

NOTES ON NATURAL WEAK FACTORIZATION SYSTEMS

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ABSTRACT. These notes describe several observations I've made since beginning serious study of natural weak factorization systems in December 2008, although I was aware of their existence and some of the recent work in this field as much as two years earlier. The first section establishes some basic notation and terminology. The second discusses the induced natural weak factorization system on a diagram category. The main result is that the induced natural weak factorization system is cofibrantly generated if the inducing one is. The third section defines a notion of a natural model structure and proves analogs of some classical results.

At the moment, these notes presuppose some familiarity with Garner's work, which is well summarized in [2], though a future version will include the necessary background here.

1. BACKGROUND/NOTATION

For any category \mathcal{M} , let $\mathcal{M}^{\mathbf{2}}$ denote the category of functors $\mathbf{2} \rightarrow \mathcal{M}$, where $\mathbf{2}$ is the ordinal category consisting of two objects and one non-identity arrow. Objects of $\mathcal{M}^{\mathbf{2}}$ are arrows in \mathcal{M} and morphisms $(u, v) : f \rightarrow g$ are commutative squares

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ f \downarrow & & \downarrow g \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

We write (\mathbb{L}, \mathbb{R}) for a natural weak factorization system on a category \mathcal{M} . Explicitly, \mathbb{L} is the comonad (L, Φ, Σ) on $\mathcal{M}^{\mathbf{2}}$ and \mathbb{R} is the monad (R, Λ, Π) on $\mathcal{M}^{\mathbf{2}}$. More precisely, \mathbb{L} is a comonad over $\text{dom} : \mathcal{M}^{\mathbf{2}} \rightarrow \mathcal{M}$, which means that $\text{dom}L = \text{dom}$, $\text{dom}\Phi = \text{id}_{\text{dom}}$, and $\text{dom}\Sigma = \text{id}_{\text{dom}}$. Similarly, \mathbb{R} is a monad over cod .

For convenience, we'll abbreviate natural weak factorization system as *nwfs*, which will also be the abbreviation for the plural. The correct interpretation should be clear from context. Similarly, we write *wfs* for the singular or plural of weak factorization system, which will always mean the ordinary kind. The wfs mentioned in this paper underlie some nwfs and are therefore functorial.

Notation. When every lifting problem of the form posed by the commutative square

(1.1)
$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ l \downarrow & \nearrow w & \downarrow r \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

Date: April 17, 2009.

has a solution w , we write $l \sqsupseteq r$ and say that l has the left lifting property with respect to r and that r has the right lifting property with respect to l . If \mathcal{A} is a class of arrows, we write \mathcal{A}^\square for the class of arrows with the right lifting property with respect to each arrow in \mathcal{A} . Similarly, we write ${}^\square\mathcal{A}$ for the class of arrows with the left lifting property with respect to each arrow in \mathcal{A} .

In general, $\mathcal{A} \subset {}^\square\mathcal{B}$ iff $\mathcal{B} \subset \mathcal{A}^\square$; in this situation, we write $\mathcal{A} \sqsupseteq \mathcal{B}$ and say that \mathcal{A} has the left lifting property with respect to \mathcal{B} and \mathcal{B} has the right lifting property with respect to \mathcal{A} . The operations $(-)^{\square}$ and ${}^\square(-)$ form a Galois connection with respect to the posets of classes of arrows of a category ordered by inclusion.

Definition 1.2. More generally, we define a pair of functors

$$(-)^{\square} : \mathbf{CAT}/\mathcal{M}^2 \xrightleftharpoons[\perp]{} (\mathbf{CAT}/\mathcal{M}^2)^{\text{op}} : {}^\square(-)$$

that are mutually right adjoint. If \mathcal{J} is a category over \mathcal{M}^2 , the objects of \mathcal{J}^\square are pairs (g, ϕ) , where g is an arrow of \mathcal{M} and ϕ is a *lifting function* that assigns each square

$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ j \downarrow & & \downarrow g \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

with $j \in \mathcal{J}$ a lift $\phi(j, u, v)$ that makes the usual triangles commute. When \mathcal{J} is indiscrete, we also require that ϕ be coherent with respect to morphisms in \mathcal{J} . Explicitly, given $(a, b) : j' \rightarrow j$, we require that $\phi(j', ua, vb) = \phi(j, u, v)b$, which says that the triangle of lifts ϕ assigns to

$$\begin{array}{ccccc} \cdot & \xrightarrow{a} & \cdot & \xrightarrow{u} & \cdot \\ j' \downarrow & & \downarrow j & & \downarrow g \\ \cdot & \xrightarrow{b} & \cdot & \xrightarrow{v} & \cdot \end{array}$$

commutes.

Morphisms $(g, \phi) \rightarrow (g', \phi')$ of \mathcal{J}^\square are arrows in \mathcal{M}^2 that preserve the lifting functions. The category \mathcal{J}^\square is equipped with an obvious functor to \mathcal{M}^2 that forgets the lifting data.

Garner provides two definitions of a cofibrantly generated nwfs [2], though his terminology more closely parallels the theory of monads. He says a nwfs (\mathbb{L}, \mathbb{R}) is *free* on a small category $\mathcal{J} : \mathcal{J} \rightarrow \mathcal{M}^2$ if there is a functor $\eta : \mathcal{J} \rightarrow \mathbb{L}\text{-Map}$ over \mathcal{M}^2 that is universal from \mathcal{J} to categories of coalgebras for the left half of a nwfs. A stronger notion is of an *algebraically-free* nwfs, for which we require that the composite functor

$$\mathbb{R}\text{-Map} \xrightarrow{\text{lift}} (\mathbb{L}\text{-Map})^\square \xrightarrow{\eta^\square} \mathcal{J}^\square$$

is an isomorphism of categories. The functor *lift* uses the algebra and coalgebra structures of \mathbb{R} -maps and \mathbb{L} -maps to construct lifting functions in a natural way. (But see Remark 1.3 below!)

We will say that the nwfs produced by Garner's small object argument are *cofibrantly generated*. Garner proves that these nwfs are both free and algebraically-free; we will find occasion to use both defining properties. This notion of cofibrantly generated is more general than the usual one. Ordinary cofibrantly generated wfs

are generated by a set of maps, rather than a category. We will refer to this as the “discrete case”, discrete small categories being simply sets.

