GENERALIZED WITT SCHEMES AND ALGEBRAIC TOPOLOGY

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Abstract. The even-periodic cohomology of the space BU and some of its relatives are analyzed using the language of formal schemes as developed by Strickland. In particular, we connect $E^0(BU)$ to the theory of Witt vectors and λ-rings. We use these connections to study the effect of the coproduct arising from the tensor product and the Adams operations on generalized Chern classes. Then we exploit this connection to give an alternate construction of the Husemoller splitting of $E^0(BU)$.

1. Introduction

TO DO: Add a bit of history of the interplay between algebraic geometry and algebraic topology. Quillen->Morava->Landweber-Ravenel-Wilson->Hopkins->Lurie.

TO DO: Add a bit about how this is an introduction to these ideas at a less rigid level then that used by Lurie.

TO DO: Add that Witt vectors appear in many places in algebraic topology (Husemoller-Witt splitting, the $K-$theory of QS^0 (Hodgkin), classification of bicommutative Hopf algebras (Goerss), formulas for formal group laws (Ravenel) . What we seek to do is use the language of algebraic geometry to see how these notions are related.

This paper picks up a thread left by Ben-Zvi [BZ95] and a thread left by Strickland [Str00] and weaves them together. More specifically, in Ben-Zvi's minor thesis he observes that the cohomology ring $H^*(BU)$ is the ring of functions on the Cartier dual of the Witt scheme. Strickland develops the proper foundations for making these kind of connections in [Str00]. There he also remarks that the formal scheme associated to $E^0(\coprod_{n\geq 0} BU(n))$, for E an even-periodic cohomology theory, is a graded λ-semiring object. Using Strickland's framework and the algebra outlined in Ben-Zvi's minor thesis we will flesh out these remarks and connect them.

Individually the results below are in the literature although they may require some translation since they appear in different contexts. For example our form of the algebraic splitting principle, Theorem 6.5 can probably be constructed from different forms of the splitting principle that appear in the literature. We have made a signifiicant effort to cite relevant sources, but some of the topics discussed below have been studied by many people for quite some time. By juxtaposing results spread out across the literature we aim to clarify these connections.

We apply this framework to give low-dimensional formulas for the generalized Chern class of a tensor product of (stable) vector bundles in Section 10. The method of divining these formulas has been known to experts but has yet to appear in the literature.

First, we will recall the relevant material about schemes and formal schemes from [Str00]. Then we will recall the theory of λ-rings and Witt vectors which can be found in [BZ95, Haz78] or the recent survey [Haz08]. Then we will describe several alternate characterizations of the schemes associated to the generalized cohomology of classifying spaces for vector bundles.

This paper is written for the general algebraic topologist. We therefore assume familiarity with some category theory, especially representable functors and the Yoneda lemma and we assume some familiarity with the theory of vector bundles and classifying spaces as laid out in [Hus94, May99]. But we do not assume the reader is familiar with the basics of affine schemes (Section 2), rngs (Section 11.1) or ind-objects (Section 11.2). To keep this paper mostly self-contained we try to give a short treatment of these topics.

1.1. **Conventions.** All of our rings will be commutative and unital. The category of rings will be denoted $\mathcal{Ring}$.

We will assume all of our binary operations are commutative and associative and dually that all of our cooperations are cocommutative and coassociative.

We denote inverse limits by lim and directed limits by colim.

Unless otherwise specified, all schemes and formal schemes will be affine.

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2. Schemes

We are interested in studying the cohomology of nice spaces that admit an associative, commutative and unital product in the homotopy category. The cohomology rings of such spaces come equipped with a coassociative, cocommutative and counital coproduct. Being more comfortable with product structures we choose to work in the opposite category of rings or more precisely, the category of affine schemes. In addition to easing our study of the comultiplicative structure on these cohomology rings, schemes provide interesting alternative characterizations of these rings.

Recall that the essential image of an embedding $F : C \to D$ is the full subcategory of D whose objects are isomorphic to some object in the image of F.

The category of affine schemes, $\mathcal{Sch}$, is defined to be the essential image of the Yoneda embedding:

$$\begin{array}{rcl} \mathrm{Spec} : \mathcal{Ring}^{\mathrm{op}} & \longrightarrow & \mathcal{Set}^{\mathcal{Ring}} \\ \mathrm{Spec}(R) : S & \mapsto & \mathcal{Ring}(R, S). \end{array}$$

By definition, $\mathcal{Sch}$ is equivalent to $\mathcal{Ring}^{\mathrm{op}}$. The value of a scheme X on a ring S, is called the set of S-points of X and denoted $X(S) = \mathcal{Sch}(\mathrm{Spec}(S), X)$.

Example 2.1. The affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$, takes a ring to its underlying set. In other words, $\mathbb{A}^1$ is isomorphic to the forgetful functor from rings to sets.

Example 2.2. The scheme $\mathbb{A}^1 \setminus \{*\} \cong \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$, takes a ring to the set of units in that ring.

Example 2.3. The scheme $\mathrm{Nil}_n \cong \mathrm{Spec}(\mathbb{Z}[x]/(x^n))$, takes a ring R to the set of x in R, such that $x^n = 0$.

To discuss various algebraic categories in $\mathcal{Sch}$ (see Section 11.1), such as group schemes, we will need finite products. The product in schemes arises from the tensor product in rings. For example, affine n-space

$$\begin{array}{rcl} \mathbb{A}^n & \cong & \left(\mathbb{A}^1\right)^{\times n} \\ & = & \mathrm{Spec}\,\mathbb{Z}[t_1] \times \cdots \times \mathrm{Spec}\,\mathbb{Z}[t_n] \\ & \cong & \mathrm{Spec}(\mathbb{Z}[t_1] \otimes \cdots \otimes \mathbb{Z}[t_n]) \\ & \cong & \mathrm{Spec}\,\mathbb{Z}[t_1, \ldots, t_n]. \end{array}$$

Remark 2.4. In fact, $\mathcal{Sch}$ is complete and cocomplete because $\mathcal{Ring}$ is complete and cocomplete, however we must comment that the colimits in affine schemes constructed using this equivalence do not generally agree with those of the larger categories of non-affine schemes or $\mathcal{Set}^{\mathcal{Ring}}$. Since the reader might have little intuition for schemes, we will try to emphasize their role as set-valued functors and clarify the differences between these perspectives.

Later we will need to work with k-algebras, for k a ring, whose scheme theoretic analogues are schemes over $\mathrm{Spec}(k)$. The category of schemes over $\mathrm{Spec}(k)$ is equivalent to the essential image of the Yoneda embedding:

$$\mathrm{Spec}_k : k\text{–}\mathcal{alg}^{\mathrm{op}} \longrightarrow \mathcal{Set}^{k\text{–}\mathcal{alg}},$$

where $k\text{--}\mathcal{alg}$ is the category of k-algebras (i.e. rings under k). The category of schemes over $\operatorname{Spec}(k)$ will be denoted $\mathcal{Sch}_k$. Note that $\mathcal{Sch} \cong \mathcal{Sch}_{\mathbb{Z}}$.

