqcup \mathbf{n}). \tag{9}$$

Here $\mathcal{I}_S$ denotes the topological category such that Sp^Σ may be identified with the category of based $\mathcal{I}_S$-diagrams, see [23] and [30]. Consider the natural transformation of $\mathcal{I}/\mathbf{m} \times \mathcal{I}/\mathbf{n}$-diagrams that to a pair of objects $\alpha\colon \mathbf{m'} \to \mathbf{m}$ and $\beta\colon \mathbf{n'} \to \mathbf{n}$ associates the map

$$S^{m-\alpha} \wedge T_1(m') \wedge S^{n-\beta} \wedge T_2(n') \longrightarrow S^{m+n-\alpha\sqcup\beta} \wedge T_1 \wedge T_2(m'+n'),$$

where we first permute the factors and then apply the universal map to the smash product $T_1 \wedge T_2$. Using that based homotopy colimits commute with smash products, the map (9) is the induced map of homotopy colimits, followed by the map of homotopy colimits induced by the concatenation functor $\mathcal{I}/\mathbf{m} \times \mathcal{I}/\mathbf{n} \to \mathcal{I}/\mathbf{m} \sqcup \mathbf{n}$. With this definition, it is clear that the natural transformation $\bar{T} \to T$ is monoidal. □

It follows from this that the replacement functor induces a functor on the category of symmetric ring spectra. However, even for symmetric ring spectra that are level-wise well-based, this functor does not in general produce flat symmetric ring spectra in the sense of Definition 8.4. For this reason we apply the usual method of replacing a symmetric ring spectrum T with the mapping cylinder

$$T' = T \cup_{S \wedge \{0\}_+} T \wedge I_+.$$

This construction gives a well-based symmetric ring spectrum as follows from the fact that T' is the homotopy colimit of the diagram $S \to T$ indexed by the category $1 \to 0$. This category has a unique strict symmetric monoidal structure with unit 1 and this gives rise to the multiplication in T'. The inclusion of the subcategory $\{1\}$ gives rise to the unit of T'. It is easy to check that if T is flat as a symmetric spectrum, then T' is a flat symmetric ring spectrum. Combining these remarks with Proposition 8.6, we get the following.

Proposition 8.9. *If T is a symmetric ring spectrum which is level-wise well-based, then the symmetric spectrum $T^c = (\bar{T})'$ is a flat symmetric ring spectrum.*

9. Implementing the axioms for symmetric spectra

In this section we verify the axioms **A1**–**A8** in the setting of $\mathcal{I}$-spaces and symmetric spectra. The basic reference for this material is the paper [30] in which the multiplicative theory of Thom spectra is developed in the category of symmetric spectra.

9.1. **Symmetric spectra and $\mathcal{I}$-spaces.** In the axiomatic framework set up in Section 3 we define $\mathcal{S}$ to be the category of symmetric spectra Sp^{Σ} and we say that a symmetric ring spectrum is flat if it satisfies the conditions in Definition 8.4. We define U to be the composite functor

$$U\colon\ Sp^{\Sigma} \xrightarrow{D} Sp^{\Sigma} \longrightarrow Sp,$$

where D is the detection functor from Section 8.1 and the second arrow represents the obvious forgetful functor. It follows from the discussion in Section 8.1 that a map of symmetric spectra is a stable model equivalence if and only if applying U gives an ordinary stable equivalence of spectra.

We define $\mathcal{A}$ to be the category $\mathcal{IU}$ of $\mathcal{I}$-spaces as defined in Section 1. As is generally the case for a diagram category of spaces over a small symmetric monoidal category, this inherits a symmetric monoidal structure in which the monoidal product $A \boxtimes B$ is defined by the Kan-extension

$$A \boxtimes B(n) = \operatorname{colim}_{\mathbf{n_1} \sqcup \mathbf{n_2} \to \mathbf{n}} A(n_1) \times B(n_2).$$

Here the colimit is over the category $\sqcup/\mathbf{n}$, where $\sqcup$ denotes the monoidal structure map $\mathcal{I}\times\mathcal{I} \to \mathcal{I}$. The unit of the monoidal structure is the constant $\mathcal{I}$-space $\mathcal{I}(0,-)$ which we denote by $*$. Given an $\mathcal{I}$-space B, let $B[n]$ be the $\mathcal{I}$-space $B(\mathbf{n}\sqcup -)$ obtained by composing with the "shift" functor $\mathbf{n}\sqcup -$ on $\mathcal{I}$. We write $\operatorname{Hom}_{\mathcal{I}}(A,B)$ for the internal Hom-object in $\mathcal{IU}$ defined by

$$\operatorname{Hom}_{\mathcal{I}}(A,B)\colon\ \mathbf{n} \mapsto \mathcal{IU}(A, B[n]).$$

This makes $\mathcal{IU}$ a closed symmetric monoidal topological category in the sense that there is a natural isomorphism

$$\mathcal{IU}(A \boxtimes B, C) \cong \mathcal{IU}(A, \operatorname{Hom}_{\mathcal{I}}(B,C)).$$

We define a *monoidal $\mathcal{I}$-space* to be a monoid in $\mathcal{IU}$. This is in accordance with the definition of a monoidal functor in [16, §XI.2]. The tensor of an $\mathcal{I}$-space A with a space K is given by the obvious level-wise cartesian product $A\times K$. We define the functor U to be the homotopy colimit over $\mathcal{I}$,

$$U\colon\ \mathcal{IU} \longrightarrow \mathcal{U}, \quad UA = \operatorname{hocolim}_{\mathcal{I}} A.$$

Here we adapt the Bousfield-Kan construction [8], such that by definition UA is the realization of the simplicial space

$$[k] \mapsto \coprod_{\mathbf{n}_0 \leftarrow \cdots \leftarrow \mathbf{n}_k} A(n_k),$$

where the coproduct is indexed over the nerve of $\mathcal{I}$. In particular, $U*$ is the the classifying space $B\mathcal{I}$ which is contractible since $\mathcal{I}$ has an initial object. That U preserves tensors and colimits follows from the fact that topological realization has this property.

In the discussion of the axioms **A1**–**A6** we consider two cases corresponding to the usual Thom spectrum functor on $\mathcal{U}/BF$ and its restriction to $\mathcal{U}/BO$. In the case of **A2** and **A3** we formulate a slightly weaker version of the axioms which hold in the $\mathcal{IU}/BF$ case and which imply the original axioms when restricted to objects in $\mathcal{IU}/BO$. In Section 9.8 we then provide additional arguments to show why the weaker form of the axioms suffices to prove the statement in Theorem 1.1. One can verify the axioms for more general families of subgroups $G(n)$, but we omit the details of this since the purpose of considering the group valued case is only to explain how the arguments simplify in this situation.