In the remaining sections, we will discuss the results of the author’s recent explorations, but before we do so, we would like to make the following remark, which elucidates some of the subtleties involved in dealing with nwfs.

Remark 1.3. In some sense the name *natural weak factorization system* is a misnomer; in most cases the lift of a map g in the right class against its left factor is not *natural*; it’s simply *chosen* and recorded in the fact that we regard g as part of an algebra and not simply an arrow. Solutions to lifting problems of the form (1.1) are constructed by combining the coalgebraic and algebraic data of l and r with a functorial factorization of the square. These lifts are not natural, at least with respect to all morphisms in the arrow category. They are natural with respect to morphisms of \mathbb{L} -Map and \mathbb{R} -Map, but these are precisely because morphisms in a category of algebras are required to preserve the algebraic structure.

In an important special case, however, there are natural lifts; namely, for the cofree and free morphisms that arise as left and right factors of arrows. The multiplication of the monad \mathbb{R} , gives any arrow of the form Rf a natural \mathbb{R} -map structure. Similarly, the arrows Lf have a natural \mathbb{L} -map structure using the comultiplication of the comonad. Of course, it may be that there are other ways to choose lifting data for morphisms Lf and Lg but the natural choices provided by the comultiplication have the nice property that any map from Lf to Lg arising from an arrow $(u, v) : f \rightarrow g$ preserves the lifting data.

2. INDUCED NATURAL WEAK FACTORIZATION SYSTEMS

One of the features of a nwfs that is not true of an ordinary wfs or even of a functorial wfs is that a nwfs on a category \mathcal{M} induces a nwfs on the diagram category $\mathcal{M}^{\mathcal{C}}$ of functors $\mathcal{C} \rightarrow \mathcal{M}$, where \mathcal{C} is any small category. The factorizations are defined levelwise and the comultiplication and multiplication maps are precisely what is needed to define natural transformations that ensure that the left and right factors have the desired lifting properties.

2.1. Details. We will give a more explicit description of what will henceforth be referred to as the *induced nwfs* on $\mathcal{M}^{\mathcal{C}}$ below, but first let’s fix some notation. We’ll write $(\mathbb{L}^{\mathcal{C}}, \mathbb{R}^{\mathcal{C}})$ for the nwfs on $\mathcal{M}^{\mathcal{C}}$ induced by the nwfs (\mathbb{L}, \mathbb{R}) on \mathcal{M} . Similarly, the induced functors and natural transformations will receive the superscript decoration.

Recall, a *functorial factorization* on a category \mathcal{M} is given by a functor $F : \mathcal{M}^2 \rightarrow \mathcal{M}^3$ that is a section of the composition functor $\mathcal{M}^3 \rightarrow \mathcal{M}^2$. Explicitly, F sends a commutative square

$$(2.1) \quad \begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ f \downarrow & & \downarrow g \\ \cdot & \xrightarrow{v} & \cdot \end{array} \quad \text{to a commutative rectangle} \quad \begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ Lf \downarrow & & \downarrow Lg \\ Ef & \xrightarrow{E(u,v)} & Eg \\ Rf \downarrow & & \downarrow Rg \\ \cdot & \xrightarrow{v} & \cdot \end{array}$$

with the left and right vertical composites being f and g . The functors $L, R : \mathcal{M}^2 \rightarrow \mathcal{M}^2$ induced by F send the square on the left (a morphism in \mathcal{M}^2) to the top and bottom squares on the right, respectively.

With this in mind, a functorial factorization on \mathcal{M}^c should consist of a functorial method for factoring arrows in \mathcal{M}^c , that is, natural transformations $\alpha : G \Rightarrow H$. Not surprisingly, this will be given levelwise by the functorial factorization F ; in other words, α_c will be factored as $R(\alpha_c) \circ L(\alpha_c)$ for each $c \in \mathcal{C}$.

[This description is for Peter May.] Explicitly, given functors $G, H : \mathcal{C} \rightarrow \mathcal{M}$ and a natural transformation $\alpha : G \Rightarrow H$, we define a functor $E' : \mathcal{C} \rightarrow \mathcal{M}$ on objects by $E'c = E\alpha_c$, where $E : \mathcal{M}^2 \rightarrow \mathcal{M}$ is as in (2.1), and on morphisms by $E'f = E(Gf, Hf)$. This is plainly functorial. The functorial factorization F^c of α will factor α as a composite of natural transformations $L\alpha : G \Rightarrow E'$ and $R\alpha : E' \Rightarrow H$, where $(L\alpha)_c$ is defined to be $L(\alpha_c)$ and likewise for the natural transformation $R\alpha$. Naturality of $L\alpha$ and $R\alpha$ follows immediately from the definition of the functor E' and the fact that the two squares on the right hand side of (2.1) commute.

[This description is for Mike Shulman.] Alternatively, we use the fact that **CAT** is cartesian closed to note that $(\mathcal{M}^c)^2 \cong \mathcal{M}^{c \times 2} \cong (\mathcal{M}^2)^c$ and similarly $(\mathcal{M}^c)^3 \cong (\mathcal{M}^3)^c$. Indeed, we often use the first isomorphism to regard a natural transformation α as a functor $\alpha : \mathcal{C} \rightarrow \mathcal{M}^2$; on objects, this functor picks out the constituent morphisms of α ; the image of a morphism in \mathcal{C} is the corresponding naturality square. Morphisms $\alpha \Rightarrow \beta$ in the category of functors $\mathcal{C} \rightarrow \mathcal{M}^2$ consist of a pair of natural transformations γ, δ in \mathcal{M}^c such that $\beta\gamma = \delta\alpha$. The category **2** is sometimes called \mathbb{I} , to emphasize its analogy with the topological interval. With this notation, the first isomorphism above explains the slogan “a homotopy is a natural transformation.”

Using these isomorphisms, we simply define F^c to be the functor

$$(\mathcal{M}^c)^2 \cong (\mathcal{M}^2)^c \xrightarrow{F_*} (\mathcal{M}^3)^c \cong (\mathcal{M}^c)^3$$

induced by post-composition by F and similarly for $L^c, R^c : (\mathcal{M}^2)^c \rightarrow (\mathcal{M}^2)^c$.