Remark 2.5. In our examples we have elected to define our schemes over $\mathbb{Z}$, but we could just as easily define their analogues over an arbitrary base ring. Rather then clutter the notation we have elected to leave it to the reader to replace the integers with their preferred base whenever they see fit.

Tensor products over k give rise to products in $\mathcal{Sch}_k$ and they agree with those in $\mathcal{Set}^{k-\mathcal{alg}}$. Using this product structure we can perform the constructions in Section 11.1. In particular, we have the category of group schemes over $\operatorname{Spec} k$, $\mathcal{GrpSch}_k$. We should remark that group schemes are the dual of a more familiar concept:

Proposition 2.6. *The categories of bicommutative Hopf algebras over k and group schemes over k are antiequivalent.*

Example 2.7. The ring $\mathbb{Z}[x]$ with augmentation $\varepsilon_+ : \mathbb{Z}[x] \to \mathbb{Z}$ determined by $\varepsilon(x) = 0$ is an augmented $\mathbb{Z}$-algebra. The maps

$$\begin{array}{rclrcl} \epsilon_+ : \mathbb{Z}[x] & \to & \mathbb{Z} & x & \mapsto & 0 \\ \Delta_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x_1, x_2] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ \chi_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x] & x & \mapsto & -x \end{array}$$

make $\mathbb{Z}[x]$ into a cocommutative cogroup (i.e. a bicommutative Hopf algebra). Applying Spec to $\mathbb{Z}[x]$ and the above maps gives us the additive group scheme $\mathbb{G}_a$. We can identify $\mathbb{G}_a$ with the forgetful functor from rings to abelian groups.

Example 2.8. The ring $\mathbb{Z}[x, x^{-1}]$ with augmentation $\varepsilon_\times(x) = 1$ can be made into a cocommutative cogroup using the maps

$$\begin{array}{rclrcl} \epsilon_\times : \mathbb{Z}[x, x^{-1}] & \to & \mathbb{Z} & x & \mapsto & 1 \\ \Delta_\times : \mathbb{Z}[x, x^{-1}] & \to & \mathbb{Z}[x_1, x_2, x_1^{-1}, x_2^{-1}] & x & \mapsto & x_1 \otimes x_2 \\ \chi_\times : \mathbb{Z}[x, x^{-1}] & \to & \mathbb{Z}[x, x^{-1}] & x & \mapsto & x^{-1}. \end{array}$$

The corresponding group scheme $\mathbb{G}_m$ is called the multiplicative group scheme since it takes a ring to its group of units.

The language of Section 11.1 allows us to combine some of the structure in Example 2.8 with that of Example 2.7 to define the identity ring scheme.

Example 2.9. The ring $\mathbb{Z}[x]$ with augmentations ϵ_+ and $\epsilon_\times$ and comultiplications Δ_+ and $\Delta_\times$ equipped with the coinverse map χ_+ make $\mathbb{Z}[x]$ into a coring. By applying Spec we obtain the ring scheme Id, which takes a ring to itself.

3. Formal Schemes

Following [Str00], we define the category of formal schemes, $\mathcal{FSch}$, as the full subcategory of objects in $\mathcal{Set}^{\mathcal{Ring}}$ which are isomorphic to a cofiltered colimit of affine schemes. This category is equivalent the category of ind-schemes whose properties are discussed in Section 11.2. By identifying a scheme with a constant ind-scheme we can embed $\mathcal{Sch}$ as a full subcategory of $\mathcal{FSch}$.

Using the equivalence with Ind-schemes, $\mathcal{FSch}$ is equivalent to the opposite category of pro-rings. A pro-ring $R = \text{“}\lim\text{”} R_i$ can be identified with the topological ring $R' = \lim R_i$ where the inverse limit is taken in topological spaces and each R_i represents a discrete topological space [Joh82]. In this description a map of pro-rings corresponds to a continuous map. When the context is clear we will identify a topological ring with its associated pro-ring.

The equivalence $\mathrm{Pro}\,\mathcal{Ring}^{\mathrm{op}}$ to $\mathcal{FSch}$ is given by the functor $\mathrm{Spf} : \text{“}\lim\text{”} R_i \mapsto \operatorname{colim} \operatorname{Spec}(R_i)$.

Remark 3.1. This generalizes the definition of formal schemes in algebraic geometry. An affine formal scheme in that context is one of the form $\mathrm{Spf}(\hat{R})$, where $\hat{R} = \text{“}\lim\text{”} R/I^n$ ([Har77, Section II.9]). Each such formal scheme has a geometric interpretation that does not always hold in our category.

Example 3.2. The formal affine line

$$\begin{aligned} \hat{\mathbb{A}}^1 &= \mathrm{Spf}(\mathbb{Z}[[x]]) \\ &= \operatorname{colim} \operatorname{Spec}(\mathbb{Z}[x]/x^n) \\ &\cong \operatorname{colim} \mathrm{Nil}_n \\ &= \mathrm{Nil}, \end{aligned}$$

takes a ring to the set of nilpotent elements of that ring.

Analogous to the informal case, we have the category of formal schemes over a given (formal) scheme X, which we denote by $\mathcal{FSch}_X$.

Since colimits commute with (cofiltered) colimits we see that $\mathcal{FSch}$ is cocomplete and its colimits arise from those in schemes. Cofiltered colimits of formal schemes are preserved under the inclusion $\mathcal{FSch} \to \mathcal{Set}^{\mathcal{Ring}}$.

Since finite products commute with cofiltered colimits (see [Bor94, Theorem 2.13.4]), the products in schemes give rise to the products in formal schemes. For example, if $X = \operatorname{colim}_{i\in I} X_i$ and $Y = \operatorname{colim}_{j\in J} Y_j$ then

$$X \times Y \cong \operatorname*{colim}_{(i,j)\in I\times J} X_i \times Y_j.$$

Dually there is a coproduct on pro-rings, called the completed tensor product. For example, $\mathbb{Z}[[x]]\hat{\otimes}\mathbb{Z}[[y]] \cong \mathbb{Z}[[x,y]]$.

Now that we have finite products we can apply Section 11.1 and define formal groups, formal rings, etc.

Example 3.3. The additive formal group $\hat{\mathbb{G}}_a$ takes a ring R to $\mathrm{Nil}^+(R)$, the additive group of nilpotent elements of R. Clearly its underlying formal scheme is isomorphic to $\hat{\mathbb{A}}^1$. Fixing an isomorphism, then the additive group structure arises from the maps:

$$\begin{array}{rclcrcl} \epsilon_+ : \mathbb{Z}[[x]] & \to & \mathbb{Z} & & x & \mapsto & 0 \\ \Delta_+ : \mathbb{Z}[[x]] & \to & \mathbb{Z}[[x_1, x_2]] & & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ \chi_+ : \mathbb{Z}[[x]] & \to & \mathbb{Z}[[x]] & & x & \mapsto & -x. \end{array}$$

Example 3.4. The multiplicative formal group $\hat{\mathbb{G}}_m$ takes a ring R to the multiplicative group $(1 + \mathrm{Nil}(R))^\times$. Identifying this set with $\mathrm{Nil}(R)$ we see that $\hat{\mathbb{G}}_m$ can also be represented by $\hat{\mathbb{A}}^1$. Fixing an isomorphism, the group structure arises from the maps:

$$\begin{array}{rclcrcl} \epsilon_\times : \mathbb{Z}[[x]] & \to & \mathbb{Z} & & x & \mapsto & 0 \\ \Delta_\times : \mathbb{Z}[[x]] & \to & \mathbb{Z}[[x_1, x_2]] & & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 + x_1 \otimes x_2 \\ \chi_\times : \mathbb{Z}[[x]] & \to & \mathbb{Z}[[x]] & & x & \mapsto & \sum_{i \geq 0} (-1)^i x^{i+1}. \end{array}$$

4. Formal Schemes Arising from the Cohomology of a Space

Now we will try to realize the above formal schemes as arising from the cohomology rings of certain spaces. We are particularly interested in those spaces and cohomology theories that are connected to formal groups (see [Ada74, Hop99, Rav00]).