9.2. **Axiom A1.** It is well-known that the correspondence $\mathbf{n} \mapsto BF(n)$ defines a commutative monoidal $\mathcal{I}$-space, see e.g. [30]. In fact, this structure is obtained by restriction from a functor defined on the category of finite dimensional inner product spaces and linear isometries. In order to be consistent with the notation used in [30], we now redefine BF to be this $\mathcal{I}$-space, and we write $BF_{h\mathcal{I}}$ for its homotopy colimit, that is, for UBF. Let $\mathcal{N}$ be the subcategory of $\mathcal{I}$ whose only morphisms are the subset inclusions and let $i\colon \mathcal{N} \to \mathcal{I}$ be the inclusion. Thus, $\mathcal{N}$ may be identified with the ordered set of natural numbers. Let $BF_{h\mathcal{N}}$ be the homotopy colimit and $BF_{\mathcal{N}}$ the colimit of the $\mathcal{N}$-diagram BF, such that $BF_{\mathcal{N}}$ is now what was denoted BF in Section 3. We then have a diagram of weak homotopy equivalences

$$BF_{h\mathcal{I}} \xleftarrow{i} BF_{h\mathcal{N}} \xrightarrow{\pi} BF_{\mathcal{N}},$$

where π is the canonical projection from the homotopy colimit to the colimit. Here i is a weak homotopy equivalence by Bökstedt's approximation lemma [20, 2.3.7], and π is a weak homotopy equivalence since the structure maps are cofibrations. Using that these spaces have the homotopy type of a CW-complex, we choose a homotopy inverse j of i and we define ζ to be the composite map

$$\zeta\colon\ BF_{h\mathcal{I}} \xrightarrow{j} BF_{h\mathcal{N}} \xrightarrow{\pi} BF_{\mathcal{N}}.$$

Similarly, using the commutative monoidal $\mathcal{I}$-space $\mathbf{n} \mapsto BO(n)$, we get a weak homotopy equivalence $\zeta\colon\ BO_{h\mathcal{I}} \xrightarrow{\sim} BO_{\mathcal{N}}$.

9.3. **Axiom A2.** The symmetric Thom spectrum functor

$$T\colon\ \mathcal{IU}/BF \longrightarrow Sp^{\Sigma}$$

is defined by applying the Thom space functor level-wise: given an object $\alpha\colon\ A \to BF$ with level maps $\alpha_n\colon\ A(n) \to BF(n)$, the nth space of the symmetric spectrum $T(\alpha)$ is given by $T(\alpha_n)$. Since colimits and tensors in $\mathcal{IU}/BF$ and Sp^{Σ} are formed level-wise, the fact that the Thom space functor preserves these constructions implies that the symmetric Thom spectrum functor has the same property. Given objects $\alpha\colon\ A \to BF$ and $\beta\colon\ B \to BF$, we may view the canonical maps

$$A(m) \times B(n) \longrightarrow A \boxtimes B(m+n)$$

as maps over $BF(m+n)$ such that we get induced maps of Thom spaces

$$T(\alpha)(m) \wedge T(\beta)(n) \longrightarrow T(\alpha \boxtimes \beta)(m+n).$$

We refer to [30] for a proof of the fact that the induced map of symmetric spectra

$$T(\alpha) \wedge T(\beta) \longrightarrow T(\alpha \boxtimes \beta)$$

is an isomorphism. This implies that T is a strong symmetric monoidal functor. We next discuss homotopy invariance. Applying the usual (Hurewicz) fibrant replacement functor level-wise we get an endofunctor Γ on $\mathcal{IU}/BF$ and we say that an object α is T-*good* if the canonical map $T(\alpha) \to T\Gamma(\alpha)$ is a stable model equivalence. It is proved in [30], that if $(A, \alpha) \to (B, \beta)$ is a map of T-good $\mathcal{I}$-spaces over BF which induces a weak homotopy equivalence of homotopy colimits $A_{h\mathcal{I}} \to B_{h\mathcal{I}}$, then the induced map of symmetric Thom spectra $T(\alpha) \to T(\beta)$ is a stable model equivalence. According to our conventions this may be reformulated as saying that the Thom spectrum functor preserves weak equivalences on the full subcategory of T-good objects in $\mathcal{IU}/BF$. In particular this implies that T preserves weak equivalences when restricted to $\mathcal{IU}/BO$.

Proposition 9.1. *Restricted to the full subcategory of T-good objects the two compositions in the diagram*

$$\begin{array}{ccc} \mathcal{IU}/BF & \xrightarrow{T} & Sp^{\Sigma} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathcal{U}/BF_{h\mathcal{I}} & \xrightarrow{T\Gamma} & Sp \end{array}$$

are related by a chain of natural stable equivalences.

Again this implies that statement in **A2** when restricted to $\mathcal{IU}/BO$. We postpone the proof of this result until we have introduced the rectification in **A6**.

9.4. **Flat replacement and Axiom A3.** We first define an analogue for $\mathcal{I}$-spaces of the flatness notion introduced for symmetric spectra in Section 8.2. Thus, an $\mathcal{I}$-space A is *flat* if for any diagram of the form (7), such that the intersection of the images of β_1 and β_2 equals the image of γ, the induced map

$$A(n_1) \cup_{A(m)} A(n_2) \longrightarrow A(n)$$

is a cofibration. By Lillig's union theorem [18], this is equivalent to the requirement that (i) any morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces a cofibration $A(m) \to A(n)$, and (ii) that the intersection of the images of $A(n_1)$ and $A(n_2)$ in $A(n)$ equals the image of $A(m)$.

Example. The $\mathcal{I}$-spaces BO and BF are flat.

The *flat replacement* of an $\mathcal{I}$-space A is the $\mathcal{I}$-space $\bar{A}$ defined by

$$\bar{A}(n) = \mathrm{hocolim}_{\alpha\colon\, \mathbf{m}\to\mathbf{n}}\, A(m),$$

where the homotopy colimit is over the category $\mathcal{I}/\mathbf{n}$. We then have the following $\mathcal{I}$-space analogue of of Proposition 8.6.

Proposition 9.2. *The $\mathcal{I}$-space $\bar{A}$ is flat and the canonical projection $\bar{A} \to A$ is a level-wise equivalence.*

We also have an $\mathcal{I}$-space analogue of Proposition 8.8.

Proposition 9.3. *The flat replacement functor is a monoidal functor on $\mathcal{IU}$ and $\bar{A} \to A$ is a monoidal natural transformation.*

Proof. The flat replacement of the unit $*$ is the $\mathcal{I}$-space $\mathbf{n} \mapsto B(\mathcal{I}/\mathbf{n})$ which receives a unique map from $*$. In order to define the monoidal structure, it suffices by the universal property of the $\boxtimes$-product to define an associative and unital natural transformation of $\mathcal{I}^2$-diagrams

$$\bar{A}(\mathbf{m}) \times \bar{B}(\mathbf{n}) \longrightarrow \overline{A \boxtimes B}(\mathbf{m} \sqcup \mathbf{n}).$$

Using that unbased homotopy colimits commute with products, we define this to be the map induced by the natural transformation of $\mathcal{I}/\mathbf{m} \times \mathcal{I}/\mathbf{n}$-diagrams

$$A(\mathbf{m}) \times B(\mathbf{n}) \longrightarrow A \boxtimes B(\mathbf{m} \sqcup \mathbf{n}),$$

followed by the map of homotopy colimits induced by the concatenation functor $\mathcal{I}/\mathbf{m} \times \mathcal{I}/\mathbf{n} \to \mathcal{I}\mathbf{m} \sqcup \mathbf{n}$. □

It follows from this that the flat replacement functor induces a functor on the category of monoidal $\mathcal{I}$-spaces. However, this functor will not in general produce well-based monoidal $\mathcal{I}$-spaces as is required for our replacement functor in **A3**. For this reason we adapt the usual method of replacing a topological monoid by one that is well-based to our setting. This is analogous to the construction for symmetric spectra discussed in Section 8.3 and may be described explicitly as follows. Let I be the unit interval, thought of as a based topological monoid with base point 0 and unit 1. Given a monoidal $\mathcal{I}$-space A, let A' be the monoidal $\mathcal{I}$-space defined by the level-wise wedge products $A'(n) = A(n) \vee I$. Here we identify the base point in $A(n)$ specified by the unit with the base point 0 in I. The monoidal multiplication

$$(A(m) \vee I) \times (A(n) \vee I) \longrightarrow A(m+n) \vee I$$

restricts to the monoidal multiplication on $A(m) \times A(n)$ and to the monoid multiplication on $I \times I$. On the subspaces $A(m) \times I$ and $I \times A(n)$ it is given by the maps

$$A(m) \times I \longrightarrow A(m) \times A(n) \longrightarrow A(m+n), \quad I \times A(n) \longrightarrow A(m) \times A(n) \longrightarrow A(m+n),$$

where I projects onto the base points of $A(m)$ and $A(n)$, respectively. The proof of the following proposition is completely analogous to the proof of the analogous space-level result, see [25, A.8].