Continuing with the perspective provided by the isomorphism $(\mathcal{M}^c)^2 \cong (\mathcal{M}^2)^c$, we can readily define the natural transformations $\Phi^c, \Sigma^c, \Lambda^c, \Pi^c$ that make L^c and R^c into a comonad and monad. Given an object α of $(\mathcal{M}^c)^2$ regarded as a functor $\mathcal{C} \rightarrow \mathcal{M}^2$, the arrow Φ_α^c is obtained by “whiskering” Φ with α , as depicted below.

$$\mathcal{C} \xrightarrow{\alpha} \mathcal{M}^2 \begin{array}{c} \xrightarrow{L} \\ \Downarrow \Phi \\ \xrightarrow{R} \end{array} \mathcal{M}^2$$

All of the other natural transformations are defined similarly. It is clear that (L^c, Φ^c, Σ^c) and (R^c, Λ^c, Π^c) are indeed a comonad and a monad because all the definitions are given by simply post-composing with the old comonad and monad.

Remark 2.2. As you may have noticed above, a functorial factorization on \mathcal{M} induces a functorial factorization on \mathcal{M}^c ; so why doesn’t a functorial wfs descend to a wfs on \mathcal{M}^c ? The answer is that the left factors in \mathcal{M}^c (which we recall are natural transformations) will consist of arrows that lift levelwise against the right class, but as these lifts are not in general *natural* they will not fit together into a natural transformation, which is what a solution of a lifting problem in the diagram category is required to be.

However, when we have a nwfs, the comultiplication gives the natural transformations $L^c\alpha$ a \mathbb{L}^c -coalgebra structure. So the left factors from the functorial factorization do indeed have the left lifting property with respect to the right factors. Similarly, the multiplication gives the maps $\mathbb{R}^c\alpha$ a canonical \mathbb{R}^c -algebra structure.

2.2. Cofibrant Generation. Given a cofibrantly generated nwfs (\mathbb{L}, \mathbb{R}) on \mathcal{M} is the resulting induced nwfs $(\mathbb{L}^c, \mathbb{R}^c)$ on \mathcal{M}^c cofibrantly generated? There are many reasons to suspect that this is not the case. For example, the \mathbb{R}^c -maps in the nwfs on \mathbf{Set}^c induced by $\mathcal{J} = \{\emptyset \rightarrow 1\}$ are epis with a natural section. Right lifting properties can be used to specify additional *structure* on a class of maps, but they are not typically known to impose *coherence* conditions.

Despite this worry, the answer is yes, the induced nwfs is always cofibrantly generated when the original one is. In retrospect, the solution to the above difficulty is obvious: the generating category \mathcal{J}_c for the induced nwfs will be *indiscrete* (at least when \mathcal{C} is)! This is the first example known to the author where the extra generality allowed in Garner's small object argument is useful.

The following proof also works when the original generating category \mathcal{J} is indiscrete, but for ease of understanding it may make sense for the reader to imagine for now that \mathcal{J} is just a set of maps in \mathcal{M} .

Theorem 2.3. *Let $J : \mathcal{J} \rightarrow \mathcal{M}^2$ be a small category and let (\mathbb{L}, \mathbb{R}) be the nwfs it generates. Define a category $\mathcal{J}_c : \mathcal{J}_c \rightarrow (\mathcal{M}^c)^2$ as follows. Objects will have the form $\mathcal{C}(a, -) \cdot j$ for $a \in \mathcal{C}$ and $j \in \mathcal{J}$. Morphisms are generated by maps $\mathcal{C}(a, -) \cdot j \rightarrow \mathcal{C}(a, -) \cdot j'$ for every $j \rightarrow j'$ in \mathcal{J} and by maps $f^* : \mathcal{C}(b, -) \cdot j \rightarrow \mathcal{C}(a, -) \cdot j$ for every $f : a \rightarrow b$ in \mathcal{C} . Then the induced nwfs $(\mathbb{L}^c, \mathbb{R}^c)$ is cofibrantly generated by \mathcal{J}_c .*

Remark 2.4. Note that the category \mathcal{J}_c has shape $\mathcal{C}^{\text{op}} \times \mathcal{J}$. Indeed, the functor J_c is the composite

$$\mathcal{C}^{\text{op}} \times \mathcal{J} \xrightarrow{y \times J} \mathbf{Set}^c \times \mathcal{M}^2 \xrightarrow{-\odot-} (\mathcal{M}^c)^2,$$

where y denotes the Yoneda embedding and $-\odot-$ denotes the copower of an arrow in \mathcal{M} by a \mathbf{Set} -valued functor from \mathcal{C} . However, we prefer the notation \mathcal{J}_c for this category over $(\mathcal{M}^c)^2$.

Proof of Theorem 2.3. We don't know a priori whether \mathcal{M}^c satisfies one of Garner's smallness conditions, but we can begin to apply his small object construction to the category \mathcal{J}_c over $(\mathcal{M}^c)^2$ nonetheless. We will show that the functors $(L^c)^0, (L^c)^1, (R^c)^1, (L^c)^2, (R^c)^2$, etc that arise at each step agree with the functors L^0, L^1, R^1 , etc levelwise. It will follow that our construction on $(\mathcal{M}^c)^2$ converges to the nwfs $(\mathbb{L}^c, \mathbb{R}^c)$, which is therefore cofibrantly generated by \mathcal{J}_c .