Notation 4.1. If E is a cohomology theory and X a space then $E^*(X)$ will always refer to the unreduced E-cohomology of X. The reduced cohomology theory will we be denoted $\tilde{E}^*(X)$.

Notation 4.2. For a multiplicative cohomology theory E, we will adopt the convention of writing E_* for $E^{-*}(S^0)$.

Definition 4.3. A cohomology theory E is called even-periodic if

(1) E is a multiplicative cohomology theory, i.e. E is a contravariant functor from spaces to graded rings.
(2) $E_{\mathrm{odd}} = 0$
(3) There exists a unit element $x \in E_2$.

The standard examples include even-periodic ordinary cohomology HPR, complex K-theory K, even-periodic Morava K-theory $\overline{K(n)}$, the Morava E-theories E_n, and even-periodic complex cobordism MP. We typically recognize these theories by their coefficient rings:

$$\begin{aligned} HPR_* &\cong R[v, v^{-1}] \\ K_* &\cong \mathbb{Z}[v, v^{-1}] \\ \overline{K(n)}_* &\cong \mathbb{F}_{p^n}[v, v^{-1}] \\ E_{n*} &\cong \mathbb{W}_p(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][v, v^{-1}] \text{ (see Section 8)} \\ MP_* &\cong \mathbb{Z}[b_1, b_2, \dots][v, v^{-1}], \end{aligned}$$

where the grading is determined by putting all of the generators in degree 0 except for v which lies in degree 2. A nice description of the properties of these cohomology theories can be found in [Rez98, Hop99].

Recall that the tensor product operation on vector bundles restricts to giving a group operation on isomorphism classes of line bundles, the unit coming from the one dimensional trivial bundle [1] and the inverse of a bundle η is given by the dual bundle η^*. Since $\mathbb{C}P^\infty$ is a model for $BU(1)$ the classifying space of 1-dimensional complex line-bundles we obtain a multiplication map

$$\mu_\otimes : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

that turns $\mathbb{C}P^\infty$ into a group object in hTop, the derived category of topological spaces.

Standard calculations (see [Hop99]) show that for an even-periodic cohomology theory E, we have

$$E^0(\mathbb{C}P^\infty) \cong E_0[[x]] \tag{4.1}$$

with the choice of isomorphism dependent on the choice of unit in Definition 4.3. We also have Kunneth isomorphisms

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E^0(\mathbb{C}P^\infty) \otimes_{E_0} \cdots \otimes_{E_0} E^0(\mathbb{C}P^\infty).$$

Fixing an isomorphism as in Equation 4.1 canonically determines an isomorphism

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E_0[[t_1, \dots, t_n]].$$

The map $\mu_\otimes$ gives rise to the coproduct

$$\begin{aligned} \Delta_\otimes : E^0(\mathbb{C}P^\infty) &\longrightarrow E^0(\mathbb{C}P^\infty) \otimes_{E_0} E^0(\mathbb{C}P^\infty) \\ E_0[[x]] &\longrightarrow E_0[[x, y]] \\ x &\mapsto F_E(x, y). \end{aligned}$$

The formal power series $F_E(x, y)$ is usually called the formal group law associated to E and the isomorphism, or orientation, in Equation 4.1. A different choices of isomorphism will give rise to a formal group law of the form $F_E(\lambda x, \lambda y)$, for some unit $\lambda \in E^0$.

Calling the power series $\Delta_\otimes(x) = F_E(x,y)$ a formal group law is a bit misleading, since the map $\Delta_\otimes$ really defines a cogroup object in pro-rings. Passing to the opposite category of formal schemes allows us to reverse the arrows and recover a group.

Definition 4.4. Given a CW-complex X and an even-periodic cohomology theory E, we define the formal scheme X_E associated to X and E by

$$X_E = \operatorname{colim} \operatorname{Spec} E^0(X_n),$$

where the filtered system defining the colimit is given by by the filtration of X by its finite CW-complexes.

Remark 4.5. Note that X_E is a covariant functor of X, making the notation convenient for studying diagrams of spaces. Also note X_E should not be confused with the Bousfield localization, $X_{\langle E\rangle}$, of X with respect to the cohomology theory E.

If X is a finite-dimensional CW-complex then X_E is defined by a finite directed system and is therefore isomorphic to the ordinary scheme $\operatorname{Spec} E^0(X)$.

Definition 4.6. The formal group associated to an even-periodic cohomology theory E, $\hat{\mathbb{G}}_E$ is the formal scheme $\mathbb{C}P^\infty_E$ over E_0, with the group structure induced by the tensor product of vector bundles.

By well-known calculations we can identify the formal group associated to K-theory with $\hat{\mathbb{G}}_m$ from Example 2.8 and the formal group associated to ordinary cohomology $HP\mathbb{Z}$ is $\hat{\mathbb{G}}_a$ from Example 2.7.

TO DO: For spaces X with even homology, we don't lose any information in passing from $E^*(X)$ to X_E, as long as we equip the formal scheme with $\mathbb{G}_m$ action to recover the grading (result of Strickland).

TO DO: For commutative H-groups X with $H_*(X)$ even and torsion free, we don't lose any information and we can apply Cartier duality.

5. Cartier Duality

Since our interest is actually in the ring that represents a given scheme, it is desirable to have a theory of duality for schemes that corresponds to taking the linear dual of the representing ring. Of course, for such a duality to exist we are going to need that the dual of the representing ring is another commutative unital ring, which means that the original scheme needed to be a group scheme. In order for the double dual of a group scheme X to be canonically isomorphic X, we are going to need that the representing ring be a dualizable module. Since we want our theory of duality to apply to formal schemes we are going to need some assumptions about the maps in pro-system that define the representing pro-ring. The classical theory of Cartier duality (see [Dem72]), once suitably extended as in [Str00, Section 6.4], suits our purposes.

Cartier duality is the the analogue of Pontryagin duality for group schemes. Classically, Cartier duality restricts to the case when we are working over a field and as a result can ignore some of the issues laid out above. But we will need algebraic restrictions. We start by considering a suitable category of objects dual to k-algebras. Identifying k-algebras with the category of commutative monoids in the category of k-modules using the tensor product structure (i.e use the theory of Section 11.1 with the Cartesian product replaced by the tensor product), we see the appropriate dual is the category of cocommutative comonoids in the category of k-modules or, equivalently, counital, cocommutative coalgebras.