Proposition 9.4. *The monoidal $\mathcal{I}$-space A' is well-based and the canonical map of monoids $A' \to A$ is a homotopy equivalence of $\mathcal{I}$-spaces. If A is commutative, then so is A'.*

We now define the functor C in **A3** by

$$C\colon \mathcal{IU}[\mathbb{T}] \longrightarrow \mathcal{IU}[\mathbb{T}], \quad A \mapsto A^c = (\bar{A})'$$

and we define the natural transformation to be the composition of the level-wise weak homotopy equivalences $(\bar{A})' \to \bar{A} \to A$. Given a monoid morphism $\alpha\colon A \to BF$, we define α^c to be the composition

$$\alpha^c\colon A^c \longrightarrow A \xrightarrow{\alpha} BF.$$

We need a technical assumption in order to ensure that the associated symmetric spectrum $T(\alpha^c)$ will be flat. In general, we say that a map of $\mathcal{I}$-spaces $\alpha\colon A \to BF$ classifies a well-based $\mathcal{I}$-space over A if the cofibrations $BF(n) \to V(n)$ pull back to cofibrations via α. This condition is automatically satisfied if α factors through BO, see [17]. Thus, restricted to such morphisms the following proposition verifies **A3** in the BO case. Recall the functor $T \mapsto T^c$ from Proposition 8.9.

Proposition 9.5. *There is an isomorphism of symmetric ring spectra*

$$T(\alpha^c) \cong T(\alpha)^c$$

and if α classifies a well-based $\mathcal{I}$-space over A, then $T(\alpha^c)$ is a flat symmetric ring spectrum.

Proof. Consider in general a monoid morphism $\alpha\colon A \to BF$ and the induced morphisms $\alpha'\colon A' \to BF$ and $\bar{\alpha}\colon \bar{A} \to BF$. We claim that there are isomorphisms of symmetric ring spectra

$$T(\alpha') \cong T(\alpha)' \quad \text{and} \quad T(\bar{\alpha}) \cong \overline{T(\alpha)}.$$

Combining these isomorphisms we get the isomorphism in the proposition. The isomorphism for α' follows directly from the fact that T preserves colimits. For the second isomorphism we first observe that the fact that Thom space functor preserves coproducts and topological realization implies that it also preserves homotopy colimits. Thus, we have degree-wise homeomorphisms

$$\begin{aligned} T(\bar{\alpha})(n) = \operatorname{hocolim}_{u\colon \mathbf{m}\to\mathbf{n}} T(A(m) \xrightarrow{u\cdot\alpha_m} BF(n)) \\ \cong \operatorname{hocolim}_{u\colon \mathbf{m}\to\mathbf{n}} S^{n-u} \wedge T(\alpha)(n) \end{aligned}$$

and since the last term is $\overline{T(\alpha)}(n)$ by definition, the claim follows. For the last statement in the proposition we observe that if α classifies a well-based $\mathcal{I}$-space over A, then the symmetric spectrum $T(\alpha)$ is level-wise well-based. The statement therefore follows from Proposition 8.9. □

Remark. As for any diagram category of spaces, the category of $\mathcal{I}$-spaces admits a cofibrantly generated model category structure in which the equivalences and fibrations are formed level-wise. Similarly, there are model structures on the categories of monoids and commutative monoids in $\mathcal{IU}$. However, there are several reasons why these model structures are not well suited for the analysis of Thom spectra. For example, it is not clear that the T-goodness condition on objects in $\mathcal{IU}/BF$ is preserved under cofibrant replacement and the Thom spectra associated to cofibrant $\mathcal{I}$-spaces will not in general be cofibrant as symmetric spectra. Using the less restrictive notion of flatness also makes it clear that many $\mathcal{I}$-space, such as for example BF, behaves well with respect to the $\boxtimes$-product even though they are not cofibrant in the model structures mentioned above.

We next formulate some further properties of the replacement functor that will be needed later. Consider the homotopy invariant version of the $\boxtimes$-product defined by

$$A_1 \boxtimes^h \cdots \boxtimes^h A_k(n) = \operatorname{hocolim}_{\mathbf{n_1}\sqcup\cdots\sqcup\mathbf{n}_k\to\mathbf{n}} A(n_1) \times \cdots \times A(n_1).$$

As for the analogous constructions on symmetric spectra the flat replacement functor is closely related to the $\boxtimes^h$-product.

Proposition 9.6. *There is a natural isomorphism of $\mathcal{I}$-spaces*

$$\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k \cong A_1 \boxtimes^h \cdots \boxtimes^h A_k$$

We also have the following $\mathcal{I}$-space analogue of Proposition 8.1. The proof is similar to but slightly easier than the symmetric spectrum version since we do not have to worry about base points here.

Proposition 9.7. *If the $\mathcal{I}$-spaces $A_1, \dots, A_k$ are flat, then the canonical projection*

$$A_1 \boxtimes^h \cdots \boxtimes^h A_k \longrightarrow A_1 \boxtimes \cdots \boxtimes A_k$$

is a level-wise equivalence.

As in Section 8.2 we conclude from this that the $\boxtimes^h$-product always represents the "correct" homotopy type and that the $\boxtimes$-product has the "correct" homotopy type for flat $\mathcal{I}$-spaces.

9.5. **Axiom A4.** We define Q to be the colimit over $\mathcal{I}$,

$$Q\colon \mathcal{IU} \longrightarrow \mathcal{U}, \quad QA = \operatorname{colim}_{\mathcal{I}} A.$$

It is clear that that Q preserves colimits and the fact that $\mathcal{U}$ is closed symmetric monoidal under the categorical product implies that it also preserves tensors. Before verifying the remaining conditions it is helpful to recall some general facts about Kan extensions. Thus, consider in general a functor $\phi\colon \mathcal{B} \to \mathcal{C}$ between small categories. Given a $\mathcal{B}$-diagram $X\colon \mathcal{B} \to \mathcal{U}$, the Kan extension is the functor $\phi_* X\colon \mathcal{C} \to \mathcal{U}$ defined by

$$\phi_* X(c) = \operatorname{colim}_{\phi/c} X \circ \pi_c$$

and the homotopy Kan extension is the functor $\phi_*^h X\colon \mathcal{C} \to \mathcal{U}$ defined by the analogous homotopy colimits,

$$\phi_*^h X(c) = \operatorname{hocolim}_{\phi/c} X \circ \pi_c.$$

Here π_c denotes the forgetful functor $\phi/c \to \mathcal{B}$, see e.g. [29]. The effect of evaluating the colimits of these functors is recorded in the following lemma.

Lemma 9.8. *There are natural isomorphisms*

$$\operatorname{colim}_{\mathcal{B}} \phi_* X \cong \operatorname{colim}_{\mathcal{A}} X, \quad \operatorname{colim}_{\mathcal{B}} \phi_*^h X \cong \operatorname{hocolim}_{\mathcal{A}} X,$$

and the canonical projection from the homotopy colimit to the colimit

$$\operatorname{hocolim}_{\mathcal{B}} \phi_*^h X \longrightarrow \operatorname{colim}_{\mathcal{B}} \phi_*^h X$$

is a weak homotopy equivalence.