The first stage of the small object argument computes the step-zero comonad $(L^c)^0$ as the left Kan extension of $J_c : \mathcal{J}_c \rightarrow (\mathcal{M}^c)^2$ along itself. Note that $(\mathcal{M}^c)^2$ is complete and cocomplete, since \mathcal{M} has a model structure. Using the familiar formula for Kan extensions

$$(L^c)^0\alpha = \int^{(a,j) \in \mathcal{J}_c \cong \mathcal{C}^{\text{op}} \times \mathcal{J}} \text{Hom}_{(\mathcal{M}^c)^2}(\mathcal{C}(a, -) \cdot j, \alpha) \cdot (\mathcal{C}(a, -) \cdot j).$$

The natural transformation $\mathcal{C}(a, -) \cdot j$ is the image of j under a functor $\mathcal{M}^2 \rightarrow (\mathcal{M}^{\mathcal{C}})^2$ that is left adjoint to evaluation at a . By this adjunction, the above coend equals

$$\begin{aligned} &= \int^{\mathcal{J}_{\mathcal{C}} \cong \mathcal{C}^{\text{op}} \times \mathcal{J}} \text{Hom}_{\mathcal{M}^2}(j, \alpha_a) \cdot (\mathcal{C}(a, -) \cdot j) \\ &= \int^{\mathcal{J}_{\mathcal{C}} \cong \mathcal{C}^{\text{op}} \times \mathcal{J}} \mathcal{C}(a, -) \cdot (\text{Sq}(j, \alpha_a) \cdot j) \end{aligned}$$

where we've written Sq to indicate that morphisms from j to α_a in \mathcal{M}^2 are commutative squares. By Fubini's theorem, we can use the isomorphism $\mathcal{J}_{\mathcal{C}} \cong \mathcal{C}^{\text{op}} \times \mathcal{J}$ to compute the coend over \mathcal{J} first, yielding

$$\begin{aligned} &= \int^{\mathcal{C}^{\text{op}}} \mathcal{C}(a, -) \cdot \left(\int^{\mathcal{J}} \text{Sq}(j, \alpha_a) \cdot j \right) \\ &= \int^{\mathcal{C}^{\text{op}}} \mathcal{C}(a, -) \cdot L^0 \alpha_a \end{aligned}$$

where L^0 is the step-zero comonad for \mathcal{J} . We now express this coend as a coequalizer

$$= \text{coeq} \left(\coprod_{f:a \rightarrow b} \mathcal{C}(b, -) \cdot L^0 \alpha_a \rightrightarrows \coprod_a \mathcal{C}(a, -) \cdot L^0 \alpha_a \right)$$

where the top arrow is induced by $f^* : \mathcal{C}(b, -) \rightarrow \mathcal{C}(a, -)$ and the bottom arrow is induced by L^0 applied to the naturality square for f , a morphism from α_a to α_b in \mathcal{M}^2 . We compute this coequalizer levelwise; by inspection at an object $c \in \mathcal{C}$, the resulting arrow in \mathcal{M} should be $L^0 \alpha_c$ with $\mathcal{C}(a, c) \cdot L^0 \alpha_a \rightarrow \alpha_c$ given by the evaluation map. This object and arrow satisfies the required universal property; the map out of $L^0 \alpha_c$ should be the one given by the identity component of the copower $\mathcal{C}(c, -) \cdot L^0 \alpha_c$.

The remaining steps in the small object argument are constructed from previous ones by applying the comonad $(L^{\mathcal{C}})^0$ and taking coproducts, which are computed levelwise. As we've shown that $(L^{\mathcal{C}})^0$ is also computed levelwise, we are done. As \mathcal{M} satisfies one of the smallness conditions, this process will converge at each arrow α_a , which means that the naturally constructed arrows from the step where process converges to the next step are isomorphisms. It follows that there is a natural isomorphism from the same step on $(\mathcal{M}^{\mathcal{C}})^2$ to the next one. This completes the proof that Garner's small object argument applied to $\mathcal{J}_{\mathcal{C}}$ will give the levelwise monad and comonad of $(\mathbb{L}^{\mathcal{C}}, \mathbb{R}^{\mathcal{C}})$. \square

As is the case for ordinary wfs, cofibrantly generated nwfs behave more nicely than generic ones. Given a nwfs (\mathbb{L}, \mathbb{R}) , let $\mathcal{L} = U(\mathbb{L}\text{-Map})$ and $\mathcal{R} = U(\mathbb{R}\text{-Map})$ be the classes of maps in \mathcal{M} which admit an \mathbb{L} - and \mathbb{R} -map structure, respectively. We say that $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ is the *underlying ordinary wfs* for (\mathbb{L}, \mathbb{R}) , where the bar denotes retract closure.

Lemma 2.5. *If the nwfs (\mathbb{L}, \mathbb{R}) is cofibrantly generated, then \mathcal{R} is closed under retracts.*

Proof. When (\mathbb{L}, \mathbb{R}) is cofibrantly generated by \mathcal{J} , we have an isomorphism of categories $\mathbb{R}\text{-Map} \cong \mathcal{J}^{\square}$ (see Definition 1.2). The forgetful functor $U : \mathbb{R}\text{-Map} \rightarrow \mathcal{M}^2$

sends $(g, \phi) \in \mathcal{J}^{\square}$ to g . We wish to show that its image is closed under retracts. Suppose h is a retract of g as shown

$$\begin{array}{ccccc} \cdot & \xrightarrow{i_1} & \cdot & \xrightarrow{r_1} & \cdot \\ \downarrow h & & \downarrow g & & \downarrow h \\ \cdot & \xrightarrow{i_2} & \cdot & \xrightarrow{r_2} & \cdot \end{array}$$

Define a lifting function ψ for h by

$$\psi(j, u, v) := r_1 \phi(j, i_1 u, i_2 v).$$

The equations from the retract diagram show that ψ is indeed a lifting function. It remains to check that ϕ is coherent with respect to morphisms $(a, b) : j' \rightarrow j$ of \mathcal{J} . But

$$\psi(j', ua, vb) = r_1 \phi(j', i_1 ua, i_2 vb) = r_1 \phi(j, i_1 u, i_2 v) b = \psi(j, u, v) b,$$

as required. \square

The upshot of Lemma 2.5 is that every arrow in the right class of the ordinary wfs underlying a cofibrantly generated nwfs (\mathbb{L}, \mathbb{R}) has an \mathbb{R} -algebra structure. We will use this result frequently in what follows.

Remark 2.6. In [2], Garner notes that when the generating category \mathcal{J} is discrete, then \mathcal{R} is closed under retracts and the wfs $(\overline{\mathcal{L}}, \mathcal{R})$ is cofibrantly generated in the usual sense by this set of maps. As a consequence, the new notion of “cofibrantly generated” agrees with the usual one, in the case where they ought to overlap. We will find uses for the strong observation of Lemma 2.5 below, which is no more difficult than Garner’s observation but wasn’t made explicit there.

3. NATURAL MODEL STRUCTURES

The reasons that most topologists care (or should care) about weak factorization systems is because they figure prominently in model categories, which are equipped with an interacting pair of them. Thanks to Garner’s small object argument, many examples of natural weak factorization systems arise from cofibrantly generated model categories. This leads to the question: is there a good notion of a *natural* model structure? What is the appropriate definition?