Definition 5.1. Suppose U is a k-coalgebra free on a basis I. We can filter I by its finite subsets, which gives rise to a filtration on U. We can also filter U by its finitely generated submodules. If this filtration is cofinal in the filtration coming from I then we say that I is a *good basis* for U. Those coalgebras which admit a good basis will be called *basic*. The full subcategory of coalgebras of basic coalgebras will be denoted $\mathcal{BCoAlg}$.

To a basic coalgebra $U = \operatorname{colim} U_i$, we define the formal scheme $\mathrm{Sch}U = \operatorname{colim} \operatorname{Spec} U_i^*$, where U_i^* is the linear dual $\mathrm{Mod}_k(U_i, k)$, which inherits its multiplicative structure from the coalgebra structure on U_i (see [Str00, 4.59]). This gives a functor from basic coalgebras to formal schemes whose image $\mathcal{CFSch}$, we call *coalgebraic formal schemes*.

We can construct an inverse functor $c : \mathcal{CFSch} \to \mathcal{BCoAlg}$ given by

$$c(\operatorname{colim} \operatorname{Spec} U_i^*) = \operatorname{colim} U_i^{**} \cong \operatorname{colim} U_i.$$

If our coalgebra U has the additional multiplicative structure making it a (commutative) Hopf algebra then we can build the group scheme $\operatorname{Spec} U$ or the formal group scheme $\mathrm{Sch}U$. Given a formal coalgebraic group scheme $\hat{\mathbb{G}} \cong \operatorname{colim} \operatorname{Spec} U_i^*$, we define the Cartier dual to be

$$\begin{aligned} D\hat{\mathbb{G}} &= \operatorname{Spec} c\hat{\mathbb{G}} \\ &\cong \operatorname{Spec} \operatorname{colim} U_i \\ &\cong \operatorname{Spec} U. \end{aligned}$$

Restricting to those coalgebraic formal schemes that are actually informal schemes we can apply D again to get

$$\begin{aligned} DD\hat{\mathbb{G}} &= D\operatorname{Spec} c\hat{\mathbb{G}} \\ &= c\operatorname{Spec} c\hat{\mathbb{G}} \\ &\cong c\operatorname{Spec} U \\ &\cong \operatorname{colim}\operatorname{Spec} U_i^* \\ &\cong \hat{\mathbb{G}}. \end{aligned}$$

This gives us a well-behaved duality on group schemes that has the effect of taking the linear dual on the representing rings.

Example 5.2. The linear dual of the the truncated polynomial algebra $\mathbb{Z}[x]/(x^n - 1)$ with x grouplike (i.e.$\Delta x = x \otimes x$) is the module

$$\bigoplus_{i=0}^{n-1} \mathbb{Z} e_i$$

with coproduct

$$\Delta e_k = \sum_{i=0}^{n-1} e_i \otimes e_{\sigma(i)}$$

where $\sigma(i) \equiv k - i \pmod{n}$ and $0 \leq \sigma(i) < n$. The algebra structure is determined by the relations $e_i e_j = \delta_{ij} e_i$. We can see $\operatorname{Spec}\mathbb{Z}[x]/(x^n - 1)$ is the group scheme whose R-points is the multiplicative group of nth roots of unity in R (which might be trivial for a given R), while its Cartier dual is the constant functor $R \to \mathbb{Z}/n$.

TO DO: For nice enough spaces we have $DX_E = \operatorname{Spec} E_0 X$. So, Cartier duality relates cohomology to homology.

TO DO: Cite Strickland's result connecting this definition of Cartier duality with that algebraic geometers are more comfortable with.

6. Symmetric Schemes

We can identify a *split* monic polynomial

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} b_{n-i} x^i \\ &= \prod_{i=0}^{n} (x - t_i) \end{aligned}$$

with coefficients in R with its *unordered* set of roots $\{t_1, \ldots, t_n\}$. Let the nth splitting functor Split_n denote the functor from rings to sets that takes a ring R to the set of split monic polynomials with coefficients in R or, alternatively, the corresponding sets of roots of those polynomials.

Remark 6.1. Note that Split_n is not a scheme, affine or otherwise, although it is related to a stack.

Let the nth representable splitting functor rSplit_n be the affine scheme determined by

$$\mathrm{rSplit}_n(R) = \left\{ f(x) = \prod_{i=1}^{n}(x - t_i),\ t_i \in R \right\}.$$

Clearly rSplit_n is isomorphic to affine n-space and we have a natural transformation $\mathrm{rSplit}_n \to \mathrm{Split}_n$ where we forget the ordering of the roots.

If a natural transformation $\mathrm{rSplit}_n \to \mathrm{rSplit}_m$ descends to give a natural transformation $f : \mathrm{Split}_n \to \mathrm{Split}_m$, we shall say f is *algebraic*. For example, we have the map $F_k : \mathrm{rSplit}_n \to \mathrm{rSplit}_{nk}$ which sends $(t_1, \dots, t_n)$ to $([k]t_1, \dots, [k]t_n)$, where $[k]t_i$ indicates repeat the root t_i k-times. Passing from $([k]t_1, \dots, [k]t_n)$ to $\{[k]t_1, \dots, [k]t_n\}$ we see that reordering the t_i's does not change the target set, so the map F_k descends to give an algebraic map. This gives a well-defined natural transformation that induces a map of monic polynomials

$$F_k : \sum_{i=0}^{n} b_{n-i}x^i \to \sum_{i=0}^{nk} b_{nk-i}x^i. \tag{6.1}$$

The coefficient b_i of the polynomial $f(x)$ can be identified (up to sign) with the elementary symmetric function $(-1)^i\sigma_i(t_1, \dots, t_n)$ in the roots $t_1, \dots, t_n$, where the elementary symmetric functions $\sigma_i(t_1, \dots, t_n)$ are given by the generating function

$$\sum_{i=0}^{n} \sigma_{n-i}x^i = \prod_{i=1}^{n}(x + t_i).$$

We see that if $i \le n$ then $\sigma_i(t_1, \dots, t_n) = \sigma_i(t_1, \dots, t_n, 0)$, giving us well-defined elementary symmetric functions σ_i on "enough" variables. The following theorem of Newton will be quite useful:

Theorem 6.2. *There is an isomorphism of algebras* $R[\sigma_1, \dots, \sigma_n] \cong R[t_1, \dots, t_n]^{\Sigma_n}$.

Proof. See [Hus94]. □

Since the coefficients b_i in Equation 6.1 are symmetric in $t_1, \dots, t_n$ they can be expressed as polynomials in the elementary symmetric functions or, equivalently, as polynomials $p_i(b_1, \dots, b_n)$ in the coefficients of f. We can now forget our splitting and just use the polynomials p_i to define operations on monic polynomials.