An explicit proof of the last statement can be found in [29], Section 1.4. Using that the $\boxtimes$-product is defined as a Kan extension, the fact that Q is strong symmetric monoidal now follows from the canonical homeomorphisms

$$\operatorname{colim}_{\mathcal{I}} A \boxtimes B \cong \operatorname{colim}_{\mathcal{I}\times\mathcal{I}} A \times B \cong \operatorname{colim}_{\mathcal{I}} A \times \operatorname{colim}_{\mathcal{I}} B,$$

where the second homeomorphism again comes from the fact $\mathcal{U}$ is closed. We define the natural transformation $U \to Q$ to be the canonical projection from the homotopy colimit to the colimit.

Lemma 9.9. *Given $\mathcal{I}$-spaces $A_1, \dots, A_k$, there is a canonical homeomorphism*

$$Q(\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k) \cong UA_1 \times \cdots \times UA_k$$

and the canonical projection gives a weak homotopy equivalence

$$U(\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k) \xrightarrow{\sim} Q(\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k).$$

Proof. Using Proposition 9.6 we may write $\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k$ as a homotopy Kan extension, hence the result follows immediately from Lemma 9.5. $\square$

When A is a monoidal $\mathcal{I}$-space, the canonical map $(\bar{A})' \to \bar{A}$ is a homotopy equivalence of $\mathcal{I}$-space. The above lemma therefore implies the last condition in **A4**.

Proposition 9.10. *Given monoidal $\mathcal{I}$-spaces $A_1, \dots, A_k$, the canonical projection*

$$U(A_1^c \boxtimes \cdots \boxtimes A_k^c) \longrightarrow Q(A_1^c \boxtimes \cdots \boxtimes A_k^c)$$

is a weak homotopy equivalence.

9.6. **Axiom A5.** In the notation of Axiom **A5**, we define BF' to be the well-based monoidal $\mathcal{I}$-space defined from BF as in Section 9.4. This is again a commutative monoidal flat $\mathcal{I}$-space and the condition in **A5** therefore follows from the following more general result.

Proposition 9.11. *If $A_1, \dots, A_k$ are flat $\mathcal{I}$-spaces, then the canonical map*

$$U(A_1 \boxtimes \cdots \boxtimes A_k) \longrightarrow UA_1 \times \cdots \times UA_k$$

is a weak homotopy equivalence.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} Q(\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k) & \xrightarrow{\simeq} & Q(\bar{A}_1) \times \cdots \times Q(\bar{A}_k) \\ \uparrow \sim & & \uparrow \sim \\ U(\bar{A}_1 \boxtimes \cdots \boxtimes \bar{A}_k) & \longrightarrow & U(\bar{A}_1) \times \cdots \times U(\bar{A}_k) \\ \downarrow \sim & & \downarrow \sim \\ U(A_1 \boxtimes \cdots \boxtimes A_k) & \longrightarrow & U(A_1) \times \cdots \times U(A_k), \end{array}$$

where the vertical maps are weak homotopy equivalences by Proposition 9.7 and Lemma 9.9. The horizontal map on the top is a homeomorphism since Q is strong monoidal and the result follows. □

9.7. **Axiom A6.** The definition of the functor in **A6** is based on the rectification functor

$$R\colon \mathcal{U}/BF_{h\mathcal{I}} \longrightarrow \mathcal{IU}/BF, \quad (X \xrightarrow{f} BF_{h\mathcal{I}}) \mapsto (R_f X \xrightarrow{R(f)} BF)$$

introduced in [30]. We shall not repeat the details of the definition here, but we remark that a similar construction can be applied to give a rectification functor with BO instead of BF. The following is proved in [30].

Proposition 9.12 ([30])**.** *The Barratt-Eccles operad $\mathcal{E}$ acts on $BF_{h\mathcal{I}}$ and if $\mathcal{C}$ is an operad that is augmented over $\mathcal{E}$, then there is an induced functor*

$$R\colon \mathcal{U}[\mathcal{C}]/BF_{h\mathcal{I}} \longrightarrow \mathcal{IU}[\mathcal{C}]/BF$$

and a natural weak homotopy equivalence of $\mathcal{C}$-spaces $(R_f X)_{h\mathcal{I}} \xrightarrow{\simeq} X$.

There is an analogous result in the BO-case. Let now $\mathcal{C}$ be the associativity operad such that the categories $\mathcal{U}[\mathcal{C}]$ and $\mathcal{IU}[\mathcal{C}]$ are the categories of topological monoids and monoidal $\mathcal{I}$-spaces, respectively. The associativity operad is augmented over the Barratt-Eccles operad and we define the functor R in **A6** to be the induced functor

$$R\colon \mathcal{U}[\mathcal{C}]/BF_{h\mathcal{I}} \longrightarrow \mathcal{IU}[\mathbb{T}]/BF$$

and similarly with BO instead of BF. The composite functor

$$\mathcal{U}[\mathcal{C}]/BF_{h\mathcal{I}} \xrightarrow{R} \mathcal{IU}[\mathbb{T}]/BF \longrightarrow \mathcal{IU}[\mathbb{T}] \xrightarrow{C} \mathcal{IU}[\mathbb{T}] \xrightarrow{Q} \mathcal{U}[\mathbb{T}] \longrightarrow \mathcal{U}[\mathcal{C}]$$

takes an object $f\colon X \to BF_{h\mathcal{I}}$ to $Q(R_f X)^c$ and we have a chain of weak homotopy equivalences in $\mathcal{U}[\mathcal{C}]$ given by

$$Q(R_f X)^c \xrightarrow{\sim} Q\overline{R_f X} \xrightarrow{\sim} (R_f X)_{h\mathcal{I}} \xrightarrow{\sim} X.$$

Here the lefthand equivalence is induced by the canonical homotopy equivalence of Proposition 9.4, the next is the homeomorphism established in Lemma 9.9, and the last equivalence is provided by the preceding Proposition. The verification of the axioms is now complete except for the proof of Proposition 9.1. For this we need the following two results from [30]. Recall that by our conventions a map in $\mathcal{IU}$ is a weak equivalence if the induced map of homotopy colimits is a weak homotopy equivalence.

Proposition 9.13 ([30])**.** *The composite functor*

$$\mathcal{IU}/BF \xrightarrow{U} \mathcal{U}/BF_{h\mathcal{I}} \xrightarrow{R} \mathcal{IU}/BF$$

is related to the identity functor on $\mathcal{IU}$ by a chain of natural weak equivalences.

The next proposition shows that the ordinary Thom spectrum functor can be recovered from the symmetric Thom spectrum functor up to stable equivalence.

Proposition 9.14 ([30])**.** *The composite functor*

$$\mathcal{U}/BF_{h\mathcal{I}} \xrightarrow{R} \mathcal{IU}/BF \xrightarrow{T\Gamma} Sp^{\Sigma} \xrightarrow{U} Sp$$

is related to the Lewis-May Thom spectrum functor by a chain of natural stable equivalences.

Since the functor R takes values in the subcategory of T-good objects in $\mathcal{IU}/BF$, the composite functor in the proposition is in fact stably equivalence to UTR. However, the above formulation is convenient for the application in the following proof.