After considerable time trying to determine how algebraic the notion of weak equivalences should be, I’ve settled on the following “minimalist” definition.

Definition 3.1. A *natural model structure* consists of a pair of nwfs on \mathcal{M} together with a comparison map

$$\alpha : (\mathbb{L}_t, \mathbb{R}) \rightarrow (\mathbb{L}, \mathbb{R}_t)$$

together with a class $\mathcal{W} \subset \mathcal{M}^2$, satisfying the 2-of-3 property, such that the underlying wfs of $(\mathbb{L}_t, \mathbb{R})$ and $(\mathbb{L}, \mathbb{R}_t)$ give the trivial cofibrations, fibrations, cofibrations, and trivial fibrations, respectively of a model structure on \mathcal{M} .

Let \mathcal{C}_t denote the underlying class of maps with a \mathbb{L}_t -coalgebra structure and define \mathcal{C} , \mathcal{F}_t , and \mathcal{F} likewise. Then by definition $(\overline{\mathcal{C}_t}, \overline{\mathcal{F}})$ and $(\overline{\mathcal{C}}, \overline{\mathcal{F}_t})$ are the underlying wfs of $(\mathbb{L}_t, \mathbb{R})$ and $(\mathbb{L}, \mathbb{R}_t)$, respectively, where the bar denotes retract closure. The triple $(\overline{\mathcal{C}}, \overline{\mathcal{F}}, \mathcal{W})$ arising from a natural model structure gives a model structure on \mathcal{M} in the ordinary sense; we call this the *underlying ordinary model structure* on \mathcal{M} .

Unsurprisingly, we say that a natural model structure is *cofibrantly generated* if the two nwfs are cofibrantly generated.

3.1. The Comparison Map. The comparison map $\alpha : (\mathbb{L}_t, \mathbb{R}) \rightarrow (\mathbb{L}, \mathbb{R}_t)$ should be a morphism of nwfs, in the sense of [2]. This means firstly that α is a morphism of functorial factorizations, i.e., a natural transformation $\alpha : F_t \Rightarrow F$ such that α whiskered with the composition functor $d_1 : \mathcal{M}^3 \rightarrow \mathcal{M}^2$ is the identity. When we whisker α with $d_2, d_0 : \mathcal{M}^3 \rightarrow \mathcal{M}^2$ we obtain natural transformations $\alpha_l : L_t \Rightarrow L$ and $\alpha_r : R \rightarrow \mathbb{R}_t$, respectively; for α to be a morphism of nwfs we require additionally that α_l be a comonad morphism and α_r be a monad morphism.

Explicitly, a morphism α of functorial factorizations consists of arrows α_f for each $f \in \mathcal{M}^2$ such that

$$\begin{array}{ccc}
 & \cdot & \\
 \swarrow \mathbb{L}_t f & & \searrow \mathbb{L} f \\
 \cdot & \xrightarrow{\alpha_f} & \cdot \\
 \searrow \mathbb{R} f & & \swarrow \mathbb{R}_t f \\
 & \cdot &
 \end{array}$$

commutes. When α is a morphism of nwfs, it induces morphisms $(\alpha_l)_* : \mathbb{L}_t\text{-Map} \rightarrow \mathbb{L}\text{-Map}$ and $(\alpha_r)^* : \mathbb{R}_t\text{-Map} \rightarrow \mathbb{R}\text{-Map}$, which provide an algebraic way to regard a trivial cofibration as cofibration and a trivial fibration as a fibration. These maps have the following nice property. Given a lifting problem between a trivial cofibration and a trivial fibration, there is a priori two ways to solve it: regard the trivial cofibration as a cofibration and use the nwfs $(\mathbb{L}, \mathbb{R}_t)$ or regard the trivial fibration as a fibration and use the nwfs $(\mathbb{L}_t, \mathbb{R})$. Naturality of α says that the lifts constructed by each method are the same!

As far as the author is aware, model category theorists have not written much about the issue of comparing the two wfs provided by an ordinary model structure, a fact that I was first made aware of through discussions with Martin Hyland. But the existence of such a comparison map is more reasonable than one might expect: Peter May notes [4] that the universal property of the colimits in Quillen's small object argument gives such a natural transformation, provided we assume that the generating trivial cofibrations \mathcal{J} are contained in the generating cofibrations \mathcal{I} .

In many cases, this admittedly untraditional assumption is quite reasonable: the generating trivial cofibrations are of course cofibrations, so including them with the generators does not change the resulting model structure. We will require this additional hypothesis at least once below, without feeling too badly about it. In light of this discussion, we make the following remark in support of our definition 3.1.

Remark 3.2. Any ordinary cofibrantly generated model structure on a category satisfying one of Garner's smallness conditions can be made into a natural model structure by replacing the generating cofibrations \mathcal{I} by $\mathcal{I} \cup \mathcal{J}$ and applying Garner's small object argument in place of Quillen's. The underlying ordinary model structure of the resulting natural model structure is the same as before, by which we mean the classes $\bar{\mathcal{C}}$, \mathcal{F} ,¹ and \mathcal{W} are unchanged. Thus the abundance of cofibrantly generated model structures (in the ordinary sense) gives rise to an abundance of

¹Recall by Lemma 2.5, the right class \mathcal{F} underlying $(\mathbb{L}_t, \mathbb{R})$ is already retract closed.

examples of natural model structures in the sense of Definition 3.1, which are then of course cofibrantly generated.

While altering the generating cofibrations does not change the underlying model structure, it does change the cofibration – trivial fibration factorization. Given that the generating cofibrations are often more natural than the generating trivial cofibrations,² we provide the following alternative method for obtaining a comparison map for a cofibrantly generated natural model structure by altering the set \mathcal{J} as opposed to \mathcal{J} . As will become clear below, this method only applies in the case where the trivial cofibrations are generated by a set as opposed to a category.

First another remark.