Definition 6.3. The nth symmetric scheme $\mathrm{symm}_n \cong \mathrm{Spec}(\mathbb{Z}[b_1, \dots, b_n])$ is the scheme with

$$\mathrm{symm}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i}x^i \mid b_i \in R, b_0 = 1 \right\}$$

and the nth formal symmetric scheme $\hat{\mathrm{symm}}_n \cong \mathrm{Spf}(\mathbb{Z}[[b_1, \ldots, b_n]])$ is the formal scheme with

$$\hat{\mathrm{symm}}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i}x^i \mid b_0 = 1 \text{ and } \ b_i \in \mathrm{Nil}(R) \text{ for } i > 0 \right\}.$$

Definition 6.4. The nth formal splitting functor $\hat{\mathrm{Split}}_n$ is the functor which has values

$$\hat{\mathrm{Split}}_n(R) = \left\{ f(x) \in \hat{\mathrm{symm}}_n(R) \mid f(x) = \prod_{i=1}^{n} (x - t_i), \ t_i \in \mathrm{Nil}(R) \right\}.$$

Theorem 6.5 (Algebraic Splitting Theorem)**.** *An algebraic transformation of functors* $\mathrm{Split}_n \to \mathrm{Split}_k$ *(respectively* $\hat{\mathrm{Split}}_n \to \hat{\mathrm{Split}}_k$*) determines a map of schemes* $\mathrm{symm}_n \to \mathrm{symm}_k$ *(respectively a map of formal schemes* $\hat{\mathrm{symm}}_n \to \hat{\mathrm{symm}}_k$*).*

An algebraic transformations of functors $\mathrm{Split}_i \times \mathrm{Split}_j \to \mathrm{Split}_k$ *(respectively* $\hat{\mathrm{Split}}_i \times \hat{\mathrm{Split}}_j \to \hat{\mathrm{Split}}_k$*) determines a map of schemes* $\mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_k$ *(respectively a map of formal schemes* $\hat{\mathrm{symm}}_i \times \hat{\mathrm{symm}}_j \to \hat{\mathrm{symm}}_k$*).*

Two maps $f_1, f_2 : \mathrm{symm}_m \to \mathrm{symm}_k$ *are equal if and only if* $f_1 c_m = f_2 c_m$ *where* $c_m : \mathrm{rSplit}_m \to \mathrm{symm}_m$ *is the forgetful map. Similarly for maps involving products of the (formal) symmetric schemes.*

Proof. The argument that F_k induces a natural transformation $\mathrm{symm}_n \to \mathrm{symm}_{nk}$ given above goes through *mutatis mutandis.* Namely, in each of these cases we see that the natural transformations on split monic polynomials define polynomial maps on the coefficients which allow us to define natural transformations on monic polynomials.

The last claim follows from the fact that c_m corresponds to the injective map on representing rings

$$\begin{array}{rcl} \mathbb{Z}[b_1, \ldots, b_m] & \to & \mathbb{Z}[t_1, \ldots, t_m] \\ b_i & \mapsto & \sigma_i(t_1, \ldots, t_m). \end{array}$$

□

The last claim allows us to deduce relations between maps by checking them on the representable (formal) splitting functors.

This allows us to define a panoply of natural transformations. Unless we say otherwise, for each of the following definitions there is an analogous “hatted” version.

Definition 6.6. The natural transformation $\oplus_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_{i+j}$ is defined by operation

$$\begin{array}{rcl} \oplus_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j & \longrightarrow & \mathrm{Split}_{i+j} \\ S \times T & \mapsto & S \coprod T. \end{array}$$

In this case we can also describe the operation explicitly as

$$\begin{array}{rcl} \mathrm{symm}_i \times \mathrm{symm}_j & \to & \mathrm{symm}_{i+j} \\ f(x) \times g(x) & \mapsto & f(x)g(x). \end{array}$$

Definition 6.7. Given an operation $\mu : \mathrm{Split}_1 \times \mathrm{Split}_1 \to \mathrm{Split}_1$ (or equivalently $\mathrm{symm}_1 \times \mathrm{symm}_1 \to \mathrm{symm}_1$) we define operations $\mu_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_{ij}$ determined by

$$\begin{array}{rcl} \mu_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j & \to & \mathrm{Split}_{ij} \\ S \times T & \mapsto & \displaystyle\coprod_{(s,t) \in S \times T} \mu(s,t). \end{array}$$

In the case $\mu(r,t) = rt$ we denote the operation $\mu_{i,j}$ as $\otimes_{i,j}$.

Theorem 6.8. *A set of operations $\mu_{i,j}$ defined as in Definition 6.7 distribute over $\oplus_{i,j}$*

Definition 6.9. We define the map $i_0 : \mathrm{symm}_0 \longrightarrow \mathrm{symm}_1$ or equivalently ($i_0 : \mathrm{Split}_0 \to \mathrm{Split}_1$) by $* \mapsto \{0\}$. Similarly we have a map $i_1 : \mathrm{symm}_0 \to \mathrm{symm}_1$ (but not a map from $\mathrm{sy\hat{m}m}_0 \to \mathrm{sy\hat{m}m}_1$) defined by $* \mapsto \{1\}$.

We have maps $\iota : \mathrm{sy\hat{m}m}_n \to \mathrm{sy\hat{m}m}_{n+1}$ defined by the composite

$$\mathrm{sy\hat{m}m}_n \cong \mathrm{sy\hat{m}m}_n \times \mathrm{sy\hat{m}m}_0 \xrightarrow{id \times i_0} \mathrm{sy\hat{m}m}_n \times \mathrm{sy\hat{m}m}_1 \xrightarrow{\oplus_{n,1}} \mathrm{sy\hat{m}m}_{n+1}.$$

This map can also be described by $f(x) \mapsto xf(x)$. Taking a colimit of these maps inverts x, since the colimit of formal schemes agrees with the colimit in $\mathcal{Set}^{\mathcal{Ring}}$ we obtain:

Definition 6.10. The formal scheme $\mathrm{sy\hat{m}m}^0 = \operatorname{colim} \mathrm{sy\hat{m}m}_n \cong \operatorname{Spf} \mathbb{Z}[[b_1, b_2, \dots]]$, is the formal scheme with

$$\mathrm{sy\hat{m}m}^0(R) = \left\{ f(z) = \sum_{i=0}^{n} b_i z^i \mid b_0 = 1,\ n \in \mathbb{N},\ b_i \in \mathrm{Nil}(R) \text{ for } i > 0 \right\}.$$

The inclusions $\hat{\text{symm}}_n \to \hat{\text{symm}}^0$ take $f(x) \mapsto x^{-n}f(x)$ and then we get a polynomial of the above form by setting $z = x^{-1}$. The compatible system of maps

$$\begin{tikzcd} \hat{\text{symm}}_i \times \hat{\text{symm}}_j \arrow[r, "\oplus_{i,j}"] \arrow[d, "\iota\times\iota"] & \hat{\text{symm}}_{i+j} \arrow[d, "\iota\circ\iota"] \\ \hat{\text{symm}}_{i+1} \times \hat{\text{symm}}_{j+1} \arrow[r, "\oplus_{i+1,j+1}"] & \hat{\text{symm}}_{i+j+2} \end{tikzcd}$$

defines an operation $\oplus : \hat{\text{symm}}^0 \times \hat{\text{symm}}^0 \to \hat{\text{symm}}^0$ which we combine with $i_0 : \hat{\text{symm}}_0 \to \hat{\text{symm}}_1 \to \hat{\text{symm}}^0$ to make $\hat{\text{symm}}^0$ a formal group scheme.