Proof of Proposition 9.1. It suffices to show that the two compositions in the diagram

$$\begin{array}{ccc} \mathcal{IU}/BF & \xrightarrow{\mathrm{T}\,\Gamma} & Sp^{\Sigma} \\ \big\downarrow U & & \big\downarrow U \\ \mathcal{U}/BF_{h\mathcal{I}} & \xrightarrow{T\Gamma} & Sp \end{array}$$

are related by a chain of stable equivalences. Composing the chain of equivalences in Proposition 9.13 with the level-wise fibrant replacement functor Γ gives a chain of equivalences of T-good objects. Therefore, applying the symmetric Thom spectrum functor to this chain, we get a chain of stable model equivalences

$$T\Gamma RU \simeq T\Gamma\colon \mathcal{IU}/BF \longrightarrow Sp^{\Sigma}.$$

Combining this with Proposition 9.14 we get the required chain of stable equivalences

$$UT\Gamma \simeq UT\Gamma RU \simeq T\Gamma U.$$

9.8. **The proof of Theorem 1.1 in the general case.** In this section we show that the weaker forms of the axioms **A2** and **A3** suffice for the proof of Theorem 1.1. The main technical difficulty is that the symmetric Thom spectrum functor only preserves weak equivalences on the full subcategory of T-good objects in $\mathcal{IU}/BF$. In order to maintain homotopical control we must therefore be careful only to apply the Thom spectrum functor to T-good objects. Recall that we say that a map $\alpha_n\colon A(n) \to BF(n)$ *classifies a well-based quasifibration* if

(i) The pullback $\alpha^* V(n) \to BF(n)$ is a quasifibration, and
(ii) The induced section is a cofibration.

An object α in $\mathcal{IU}/BF$ is said to classify well-based quasifibrations if the level maps α_n do. This condition implies that α is T-good and is sometimes technically convenient as in the following lemma from [30].

Lemma 9.15 ([30]). *Let Λ be a small category and let $f_\lambda\colon X_\lambda \to BF(n)$ be a Λ-diagram in $\mathcal{U}/BF(n)$ such that each f_λ classifies a well-based quasifibration. Then the induced map*

$$f\colon \operatorname{hocolim}_\Lambda X_\lambda \longrightarrow BF(n)$$

also classifies a well-based quasifibration.

Let now $f\colon X \to BF_{h\mathcal{I}}$ be a map of topological monoids and let $\alpha\colon A \to BF$ be the object in $\mathcal{IU}[\mathbb{T}]/BF$ obtained by applying the rectification functor R. The first step is to ensure that the symmetric Thom spectra $T(\alpha)$ and $T(\alpha^c)$ have the correct homotopy type.

Lemma 9.16. *The objects $\alpha\colon A \to BF$ and $\alpha^c\colon A^c \to BF$ classify well-based quasifibrations.*

Proof. It follows from the definition of the functor R in [30] that the composition

$$A(m) \xrightarrow{\alpha_m} BF(m) \xrightarrow{u} BF(n)$$

classifies a well-based quasifibration for each morphism $u\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$. Thus, α classifies well-based quasifibrations and by Lemma 9.15 the same holds for $\bar{\alpha}$. Since A^c is the homotopy colimit of the diagram $S \to \bar{A}$, the result follows. □

It now follows from Lemma 9.5 that $T(\alpha^c)$ is a flat symmetric ring spectrum and therefore that the cyclic bar construction $B^{\mathrm{cy}}(T(\alpha^c))$ represents the topological Hochschild homology of $T(\alpha)$. It remains to analyze the map $B^{\mathrm{cy}}(\alpha^c)$.

Lemma 9.17. *The object $B^{cy}(\alpha^c)$ in $\mathcal{IU}/BF$ is T-good.*

In preparation for the proof we consider in general a simplicial object $f_\bullet\colon X_\bullet \to BF(n)$ in $\mathcal{U}/BF(n)$ with topological realization $f\colon X \to BF(n)$. Evaluating the Thom spaces degree-wise we get a simplicial based space $T(f_\bullet)$ whose realization is isomorphic to $T(f)$ as follows from Lemma 4.1.

Lemma 9.18. *Suppose that $f_\bullet\colon X_\bullet \to BF(n)$ is degree-wise T-good and that the simplicial spaces $X_\bullet$ and $T(f_\bullet)$ are good. Then the topological realization $f\colon X \to BF(n)$ is also T-good.*

Proof. We must show that the canonical map $X \to \Gamma_f(X)$ induces a weak homotopy equivalence of Thom spaces $T(f) \to T\Gamma(f)$. Using that Γ preserves tensors and colimits, we may view the latter map as the realization of the simplicial map $T(f_\bullet) \to T\Gamma(f_\bullet)$. The assumption that f be degree-wise T-good is the statement that this is a degree-wise weak homotopy equivalence. Since the simplicial space $T(f_\bullet)$ is good by assumption, it remains to show that the goodness assumption on $X_\bullet$ implies that $T\Gamma(f_\bullet)$ is also good. For this we use [17], IX, Proposition 1.11, which implies that the degeneracy maps $\Gamma(X_k) \to \Gamma(X_{k+1})$ are fibrewise cofibrations over $BF(n)$. The induced maps of Thom spaces are therefore also cofibrations. □

Proof of Lemma 9.17. By the lemma just proved it suffices to show that the simplicial map $B^{\mathrm{cy}}_\bullet(\alpha^c)(n) \to BF(n)$ is degree-wise T-good for each n. In simplicial degree k this is the composition

$$A^c \boxtimes \cdots \boxtimes A^c(n) \longrightarrow \bar{A} \boxtimes \cdots \boxtimes \bar{A}(n) \longrightarrow BF(n)$$

and since the first map is a fibrewise homotopy equivalence over $BF(n)$ it suffices to show that the second map is T-good. Using Proposition 9.6 we write the latter as a map of homotopy colimits

$$\operatorname{hocolim}_{\mathbf{n}_0 \sqcup \cdots \sqcup \mathbf{n}_k \to \mathbf{n}} A(n_0) \times \cdots \times A(n_k) \longrightarrow BF(n)$$

and since the maps in the underlying diagram classify well-based quasifibrations by construction, the result follows from Lemma 9.15.

Proof of Theorem 1.1. Given a map of $\mathcal{C}$-spaces $f\colon X \to BF_{h\mathcal{I}}$, we again write $\alpha\colon A \to BF$ for the associated monoid morphism. It follows from the above discussion that the topological Hochschild homology of $T(f)$ is represented by $B^{\mathrm{cy}}(T(\alpha^c))$ which in turn is isomorphic to the symmetric Thom spectrum $T(B^{\mathrm{cy}}(\alpha^c))$. Since $B^{\mathrm{cy}}(\alpha^c)$ is T-good, the weaker version of **A2** suffices to give a stable equivalence

$$UT(B^{\mathrm{cy}}(\alpha^c)) \simeq T\Gamma(UB^{\mathrm{cy}}(\alpha^c)).$$

From here the argument proceeds as in Section 4.3 and we get a stable equivalence

$$UT(B^{\mathrm{cy}}(\alpha^c)) \simeq T\Gamma(L^\eta(Bf))$$

which is the content of Theorem 1.1.

Proceeding as in Section 4.3 one finally deduces the general case of Theorem 1.2 and Theorem 1.3 from this result.