Remark 3.3. A comparison map between two cofibrantly generated nwfs $(\mathbb{L}_t, \mathbb{R})$ and $(\mathbb{L}, \mathbb{R}_t)$ exists if and only if there is a morphism $\gamma : \mathcal{J} \rightarrow \mathbb{L}\text{-Map}$ over \mathcal{M}^2 . This is because Garner’s small object argument produces a canonical map $\eta : \mathcal{J} \rightarrow \mathbb{L}_t\text{-Map}$ in CAT/\mathcal{M}^2 that is universal among arrows from \mathcal{J} to a category of coalgebras for the left half of a nwfs on \mathcal{M}^2 , in the sense that every such morphism γ factors uniquely as $(\alpha_t)_* \circ \eta$, where α is a morphism of nwfs. See [2, §3].

Hence, to obtain a comparison map for a pair of cofibrantly generated nwfs whose underlying classes form a model structure, it suffices to replace \mathcal{J} by a set \mathcal{J}' of arrows with a \mathbb{L} -coalgebra structure. This is accomplished by the following lemma, which is a close analog of [3, Proposition 11.2.9].

Lemma 3.4. *Suppose \mathcal{J} is a set and \mathcal{J} is a category over \mathcal{M}^2 such that the underlying wfs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ of the nwfs $(\mathbb{L}_t, \mathbb{R})$ and $(\mathbb{L}, \mathbb{R}_t)$ that they generate give a model structure on \mathcal{M} , in the ordinary sense. Then \mathcal{J} can be replaced by an isomorphic set \mathcal{J}' such that*

- (i) *There is a morphism $\mathcal{J}' \rightarrow \mathcal{J}$ over \mathcal{M}^2 that is bijective on the underlying sets of arrows. In particular, \mathcal{J}' and \mathcal{J} generate the same underlying wfs.*
- (ii) *There is a morphism $\gamma : \mathcal{J}' \rightarrow \mathbb{L}\text{-Map}$ over \mathcal{M}^2 .*

The universal property of \mathcal{J}' then says that \mathcal{J} and \mathcal{J}' generate a natural model structure on \mathcal{M}^2 with the same underlying model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$.

Proof. Define $\mathcal{J}' \xrightarrow{J'} \mathcal{M}^2$ to be the composite $\mathcal{J} \xrightarrow{J} \mathcal{M}^2 \xrightarrow{L} \mathcal{M}^2$ where L is the comonad generated by \mathcal{J} . For each $j \in \mathcal{J}$, the corresponding element of \mathcal{J}' is its left factor Lj . We claim that $\mathcal{J}' = \{Lj \mid j \in \mathcal{J}\}$ satisfies conditions (i) and (ii) above. For (ii), we define γ to be the map that assigns each Lj its canonical free coalgebra structure (Lj, σ_j) .

For (i), we note that the lifting problem

$$\begin{array}{ccc}
 \cdot & \xrightarrow{Lj} & \cdot \\
 j \downarrow & \nearrow s & \downarrow R_t j \\
 \cdot & & \cdot \\
 & & \text{---} \\
 & & \cdot
 \end{array}$$

²Indeed, there are examples such as Joyal’s model structure for quasi-categories on \mathbf{sSet} where the generating cofibrations are not given explicitly.

has a solution s , because $R_t j \in \mathcal{F}_t \subset \overline{\mathcal{F}}$ and so by Lemma 2.5 has some \mathbb{R} -algebra structure. This lift gives us a retract diagram

$$\begin{array}{ccc} \cdot & \xlongequal{\quad} & \cdot \\ j \downarrow & & \downarrow Lj \\ \cdot & \xrightarrow{\quad s \quad} & \cdot \\ & & \downarrow R_t j \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

which we use to define the functor $\mathcal{J}'^{\square} \rightarrow \mathcal{J}^{\square}$. On objects, define

$$\mathcal{J}'^{\square} \ni (g, \psi) \mapsto (g, \phi) \in \mathcal{J}^{\square},$$

where $\phi(j, h, k) := \psi(Lj, h, k \cdot R_t j) \cdot s$ for all lifting problems

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad h \quad} & \cdot \\ j \downarrow & & \downarrow g \\ \cdot & \xrightarrow{\quad k \quad} & \cdot \end{array}$$

Because \mathcal{J} is discrete, the lifting function ϕ need not satisfy any coherence conditions. Given a morphism $(u, v) : (g, \psi) \rightarrow (g', \psi')$ in \mathcal{J}'^{\square} , it follows that

$$\begin{aligned} \phi'(j, u \cdot h, v \cdot k) &= \psi'(Lj, u \cdot h, v \cdot k \cdot R_t j) \cdot s \\ &= u \cdot \psi(Lj, h, k \cdot R_t j) \cdot s \\ &= u \cdot \phi(j, h, k), \end{aligned}$$

which says precisely that $(u, v) : (g, \phi) \rightarrow (g', \phi')$ is a morphism in \mathcal{J}^{\square} .

It remains to show that this functor is surjective on the underlying arrows of \mathcal{J}'^{\square} and \mathcal{J}^{\square} . Given g such that there is some $(g, \phi) \in \mathcal{J}^{\square}$, by definition $g \in \mathcal{F}$. By the 2-of-3 property, $Lj \in \mathcal{C} \cap \mathcal{W} \subset \overline{\mathcal{C}_t}$, so $Lj \square g$. As \mathcal{J}' is discrete, any choice of lifts against the Lj can be used to define a lifting function ψ so that $(g, \psi) \in \mathcal{J}'^{\square}$. Of course the functor defined above need not map (g, ψ) to (g, ϕ) but it does mean that g is in the image when we forget down to \mathcal{M}^2 , which is all that we claimed. \square

Note that corresponding arrows in the sets \mathcal{J} and \mathcal{J}' have the same domain. The reason that the result of Lemma 3.4 does not seem to extend to indiscrete categories \mathcal{J} has to do with the fact that the \mathbb{R} -algebra structures of the $R_t j$ are *chosen*, and not *natural* with respect to morphisms in \mathcal{J} ; see Remark 1.3.

3.2. Cofibrantly Generated Natural Model Structures. Many cofibrantly generated model structures are produced from previously known ones by passing the generating sets across an adjunction. It would be nice if we could repeat this trick for the more general cofibrantly generated natural model structures. This is possible due to the following extension of a well-known theorem due to Kan.