TO DO: Provide argument for group structure.

TO DO: In aff products commute with finite coproducts. Finite products commute with filtered colimits. This allows us to use the above maps to define:

Definition 6.11. The positive symmetric scheme is the scheme $\hat{\text{symm}}^+ = \coprod_{i\geq 0} \hat{\text{symm}}_i = \operatorname{colim} \coprod_{0\leq i\leq n} \hat{\text{symm}}_i$equipped with the semiring structure defined by the maps $\oplus_{i,j}$ and $\otimes_{i,j}$ with the additive identify given by the inclusion $\hat{\text{symm}}_0 \to \hat{\text{symm}}^+$ and the multiplicative identify given by i_0 followed by the inclusion $\hat{\text{symm}}_1 \to \hat{\text{symm}}^+$.

TO DO: Define sym as colim of sym_+ where the maps are "add root 0".

TO DO: Show sym^0 is the 0 component of sym and therefore inherits a multiplication.

TO DO: There's nothing special about this multiplication. We can take any group operation on sym_1 with unit i_0 and extend it to define semiring schemes, ring schemes and rng schemes.

Theorem 6.12. *A formal group structure* $F : \hat{\text{symm}}_1 \times \hat{\text{symm}}_1 \to \hat{\text{symm}}_1$ *determines a map*

$$F^s : \hat{\text{symm}}^0 \times \hat{\text{symm}}^0 \to \hat{\text{symm}}^0$$

that makes $\hat{\text{symm}}^0$ *into a formal rng scheme (see Section 11.1).*

Proof. Using Definition 6.7 we can define multiplications $F_{i,j} : \hat{\text{symm}}_i \times \hat{\text{symm}}_j \to \hat{\text{symm}}_{ij}$. □

Theorem 6.13. *A formal group structure* $F : \hat{\text{symm}}_1 \times \hat{\text{symm}}_1 \to \hat{\text{symm}}_1$ *determines a map*

$$F^s : \hat{\text{symm}}^0 \times \hat{\text{symm}}^0 \to \hat{\text{symm}}^0$$

that makes $\hat{\text{symm}}^0$ *into a formal rng scheme (see Section 11.1).*

Proof. Using Definition 6.7 we can define multiplications $F_{i,j} : \hat{\text{symm}}_i \times \hat{\text{symm}}_j \to \hat{\text{symm}}_{ij}$. □

TO DO: Define Frobenius and Verschiebung maps and describe them on the split schemes.

TO DO: State that the we can realize some of the "twisted" symmetric schemes defined by the theorem above using the schemes BU_E and their relatives.

7. Lambda Schemes

In this section we will examine the scheme Λ and its dual. The scheme Λ plays an important role in the theory of λ-rings which are important structures in representation theory and algebraic topology, e.g.; see [AT69, Knu73]).

Definition 7.1. The Lambda-scheme Λ is the ring scheme whose underlying additive group scheme is defined by $\Lambda(R) = (1 + tR[[z]])^\times$. The multiplicative structure is more complicated and we will explain it below. This scheme can be represented by the ring $\mathrm{Sym} = \mathbb{Z}[b_1, b_2, \ldots]$, since a homomorphism $f : \mathrm{Sym} \longrightarrow R$ is determined by where the b_i are mapped to under f. These elements determine a power series

$$\sum_{i \geq 0} f(b_i) z^i, \tag{7.1}$$

where we adopt the useful convention $b_0 = 1$ and therefore $f(b_0) = 1$. Under this correspondence the additive group is described by

$$\begin{array}{rclcrcl} \epsilon_+ : \mathrm{Sym} & \to & \mathbb{Z} & & b_n & \mapsto & 0 \\ \Delta_+ : \mathrm{Sym} & \to & \mathrm{Sym} \otimes \mathrm{Sym} & & b_n & \mapsto & \sum_{i=0}^{n} b_i \otimes b_{n-i} \\ \chi_+ : \mathrm{Sym} & \to & \mathrm{Sym} & & b_n & \mapsto & -\sum_{i=0}^{n-1} \chi_+(b_i) b_{n-i} \end{array}$$

for all $n \geq 1$.

.

Since Λ takes values in formal rings we might expect it to be an inverse limit of regular schemes and in fact it is. Since $\mathrm{Sym} \cong \operatorname{colim} \mathrm{Sym}_n$ where $\mathrm{Sym}_n = \mathbb{Z}[b_1, \ldots, b_n]$, we obtain

$$\mathrm{symm} = \operatorname{Spec} \mathrm{Sym} = \mathcal{Ring}(\operatorname{colim} \mathrm{Sym}_n, -) \cong \lim \mathcal{Ring}(\mathrm{Sym}_n, -) = \lim \operatorname{Spec} \mathrm{Sym}_n.$$

While it is clear that the Λ is an informal analogue of $\mathrm{sy\hat{m}m}^0$, it is surprising that

Theorem 7.2. *The Cartier dual of the* $\mathrm{sy\hat{m}m}^0$ *is* Λ.

Proof. TODO: It's preferable to find a source for this.

To underscore this relationship we will write $\hat{\Lambda} = \mathrm{sy\hat{m}m}^0$. □

I'll show that the Frobenius and Verschiebung are switched in the dual.

8. Witt Schemes

Witt schemes appear in many areas of mathematics, from starring roles in the classification of commutative group schemes and p-divisible groups ([Dem72]), to class field theory for fields of characteristic p (Witt's original purpose) and to cameo appearances in the Teichmuller embedding of finite fields into rings of characteristic zero. The role that Witt schemes (or more precisely, the truncated Witt schemes), play in commutative group schemes is reflected in the classification of bicommutative Hopf algebras and their characterization by Dieudonne modules ([Goe98, Sch70]) . Their role in constructing characteristic 0 lifts of finite fields provokes them to appear in the construction of Landweber exact formal group laws and their associated cohomology theories ([Rez98]).

The Witt scheme is a ring scheme whose underlying scheme is isomorphic to $\mathbb{A}^\infty$, just like Λ. In fact, there is an isomorphism of ring schemes between them. We will exploit this fact to circumvent defining the Witt scheme's ring structure directly and save us the trouble of restating a number integrality lemmas (see [Haz78]). On representing rings, this isomorphism reflects a different choice of generators which are more convenient for some purposes. For example, the formulas for the primitive elements are simpler and satisfy some useful congruences. In Section 10 we will give formulas relating the choices of generators.

Definition 8.1. The Witt scheme $\mathbb{W}$ has the underlying scheme $\mathrm{Spec}(\mathbb{Z}[\theta_1, \theta_2 \dots])$, and a ring scheme structure will be given in Corollary 8.3. We identify an element of $f \in \mathbb{W}(R)$ with the power series

$$\prod (1 - f(\theta_n) t^n)^{-1} = 1 + p_1(f)t + p_2(f)t^2 + \cdots \in 1 + tR[[t]].$$

For example,

$$\begin{aligned} p_1(f) &= f(\theta_1) \\ p_2(f) &= f(\theta_1)^2 + f(\theta_2) \\ p_3(f) &= f(\theta_1)^3 + f(\theta_1)f(\theta_2) + f(\theta_3). \end{aligned}$$

Examining these formulas for the coefficients and comparing them to Equation 7.1 defining the R-points of Λ, we can find a formula for $f(\theta_i)$ in terms of the $f(b_i)$, and conversely, inductively. This leads us to the following theorem (cf. [Haz78]):

Theorem 8.2. *There is an isomorphism of schemes* $\mathbb{W} \cong \Lambda$.