10. Commutativity in the context of $\mathcal{I}$-spaces

In outline, the proof of Theorem 1.4 is quite simple. Given an E_∞-map $f\colon X \to BF_{h\mathcal{I}}$, we first rectify this to a map of commutative monoidal $\mathcal{I}$-spaces $f\colon A \to BF$. Since A is commutative, $B^{\mathrm{cy}}_\bullet(A)$ and $B_\bullet(A)$ are simplicial commutative $\mathcal{I}$-spaces and the sequence

$$A \longrightarrow B^{\mathrm{cy}}_\bullet(A) \longrightarrow B_\bullet(A)$$

is a sequence of monoid morphisms. Furthermore, the first morphism has a retraction $r\colon B^{\mathrm{cy}}_\bullet(A) \to A$, such that we get a monoid morphism of $\mathcal{I}$-spaces over BF,

$$\begin{array}{ccc} B^{\mathrm{cy}}(A) & \xrightarrow{r\times p} & A \times B(A) \\ \big\downarrow{\scriptstyle B^{\mathrm{cy}} f} & & \big\downarrow{\scriptstyle f\circ p} \\ BF & = & BF. \end{array}$$

Here p denotes the projection onto $B(A)$. Suppose now that X is grouplike. Then A is also grouplike and if the upper map in the diagram is an equivalence **??**, then Theorem 1.4 follows by applying the symmetric Thom spectrum functor (or $T\Gamma$ if the maps in question are not T-good). However, since the $\boxtimes$-product is not homotopically well-behaved without flatness assumptions on A, this will not be the case in general. The problem with applying the flat replacement functor at this point is that it does not preserve strict commutativity. One might also try to apply a cofibrant replacement in the model category structure on commutative monoidal $\mathcal{I}$-spaces [**?**], but it is neither clear that this replacement preserves the T-goodness assumption, nor that the associated Thom spectra are flat. Our solution to these difficulties is to use the Bökstedt-Hesselholt-Madsen model of topological Hochschild homology which, given a symmetric ring spectrum T, we shall write as $\mathrm{T\hat{H}H}(T)$. We briefly recall the details most relevant for our purposes, referring the reader to [**?**] for a full discussion. Notice first, that the correspondence $\mathbf{n} \mapsto \mathcal{I}^n$ defines an $\mathcal{I}$-diagram of symmetric monoidal categories if we let a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ act by

$$\alpha_*\colon \mathcal{I}^m \longrightarrow \mathcal{I}^n, \quad \alpha_*(\mathbf{s}_1, \dots, \mathbf{s}_m) = (\mathbf{s}_{\alpha^{-1}(1)}, \dots, \mathbf{s}_{\alpha^{-1}(n)}).$$

Here $\mathcal{I}^n$ is the n-fold product of the symmetric monoidal category $\mathcal{I}$, and if i is not in the image of α, then we define $\mathbf{s}_{\alpha^{-1}(i)}$ to be the initial object $\mathbf{0}$. Using the symmetric monoidal structure, we define an $\mathcal{I}$-diagram of cyclic categories $[k] \mapsto \mathcal{I}^{n(k+1)}$, where the cyclic structure maps are defined as for the cyclic bar construction of a monoid. The point of introducing the extra coordinates is that we now have a natural pairing of cyclic categories

$$\mathcal{I}^{m\bullet} \times I^{n\bullet} \longrightarrow I^{(m+n)\bullet}, \tag{10}$$

induced by the identification of $\mathcal{I}^m \times \mathcal{I}^n$ with $\mathcal{I}^{m+n}$. Given a convergent symmetric ring spectrum T, the spectrum $\mathrm{T\hat{H}H}(T)$ is the realization of the cyclic spectrum whose nth space $\mathrm{T\hat{H}H}(T, n)$ is defined by

$$[k] \mapsto \mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} F(S^{\sqcup \vec{\mathbf{n}}_0} \wedge \cdots \wedge S^{\sqcup \vec{\mathbf{n}}_k}, T(\sqcup \vec{\mathbf{n}}_0) \wedge \cdots \wedge T(\sqcup \vec{\mathbf{n}}_k) \wedge S^n).$$

Here $F(-,-)$ is the space of based mappings, each $\vec{\mathbf{n}}_i$ denotes an object of $\mathcal{I}^n$, and $\sqcup \vec{\mathbf{n}}_i$ is the object of $\mathcal{I}$ obtained by concatenation. Since T is assumed to be convergent, it follows from [**?**] and [**?**] that $\mathrm{T\hat{H}H}(T)$ represents the homotopy type of the topological Hochschild homology spectrum as long as T is level-wise well-based. Furthermore, if T is commutative, then the pairings (10) give rise to natural maps

$$\mathrm{T\hat{H}H}(T, m) \wedge \mathrm{T\hat{H}H}(T, n) \longrightarrow \mathrm{T\hat{H}H}(T, m + n),$$

giving $\mathrm{T\hat{H}H}(T)$ the structure of a commutative symmetric ring spectrum. Consider in particular the commutative symmetric ring spectrum $\mathrm{T\hat{H}H}_0(T)$ in simplicial degree 0. We shall later need the following result.

Lemma 10.1. *If T is a commutative symmetric ring spectrum which is convergent, then there is a chain of level-wise weak homotopy equivalences of commutative symmetric ring spectra relating T and* $\mathrm{T\hat{H}H}_0(T)$.

Proof. Consider the commutative symmetric ring spectrum $\mathrm{T\check{H}H}_0(T)$ with nth space

$$\mathrm{T\check{H}H}_0(T, n) = \mathrm{hocolim}_{\mathcal{I}^n} F(S^{\sqcup \vec{\mathbf{n}}_0}, T(\sqcup \vec{\mathbf{n}}_0 \sqcup \mathbf{n})).$$

We then have a diagram of commutative symmetric ring spectra

$$\mathrm{T\hat{H}H}_0(T) \longrightarrow \mathrm{T\check{H}H}_0(T) \leftarrow T,$$

where the maps are level-wise weak homotopy equivalences by Bökstedt's approximation lemma [20, 2.3.7]. □

Given a monoidal $\mathcal{I}$-space A, one may define an $\mathcal{I}$-space analogue of the above by letting $\hat{B}^{\mathrm{cy}}(A)$ be the realization of the cyclic $\mathcal{I}$-space defined by

$$\hat{B}^{\mathrm{cy}}_k(A)(n) = \mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} A(\sqcup\vec{\mathbf{n}}_0) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k).$$

Using the pairings (10), one checks that if A is commutative, then this is again a commutative monoidal $\mathcal{I}$-space. In fact, we shall not use this construction directly, but rather the analogous bar construction $\hat{B}(A)$, which we define to be the realization of the simplicial $\mathcal{I}$-space

$$\hat{B}_k(A)(n) = \mathrm{hocolim}_{I^{nk}} A(\sqcup\vec{\mathbf{n}}_1) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k).$$

By definition, $\hat{B}(A)(0)$ is the usual bar construction of the topological monoid $A(0)$. Notice, that if A is convergent, then the $\mathcal{I}$-structure maps are weak homotopy equivalences in positive degrees. Again one checks thet $\hat{B}(A)$ is a commutative monoidal $\mathcal{I}$-space if A is. Theorem 1.4 now follows from the following result. In general, given a symmetric ring spectrum T and a monoidal $\mathcal{I}$-space A, we define $T \wedge A_+$ to be the symmetric ring spectrum with nth space $T(n) \wedge A(n)_+$.