To make sense of the following theorem, we need a few simple observations. An adjunction $F : \mathcal{M} \xrightleftharpoons[\perp]{} \mathcal{K} : G$ lifts to an adjunction on the arrow categories \mathcal{M}^2 and \mathcal{K}^2 . We denote these functors by F and G as well. In particular, a small category $J : \mathcal{J} \rightarrow \mathcal{M}^2$ over \mathcal{M}^2 becomes a small category $FJ : \mathcal{J} \rightarrow \mathcal{K}^2$. Assuming, \mathcal{K} satisfies one of the conditions for Garner's small object argument, FJ generates a nwfs on \mathcal{K} . As above, we write \mathcal{C}_t for the underlying set of morphisms in \mathcal{K} which have some coalgebra structure for the resulting comonad.

Theorem 3.5. *Let \mathcal{M} have a natural model structure, cofibrantly generated by $I : \mathcal{J} \rightarrow \mathcal{M}^2$ and $J : \mathcal{J} \rightarrow \mathcal{M}^2$ and with weak equivalences $\mathcal{W}_{\mathcal{M}}$, and such that there is a morphism $\mathcal{J} \rightarrow \mathcal{J}$ over \mathcal{M}^2 . Let $F : \mathcal{M} \xrightarrow{\perp} \mathcal{K} : G$ be an adjunction. Suppose \mathcal{K} satisfies one of Garner's smallness conditions and also that*

$$(\star) G(\mathcal{C}_t) \subset \mathcal{W}_{\mathcal{M}}.$$

Then FJ and FI generate a natural model structure on \mathcal{K} with $\mathcal{W}_{\mathcal{K}} = G^{-1}(\mathcal{W}_{\mathcal{M}})$.

Remark 3.6. The surprising reason to require that $\mathcal{J} \subset \mathcal{J}$ is that in general the monads and comonads of the nwfs on \mathcal{K} do not seem to correspond well to the ones on \mathcal{M} ; without this requirement, it is unclear that we could expect a comparison map between the nwfs on \mathcal{K} . However, lifting properties behave nicely with respect to adjunctions, which is why we can prove the result without much information on the new factorizations.

Proof of Theorem 3.5. By the small object argument, FJ and FI generate nwfs with underlying wfs $(\overline{\mathcal{C}}_t, \mathcal{F})$ and $(\overline{\mathcal{C}}, \mathcal{F}_t)$.³ Composing with F induces a morphism $\mathcal{J} \rightarrow \mathcal{J}$ over \mathcal{K}^2 , which gives the comparison map between the resulting nwfs.

It remains to show that $\overline{\mathcal{C}}_t = \overline{\mathcal{C}} \cap \mathcal{W}_{\mathcal{K}}$ and $\mathcal{F}_t = \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$. In fact, by [3, 11.3.1] we only need to show three of these four inclusions.

The inclusion $\mathcal{C}_t \subset \mathcal{C}$ is immediate, since the comparison map explicitly provides each trivial cofibration with a cofibration structure; taking retract closures $\overline{\mathcal{C}}_t \subset \overline{\mathcal{C}}$. The hypothesis (\star) says that $\mathcal{C}_t \subset \mathcal{W}_{\mathcal{K}}$ and $\mathcal{W}_{\mathcal{K}}$ is retract closed by functoriality of G , so $\overline{\mathcal{C}}_t \subset \mathcal{W}_{\mathcal{K}}$. So $\overline{\mathcal{C}}_t \subset \overline{\mathcal{C}} \cap \mathcal{W}_{\mathcal{K}}$.

Similarly, the comparison map guarantees that $\mathcal{F}_t \subset \mathcal{F}$. If $g \in \mathcal{F}_t$ then it has some algebra structure $(g, \phi) \in F\mathcal{J}^{\square}$. By adjunction $(Gg, \phi^{\sharp}) \in \mathcal{J}^{\square}$, so Gg is a trivial fibration for the model structure on \mathcal{M} . In particular, $Gg \in \mathcal{W}_{\mathcal{M}}$, which says that $g \in \mathcal{W}_{\mathcal{K}}$. So $\mathcal{F}_t \subset \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$.

It remains to show that $\mathcal{F} \cap \mathcal{W} \subset \mathcal{F}_t$; we will appeal to Lemma 2.5 on two occasions. Suppose $f \in \mathcal{F} \cap \mathcal{W}_{\mathcal{K}}$; as above, f has some algebra structure $(f, \psi) \in F\mathcal{J}^{\square}$ and by adjunction $(Gf, \psi^{\sharp}) \in \mathcal{J}^{\square}$. As $f \in \mathcal{W}_{\mathcal{K}}$, Gf is a trivial fibration in the natural model structure on \mathcal{M} ; by Lemma 2.5, it follows that there is some algebra structure ξ such that $(Gf, \xi) \in \mathcal{J}^{\square}$. By adjunction, $(f, \xi^{\flat}) \in F\mathcal{J}^{\square}$, which says that $f \in \mathcal{F}_t$, as desired. \square

Remark 3.7. In the proof of Theorem 3.5, we frequently made use of the fact that our nwfs were cofibrantly generated, which by Lemma 2.5 says that every fibration has an \mathbb{R} -algebra structure. However, we believe it worth noting that to prove the inclusion $\overline{\mathcal{F}} \cap \mathcal{W} \subset \overline{\mathcal{F}}_t$ for any pair of nwfs, it actually suffices to show that $\mathcal{F} \cap \mathcal{W} \subset \overline{\mathcal{F}}_t$ – an observation that was not needed here but may prove useful in the future.

Why does this suffice? Morphisms in $\overline{\mathcal{F}}$ have the right lifting property with respect to \mathbb{L}_t -maps, simply because \mathcal{F} does and retracts preserve lifting properties. These lifts don't necessarily give the morphism in $\overline{\mathcal{F}}$ an \mathbb{R} -algebra structure, but in any case they are enough to make $f \in \overline{\mathcal{F}}$ a retract of Rf . From the retract diagram and 2 of 3 it is clear that if $f \in \mathcal{W}$ (here, some generic class of weak equivalences) then $Rf \in \mathcal{F} \cap \mathcal{W}$, so $\overline{\mathcal{F}} \cap \mathcal{W} \subset \overline{\mathcal{F}} \cap \mathcal{W}$ since f is therefore a retract of a trivial fibration Rf (which has the free \mathbb{R} -algebra structure). Hence, the inclusion $\mathcal{F} \cap \mathcal{W} \subset \overline{\mathcal{F}}_t$ implies that $\overline{\mathcal{F}} \cap \mathcal{W} \subset \overline{\mathcal{F}}_t$.