Proof. By the Yoneda lemma, the maps from $\mathbb{W}$ to Λ are in bijection with $\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots]) \cong \mathcal{Ring}(\mathbb{Z}[b_1, b_2, \dots], \mathbb{Z}[\theta_1, \theta_2, \dots])$. The power series

$$\prod (1 - \theta_n t^n)^{-1}$$

defines an element of $\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots])$ and hence a map f. This map gives rise to maps $f_n : \mathbb{Z}[b_1, \dots, b_n] \to \mathbb{Z}[\theta_1, \dots, \theta_n]$. Each of these algebras admits an augmentation that sends each of the polynomial generators to 0. The induced map on indecomposables

$$f_n : (b_1, \dots, b_n)/(b_1, \dots, b_n)^2 \to (\theta_1, \dots, \theta_n)/(\theta_1, \dots, \theta_n)^2$$

is an isomorphism since modulo decomposables

$$\begin{aligned} \prod_{1 \le i \le n} (1 - \theta_i t^i)^{-1} &\equiv \prod_{1 \le i \le n} (1 + \theta_i t^i) \\ &\equiv 1 + \sum_{1 \le i \le n} \theta_i t^i. \end{aligned}$$

Filtering by the augmentation ideal this induces an isomorphism on the associated graded algebras $E^0(\mathbb{Z}[b_1, \dots, b_n]) \to E^0(\mathbb{Z}[\theta_1, \dots, \theta_n])$. With such a filtering the associated graded of a polynomial algebra is the algebra itself so this gives us an isomorphism between the truncated algebras $\mathbb{Z}[b_1, \dots, b_n]$ and $\mathbb{Z}[\theta_1, \dots, \theta_n]$. Taking colimits over the truncated algebras gives the desired isomorphism. $\square$

Corollary 8.3. *$\mathbb{W}$ can be given the structure of a ring-scheme such that the map in Theorem 8.2 is an isomorphism of ring-schemes.*

Remark 8.4. We can consider this to be a definition of the ring-scheme structure on $\mathbb{W}$.

Corollary 8.5. *There is an isomorphism of formal group schemes $\hat{\mathbb{W}} \cong \mathrm{sy\hat{m}m}^0$.*

TO DO:I will describe how the Frobenius and Verschiebung act in this context.

TO DO: We'll show that the identification BU_E with $\hat{\mathbb{W}}_E$ when E is p-local gives a generalized form of the splitting result in [Hus71] with significantly less fuss.

9. Adams Operations, The Frobenius Map and J.

Here we identify the Frobenius operation on our Witt scheme with the Adams operation thought of as a map from BU to itself.

Next we will recall the work of [RWY98], which shows that applying E-cohomology (for E satisfying their reasonable restrictions) to the p-local fibration sequence:

$$J \to BU^{\oplus} \xrightarrow{\psi^l - 1} BU^{\oplus}$$

yields a short exact sequence of Hopf algebras. We recharacterize this as saying we have a short exact sequence of abelian group schemes. Using our description of ψ^l on $\hat{\mathbb{W}}$ we characterize the kernel J_E.

Now I will take $E = E_1$ which is p-complete K-theory and add the d-invariant $QS^0_0 \to BU^{\oplus}$ to the above diagram. Then we'll rephrase the results of [Hod72,

$$\begin{tikzcd} X \times X \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & X \times X \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

FIGURE 11.1. Commutativity Relation

$$\begin{tikzcd} (X \times X) \times X \arrow[rr, "\sim"] \arrow[d, "a \times Id_X"'] & & X \times (X \times X) \arrow[d, "Id_X \times \mu"] \\ X \times X \arrow[r, "\mu"] & X & X \arrow[l, "\mu"'] \end{tikzcd}$$

FIGURE 11.2. Associativity Relation

HS78] in our context. I haven't done this calculation yet, but I want to see how the formulas which Hodgkin uses for the elements in $R(\Sigma_\infty)$ that survive completion relate to the formulas for the generators of $\hat{\mathbb{W}}$. His formulas are similar to those defining the ghost components of the Witt vectors and there is likely to be a relationship. The only potential hitch is that he is studying $QS^0 \to BU \times \mathbb{Z}$ and this fudges many of the formulas.

10. Formulas and $c(\xi_1 \otimes \xi_2)$

Give sample formulas for the various coordinate changes.

Recall how Chern classes can be defined on universal bundles and for virtual bundles.

Give some low level formulas for the Chern class of a tensor product of two bundles.

11. Appendix

11.1. **Universal Algebra.** Suppose $\mathcal{C}$ is a category with finite products. The empty product will be denoted $*$.

A binary operation $\mu : X \times X \to X$ is commutative if the diagram in Figure 11.1 commutes, where τ is the isomorphism that interchanges the two products. We will say that μ is associative if Figure 11.2 commutes, where $a : (X \times X) \times X \to X \times (X \times X)$ is the canonical associativity isomorphism. A map $\eta : * \to X$ is a unit map for the operation μ if the diagram in Figure 11.3 commutes. A unital binary operation admits an inverse map $\chi : X \to X$, if the diagram in Figure 11.4 commutes. *From here on each binary operation is assumed to be associative and commutative.*

$$\begin{tikzcd}
* \times X \arrow[r, "\eta \times Id_X"] & X \times X & X \times * \arrow[l, "Id_X \times \eta"'] \\
& X \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

FIGURE 11.3. Unit Relation

$$\begin{tikzcd}
X \arrow[rr, "Id_X \times \chi"] & & X \times X \arrow[d, "\mu"'] \\
X \arrow[u, "\Delta"'] \arrow[r] & * \arrow[r, "\eta"] & X
\end{tikzcd}$$

FIGURE 11.4. Inverse Relation

$$\begin{tikzcd}
X \times X \times X \times X \arrow[r, "\tau \times Id_X \times Id_X"] & X \times X \times X \times X \arrow[r, "\mu_\times \times \mu_\times"] & X \times X \arrow[d, "\mu_+"] \\
X \times X \times X \arrow[u, "\Delta \times Id_X \times Id_X"] \arrow[r, "Id_X \times \mu_+"] & X \times X \arrow[r, "\mu_\times"] & X
\end{tikzcd}$$

FIGURE 11.5. Distributivity Relations

Given a binary operation μ_+ then we say that $\mu_\times$ distributes over μ_+ if the diagram in Figure 11.5 commutes.