Theorem 10.2. *Let $f\colon A \to BF$ be a T-good morphism of commutative monoidal $\mathcal{I}$-spaces and suppose that A is convergent. Then there is a chain of equivalences of commutative symmetric ring spectra*

$$\mathrm{T\hat{H}H}(T(f)) \simeq T(f) \wedge \hat{B}(A)_+.$$

Before giving the proof, we describe the chain of equivalences in the theorem. Consider first the commutative diagram of $\mathcal{I}^{n(k+1)}$-spaces

$$\begin{array}{ccc} A(\sqcup\vec{\mathbf{n}}_0) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k) & \longrightarrow & A(\sqcup_i(\sqcup\vec{\mathbf{n}}_i)) \times A(\sqcup\vec{\mathbf{n}}_1) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k) \\ \downarrow & & \downarrow \\ BF(\sqcup_i(\sqcup\vec{\mathbf{n}}_i)) & = & BF(\sqcup_i(\sqcup\vec{\mathbf{n}}_i)), \end{array} \tag{11}$$

where the first component of the upper horizontal map is defined using the monoidal product of A and the remaining components are the obvious projections. The vertical map on the the right is defined by first projecting onto the first factor and then applying f. Since the Thom space functor takes cartesian products to smash products, the upper map induces a map of Thom spaces

$$T(\sqcup\vec{\mathbf{n}}_0) \wedge \cdots \wedge T(\sqcup\vec{\mathbf{n}}_k) \longrightarrow T(\sqcup_i(\sqcup\vec{\mathbf{n}}_i)) \wedge \big(A(\sqcup\vec{\mathbf{n}}_1) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k)\big)_+. \tag{12}$$

By definition, $\mathrm{T\hat{H}H}(T(f))$ is the realization of the simplicial spectrum with nth space

$$[k] \mapsto \mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} F(S^{\sqcup\vec{\mathbf{n}}_0} \wedge \cdots \wedge S^{\sqcup\vec{\mathbf{n}}_k}, T(\sqcup\vec{\mathbf{n}}_0) \wedge \cdots \wedge T(\sqcup\vec{\mathbf{n}}_k) \wedge S^n),$$

and using the natural map in (12), we get a simplicial map from this to the simplicial spectrum

$$\mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} F(S^{\sqcup\vec{\mathbf{n}}_0} \wedge \cdots \wedge S^{\sqcup\vec{\mathbf{n}}_k}, T(\sqcup_i(\sqcup\vec{\mathbf{n}}_i)) \wedge (A(\sqcup\vec{\mathbf{n}}_1) \times \cdots \times A(\sqcup\vec{\mathbf{n}}_k))_+ \wedge S^n).$$

If we instead put the $A(\sqcup\vec{\mathbf{n}}_i)$ factors outside the function space we get the simplicial spectrum

$$\mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} F(S^{\sqcup\vec{\mathbf{n}}_0}\wedge\cdots\wedge S^{\sqcup\vec{\mathbf{n}}_k}, T(\sqcup_i(\sqcup\vec{\mathbf{n}}_i))\wedge S^n)\wedge(A(\sqcup\vec{\mathbf{n}}_1)\times\cdots\times A(\sqcup\vec{\mathbf{n}}_k))_+$$

and there is a canonical simplicial map from this to the above. We now want to separate the $A(\sqcup\vec{\mathbf{n}}_i)$-factors and we do this by mapping into the simplicial spectrum

$$\mathrm{hocolim}_{\mathcal{I}^{n(k+1)}\times\mathcal{I}^{nk}} F(S^{\sqcup\vec{\mathbf{m}}_0}\wedge\cdots\wedge S^{\sqcup\vec{\mathbf{m}}_k}, T(\sqcup_i(\sqcup\vec{\mathbf{m}}_i))\wedge S^n)\wedge(A(\sqcup\vec{\mathbf{n}}_1)\times\cdots\times A(\sqcup\vec{\mathbf{n}}_k))_+.$$

The map is induced by the functor from $\mathcal{I}^{n(k+1)}$ to $\mathcal{I}^{n(k+1)}\times\mathcal{I}^{nk}$ whose first factor is the identity and whose second factor is the projection away from the 0th $\mathcal{I}^n$-factor. Since based homotopy colimits commute with smash products, we see that this is isomorphic to the simplicial spectrum

$$\mathrm{hocolim}_{\mathcal{I}^{n(k+1)}} F(S^{\sqcup\vec{\mathbf{m}}_0}\wedge\cdots\wedge S^{\sqcup\vec{\mathbf{m}}_k}, T(\sqcup_i(\sqcup\vec{\mathbf{m}}_i))\wedge S^n)\wedge\hat{B}_k(A)_+.$$

It remains to identify the first factor, and we observe that this is a simplicial spectrum which in simplicial degree zero is the spectrum $\hat{\mathrm{THH}}_0(T)$ considered in Lemma 10.1. If we view the latter as a constant simplicial spectrum, we therefore get a map from the simplicial spectrum $\hat{\mathrm{THH}}_0(T)\wedge\hat{B}_\bullet(A)_+$. Finally, it follows from Lemma 10.1, that the latter is related to the simplicial spectrum $T\wedge\hat{B}_\bullet(A)_+$ by a chain of level-wise weak homotopy equivalences.

Proof of Theorem 10.2. It is straight forward to check that the maps defined above are all maps of commutative symmetric ring spectra, and except from the first step, it easily follows from Bökstedt's approximation lemma [20, 2.3.7], that they are weak homotopy equivalences. Thus, it remains to prove that the simplicial map induced by the map of Thom spaces (12) is a level-wise weak homotopy equivalence. We claim that there exists an unbounded non-decreasing sequence of integers λ_n, such that if each $\sqcup\vec{\mathbf{n}}_i$ has cardinality greater than or equal to n, then the connectivity of (12) is greater than or equal to $\lambda_n+\Sigma_i|\sqcup\vec{\mathbf{n}}_i|$. For this it suffices by the usual connectivity properties of the Thom space functor that there exists a sequence λ_n, such that the underlying map of spaces in the diagram (11) is at least λ_n-connected. Writing A' for the product of the spaces $A(\sqcup\vec{\mathbf{n}}_i)$ for $i\geq 1$, the induced map of homotopy groups is represented by a triangular matrix of the form

$$\begin{bmatrix} i & * \\ 0 & \mathrm{id}\end{bmatrix}:\ \pi_kA(\sqcup\vec{\mathbf{n}}_0)\times\pi_kA'\longrightarrow\pi_kA(\sqcup_i(\sqcup\vec{\mathbf{n}}_i))\times\pi_kA',$$

where i is the map induced by the inclusion of $\sqcup\vec{\mathbf{n}}_0$ in $\sqcup_i(\sqcup\vec{\mathbf{n}}_i)$. The claim now follows since A is assumed to be convergent, and we conclude by Bökstedt's approximation lemma [20, 2.3.7], that the induced map of simplicial spectra is a level-wise weak homotopy equivalence in each simplicial degree.

11. Computation of the THH of Eilenberg-MacLane spectra

By an old observation of Mahowald, we can realize certain Eilenberg-MacLane spectra as Thom spectra [21, 10]. In this section we discuss the application of elaborations of this observation to computations of THH of $H\mathbb{Z}/2$, $H\mathbb{Z}/p$ (for p odd), and $H\mathbb{Z}$.

First, recall that $H\mathbb{Z}/2$ is the Thom spectrum associated to a certain map $\alpha:\Omega^2S^3\to BO$. We can regard this map as obtained by looping down the generator

$S^3 \to BO$ in $\pi_3(BO)$. This is only a two-fold loop map, and as such we cannot directly apply our splitting result. Nonetheless, we can proceed as follows. Since $THH(H\mathbb{Z}/2)$ is a module over $H\mathbb{Z}/2$, it splits as a wedge of Eilenberg-MacLane spectra and therefore it will suffice to compute the homology in order to identify the homotopy type. Furthermore, we can work 2-locally. Thus, we can use the Thom isomorphism. Our identification of $THH(H\mathbb{Z}/2)$ as the Thom spectrum of a certain bundle over the free loop space $L(\Omega S^3)$ implies that there is an isomorphism

$$H_*(THH(H\mathbb{Z}/2);\mathbb{Z}/2) \cong H_*(L(\Omega S^3);\mathbb{Z}/2).$$

Using the standard splitting of the free loop space, we can reduce this to

$$H_*(\Omega S^3 \times \Omega^2 S^3;\mathbb{Z}/2) \cong H_*(H\mathbb{Z}/2 \wedge \Omega S^3;\mathbb{Z}/2).$$

Therefore we have the identification $THH(H\mathbb{Z}/2) \simeq H\mathbb{Z}/2 \wedge \Omega S^3$, as desired.

The calculation of $THH(H\mathbb{Z}/p)$, for p an odd prime, is substantially similar, once we have a construction of $H\mathbb{Z}/p$ as a Thom spectrum. One knows that any Thom spectrum over BF will necessarily have π_0 either $\mathbb{Z}$ or $\mathbb{Z}/2$ (depending on the existence of an orientation), and so we must work at the prime p. One can construct a space $BF_{(p)}$ which classifies p-local stable spherical fibrations by replacing self-equivalences of spheres with self-equivalences of p-local spheres in the definition of F, and it is an insight of Hopkins that $H\mathbb{Z}/p$ is the Thom spectrum associated to a certain map $\alpha_p : \Omega^2 S^3 \to BF_{(p)}$.

Specifically, we define the $\mathscr{I}_c$-space $F_{(p)}$ which takes V to the space of self-equivalences of the p-localization of S^V. Associated to this we have levelwise bar constructions defining $BF_{(p)}$ and $EF_{(p)}$, and a projection map $(\pi_V)_{(p)}\colon EF_{(p)} \to BF_{(p)}$ with fiber $(S^V)_{(p)}$. We can then carry out the construction of Thom spectra in this setting, and a similar argument to the one above for $p = 2$ (described in [22]) shows that we can obtain $H\mathbb{Z}/p$ as a Thom spectrum over $\Omega^2 S^3$. Note that one needs a continuous localization functor to make $F_{(p)}$ as an $\mathscr{I}_c$-FCP [28, 14].

Finally, we consider the case of $H\mathbb{Z}$. This can be modeled as a Thom spectrum over $\Omega^2 S^3 \langle 3\rangle$. However, the construction of $H\mathbb{Z}$ as a Thom spectrum studied in [10] does not have sufficient multiplicative structure; only an H-space structure is constructed. Nonetheless, in work of the first author [4] a new construction realizing $H\mathbb{Z}$ as an E_2-ring spectrum is given, and this suffices for us to perform a calculation analogous to the one given above. The end result is that $THH(H\mathbb{Z}) \simeq H\mathbb{Z} \wedge \Omega S^3 \langle 3\rangle$.

Remark. One might wonder about whether other Eilenberg-MacLane spectra, notably $H\mathbb{Z}/p^n$, can be realized as Thom spectra. Unfortunately, these cannot be recovered as Thom spectra with multiplicative structure in this fashion; a simple argument involving the Dyer-Lashof operations produces a contradiction. See [4] for a more comprehensive discussion.

The careful reader will note that the kind of multiplicative structure we are describing above is not in the form accepted by either of our implementations of the rigid Thom spectrum functor $\mathrm{T}_{\mathcal{A}}$. At least for the example of $H\mathbb{Z}/2$, one can explicitly construct input in the form accepted our various implementations as follows. Let $\mathcal{C}_2$ be an E_2-operad augmented over the linear isometries operad (e.g. take the product of the little 2-cubes operad and the linear isometries operad). This acts on BO, and we can realize α as the free $\mathcal{C}_2$ map associated to the generator

$S^1 \to BO$ of $\pi_1(BO)$. An identical argument also works for the Barratt-Eccles operadic model of BF.

Remark. As a historical note, it is interesting to recall that Lewis' original investigations into the behavior of the Thom spectrum functor with respect to multiplicative maps were spurred in part by Mahowald's curiosity about this example.

More generally, we can employ "change of operad" techniques. We will first discuss examples generated by "looping" maps to deloopings of BF as in the cases which give rise to Eilenberg-Mac Lane spectra, and then proceed to handle the general case. Above, the initial input was maps $X \to B^n(BF)$, which were looped down to produce n-fold loop maps $\Omega^n X \to \Omega^n B^n(BF)$. To specify the multiplicative structure carefully, we need to choose a precise model of the delooping B. Let us assume we are working with a specified choice of BF where the E_∞ structure is described by an action of the linear isometries operad $\mathcal{L}$. As will be clear, the argument is completely analogous in the case of an action by the Barratt-Eccles operad. By pullback, we regard this as a space structured by the product operad $\mathcal{C}_n \times \mathcal{L}$, where $\mathcal{C}_n$ is the little n-cubes operad. Denote by $\mathbb{D}$ the monad associated to this operad. Following [25, 13.1], we have the following diagram

$$X \xleftarrow{\simeq} B(\mathbb{D}, \mathbb{D}, X) \xrightarrow{\simeq} \Omega^n B(\Sigma^n, \mathbb{D}, X)$$

in which the maps are maps of $\mathbb{D}$-spaces, and the action of $\mathbb{D}$ on $\Omega^n\Sigma^n$ comes from the augmentation of $\mathbb{D}$ over the monad associated to the little n-cubes operad. The $\mathbb{D}$-space action on $\Omega^n B(\Sigma^n, D, X)$ is produced by pullback from the $\mathcal{C}_n$ action on $B(\Omega^n\Sigma^n, D, X)$. Thus, we use $B(\Sigma^n, D, BF)$ as our model of $B^n BF$.

Given a map $X \to B(\Sigma^n, D, BF)$, the associated map $\Omega^n X \to \Omega^n B(\Sigma^n, D, BF)$ is a map of $\mathbb{D}$-spaces with regard to the geometric action of the little n-cubes operad — and on $\Omega^n B(\Sigma^n, D, BF)$, this is precisely the action that arises in the diagram above. Replacing the map by a fibration and pulling back back, we get a map of $\mathbb{D}$-spaces $X' \to B(D, D, BF)$, and pushing forward we get a map of $\mathbb{D}$-spaces $X' \to BF$ where the $\mathbb{D}$ action on BF comes from the augmentation over the linear isometries operad. This is precisely the data we need to apply our implementations.

To extend to a more general case, it is useful to clarify the role of the bar construction that arises in the previous discussion. Given a map of operads $f\colon \mathcal{C} \to \mathcal{D}$, there is an associated map of monads $f\colon \mathbb{C} \to \mathbb{D}$. The categories of algebras over $\mathbb{C}$ and $\mathbb{D}$ are connected by an adjoint pair of functors (f_*, f^*). Here f^* is the pullback functor, and f_* is defined as a coequalizer which is typically denoted as a tensor product $\mathbb{C} \otimes_{\mathbb{D}} (-)$ [12]. In order to avoid the need for cofibrant replacement to control the left adjoint f_*, May introduced the bar construction (which can precisely be thought of as a homotopy colimit) $B(\mathbb{C}, \mathbb{D}, X)$. This gives homotopically well-behaved ways to change operad structures along maps of operads. In our case, there is the added wrinkle that we are working with parameterized spaces; this necessitates the use of "change of base" functors as well.

Thus, we assume that we have a space Z and an E_∞ operad $\mathcal{E}$ such that there is a zigzag of weak equivalences of operads from $\mathcal{E}$ to $\mathcal{L}$ and the image of Z under the associated change of operad functors is connected by a natural zig-zag of weak equivalences to our preferred model of BF. It is then straightforward to transfer an operadic map $X \to Z$ to an $\mathcal{L}$-map $X \to BF$; we leave the details to the interested reader.

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