The category $\mathcal{Based}_\mathcal{C}$ of based objects in $\mathcal{C}$ is the category of objects under $*$. In other words, the object (X, η) of $\mathcal{Based}_\mathcal{C}$ is an object X of $\mathcal{C}$ equipped with a map $\eta : * \to \mathcal{C}$. The morphisms from (X, η_1) to (Y, η_2) correspond to commuting diagrams of the form:

$$\begin{tikzcd}
& * \arrow[dl, "\eta_1"'] \arrow[dr, "\eta_2"] & \\
X \arrow[rr] & & Y
\end{tikzcd}$$

Since $*$ is terminal, $\mathcal{Based}_\mathcal{C}$ is tautologically equivalent to the category of objects over and under $*$ (defined similarly).

The category $\mathcal{SemiGp}_\mathcal{C}$ of semigroups in $\mathcal{C}$, is the category whose objects are pairs (X, μ) where μ is a binary operation on X, and whose morphisms from (X, μ_1) to (Y, μ_2) are given by a morphism $f : X \to Y$ in $\mathcal{C}$ such that $f\mu_1 = \mu_2(f \times f)$.

The category $\mathcal{Mon}_\mathcal{C}$ of monoids in $\mathcal{C}$, is the category whose objects are tuples (X, μ, η) such that η is unit with respect to μ and whose morphism make the appropriate diagrams commute.

The category $\mathcal{Gp}_{\mathcal{C}}$ of (abelian) groups in $\mathcal{C}$, is the category whose objects are tuples (X, μ, η, χ) such that η and χ are unit and inverse maps with respect to μ and whose morphisms make the appropriate diagrams commute.

The category $\mathcal{Rig}_{\mathcal{C}}$ of rigs in $\mathcal{C}$ (Rings without negatives), is the category whose objects are tuples $(X, \mu_+, \eta_+, \mu_\times, \eta_\times)$ such that: η_+ is the unit with respect to μ_+, $\eta_\times$ is the unit with respect to $\mu_\times$, $\eta_\times$ distributes over η_+, and whose morphisms make the appropriate diagrams commute.

The category $\mathcal{Rng}_{\mathcal{C}}$ of rngs (pronounced rüngs) in $\mathcal{C}$ (Rings without units), is the category whose objects are tuples $(X, \mu_+, \eta_+, \chi_+, \mu_\times)$ such that: η_+ and χ_+ are the unit and inverse maps with respect to μ_+, $\mu_\times$ distributes over μ_+, and whose morphisms make the appropriate diagrams commute.

The category $\mathcal{Ring}_{\mathcal{C}}$ of rings in $\mathcal{C}$ (Rings without units), is the category whose objects are tuples $(X, \mu_+, \eta_+, \chi_+, \mu_\times)$ such that: η_+ and χ_+ are the unit and inverse maps with respect to μ_+, $\eta_\times$ is the unit with respect to $\mu_\times$, $\mu_\times$ distributes over μ_+, and whose morphisms make the appropriate diagrams commute.

These categories are connected by an obvious diagram of forgetful functors.

There is an analogous dual picture making use of a coproduct on $\mathcal{C}$, in which we can define the notion of a comonoid, cogroup, etc.

11.2. **Ind-Objects and Pro-Objects.** A more thorough treatment of ind-objects and pro-objects can be found in [Joh82, Gro64].

Definition 11.1. A small category $\mathcal{D}$ is called *cofiltered* if

(1) $\mathcal{D}$ is non-empty.
(2) For every $X, Y \in \mathcal{D}$, there exists an object $Z \in \mathcal{D}$ and morphisms $f : Z \to X$ and $g : Z \to Y$.
(3) For every two arrows $f, g : X \to Y$ there exists an object $Z \in \mathcal{D}$ and a morphism $h : Z \to X$ such that $fh = gh$.

It follows immediately from the definition that any product of cofiltered categories is cofiltered.

Given a category $\mathcal{C}$ with small hom-sets and a functor $F : \mathcal{D} \to \mathcal{C}$ from a cofiltered category $\mathcal{D}$, we define the ind-object "colim"$F \in \mathcal{Set}^{\mathcal{C}^{op}}$ to be $\mathrm{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i))$, where the colimit is computed in $\mathcal{Set}^{\mathcal{C}^{op}}$. We can compute the morphisms between two ind-objects to be

$$\begin{aligned} \mathcal{Set}^{\mathcal{C}^{op}}(\text{“colim”}F, \text{“colim”}G) &= \mathcal{Set}^{\mathcal{C}^{op}}(\operatorname*{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i)), \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(-, G(j))) \\ &\cong \lim_{i \in \mathcal{D}} \mathcal{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, F(i)), \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(-, G(j))) \\ &\cong \lim_{i \in \mathcal{D}} \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(F(i), G(j))). \end{aligned}$$

The definition of a colimit gives us the first isomorphism and the the second is given by the Yoneda lemma and that colimits in $\mathcal{S}et^{\mathcal{C}^{op}}$ are computed pointwise. From this we can see that the full subcategory of $\mathcal{S}et^{\mathcal{C}^{op}}$ consisting of objects isomorphic to an ind-object has small hom-sets. This is the category $\operatorname{Ind}\mathcal{C}$ of ind-objects in $\mathcal{C}$.

The functor category $\mathcal{S}et^{\mathcal{C}^{op}}$ is complete because $\mathcal{S}et$ is complete. If $\mathcal{C}$ has finite products then we can see that the product "colim"$F \times$ "colim"$G \in \mathcal{S}et^{\mathcal{C}^{op}}$ can be realized as an ind-object:

$$\begin{aligned} \text{"colim"}F \times \text{"colim"}G &= \operatorname*{colim}_{i\in\mathcal{D}} \mathcal{C}(-,F(i)) \times \operatorname*{colim}_{j\in\mathcal{E}} \mathcal{C}(-,G(j)) \\ &\cong \operatorname*{colim}_{(i,j)\in\mathcal{D}\times\mathcal{E}} \mathcal{C}(-,F(i)\times G(j)), \end{aligned}$$

where the isomorphism follows from cofiltered colimits commuting with finite products in $\mathcal{S}et$ (see 2.13.4 in [Bor94]). Using this we see that the algebraic constructions in $\mathcal{C}$, given in Section 11.1 give rise to constructions in $\operatorname{Ind}\mathcal{C}$.

If $\mathcal{C}$ is equivalent to a category of representable functors then we can think of $\operatorname{Ind}\mathcal{C}$ as adjoining cofiltered colimits to $\mathcal{C}$:

Theorem 11.2. *Suppose that $\mathcal{D}$ is the subcategory of functors in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ such that for any object $X \in Obj(\mathcal{D})$, $X \cong \mathcal{C}(-,Y)$ for some Y. Then $\operatorname{Ind}\mathcal{D}$ is equivalent to the subcategory $\mathcal{E}$ of functors in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$, such that for all $\overline{X} \in Obj(\mathcal{E})$, $\overline{X} = \operatorname{colim} Y_i$. where $Y_i \in \mathcal{D}$.*

Proof. See [Joh82, Section VI]. □

Dual to the notion of ind-objects is the notion of pro-objects. The pro-objects in a category $\mathcal{C}$ are denoted $\operatorname{Pro}\mathcal{C}$. We have the tautological equivalence $\operatorname{Pro}\mathcal{C} \simeq \operatorname{Ind}\mathcal{C}^{op}$.

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