³Again recall Lemma 2.5.

Remark 3.8. By a similar argument, $\mathcal{C} \cap \mathcal{W} \subset \overline{\mathcal{C}_t}$ implies that $\overline{\mathcal{C}} \cap \mathcal{W} \subset \overline{\mathcal{C}_t}$, although in general the right hand left halves of a nwfs behave rather differently, especially when the nwfs is cofibrantly generated. Thus, it is not in general safe to assume that some property of the right class is also true of the left.

Using Theorem 3.5, we can prove that any category \mathcal{M} with a cofibrantly generated natural model structure induces a cofibrantly generated *projective* natural model structure on the diagram category $\mathcal{M}^{\mathcal{C}}$. The nwfs of this model structure are not the induced nwfs on $\mathcal{M}^{\mathcal{C}}$; these are discrete, and least when the original generators \mathcal{J} and \mathcal{J} are. This model structure agrees with the usual projective model structure on a diagram category: weak equivalences are levelwise weak equivalences and fibrations are levelwise fibrations.

The generating categories $\mathcal{J}_{\text{proj}}$ and $\mathcal{J}_{\text{proj}}$ for the projective model structure look familiar; in the case where \mathcal{J} and \mathcal{J} are discrete these are the usual generating sets in the classical theory. Objects of $\mathcal{J}_{\text{proj}}$ are functors $\mathcal{C}(a, -) \cdot i$, for all $a \in \mathcal{C}$ and $i \in \mathcal{J}$. Each morphism $i \rightarrow i'$ in \mathcal{J} gives rise to a morphism $\mathcal{C}(a, -) \cdot i \rightarrow \mathcal{C}(a, -) \cdot i'$ in $\mathcal{J}_{\text{proj}}$; there are no others. The category $\mathcal{J}_{\text{proj}}$ is defined similarly.

Theorem 3.9. *Let \mathcal{M} have a natural model structure, cofibrantly generated by $I : \mathcal{J} \rightarrow \mathcal{M}^2$ and $J : \mathcal{J} \rightarrow \mathcal{M}^2$ and with weak equivalences $\mathcal{W}_{\mathcal{M}}$, and such that there is a morphism $\mathcal{J} \rightarrow \mathcal{J}$ over \mathcal{M}^2 . Then the categories $\mathcal{J}_{\text{proj}}$ and $\mathcal{J}_{\text{proj}}$ give rise to a cofibrantly generated natural model structure on $\mathcal{M}^{\mathcal{C}}$, which we will call the projective natural model structure.*

Proof. Write \mathcal{C}_0 for the discrete subcategory of objects of \mathcal{C} . We first show that the natural model structure on \mathcal{M} induces a model structure on the diagram category $\mathcal{M}^{\mathcal{C}_0}$. We then use an adjunction to pass this across to the projective model structure on $\mathcal{M}^{\mathcal{C}}$.

Arrows of $\mathcal{M}^{\mathcal{C}_0}$ are natural transformations with no naturality conditions, i.e., collections α_c of morphisms in \mathcal{M} for each $c \in \mathcal{C}_0$. The categories \mathcal{J} and \mathcal{J} induce a pair of nwfs on $\mathcal{M}^{\mathcal{C}_0}$ by factoring these morphisms levelwise. By Theorem 2.3, these nwfs are cofibrantly generated by $\mathcal{J}_{\mathcal{C}_0}$ and $\mathcal{J}_{\mathcal{C}_0}$ ⁴. The morphism $\mathcal{J} \rightarrow \mathcal{J}$ over \mathcal{M}^2 gives rise to a morphism $\mathcal{J}_{\mathcal{C}_0} \rightarrow \mathcal{J}_{\mathcal{C}_0}$ over $(\mathcal{M}^{\mathcal{C}_0})^2$, which induces a comparison map between the nwfs. As in the proof of Theorem 3.5, it follows from the existence of this comparison map implies that trivial cofibrations are cofibrations and trivial fibrations are fibrations. Let \mathcal{W}_0 be the class of morphisms of $\mathcal{M}^{\mathcal{C}_0}$ that are levelwise weak equivalences. With this definition it is clear that trivial cofibrations and trivial fibrations are weak equivalences. So to show that $\mathcal{J}_{\mathcal{C}_0}$ and $\mathcal{J}_{\mathcal{C}_0}$ give rise to a natural model structure, it remains only to show that fibrations that are weak equivalences are trivial fibrations.

The language of the above paragraph was a bit colloquial. What we really need to show is that algebras for the monad induced by $\mathcal{J}_{\mathcal{C}_0}$ that are levelwise weak equivalences have an algebra structure for the monad induced by $\mathcal{J}_{\mathcal{C}_0}$. Since the category \mathcal{C}_0 is discrete, a collection of morphisms α_c has an algebra structure for the monad induced by $\mathcal{J}_{\mathcal{C}_0}$ just when each α_c is an algebra for the monad induced by \mathcal{J} . In this case, each α_c is a trivial fibration for the natural model structure on \mathcal{M} , which means that it has an algebra structure for the monad induced by \mathcal{J} . Again because \mathcal{C}_0 is discrete, this means that the collection α has an algebra structure for

⁴Of course, it is also possible to prove this directly; this is an easier special case of Theorem 2.3.

the monad induced by $\mathcal{J}_{\mathcal{C}_0}$, which is what we wanted to show. So the categories $\mathcal{J}_{\mathcal{C}_0}$ and $\mathcal{J}_{\mathcal{C}_0}$ generate a natural model structure on $\mathcal{M}^{\mathcal{C}_0}$.

Let $i : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be the canonical inclusion. Then left Kan extension along i gives rise to an adjunction

$$\text{Lan}_i : \mathcal{M}^{\mathcal{C}_0} \xrightleftharpoons[\perp]{} \mathcal{M}^{\mathcal{C}} : i^*$$

Here i^* might be thought of as an “evaluation” map; it takes a functor G to the collection of objects in its image and a natural transformation α to its collection of constituent arrows. Using the usual formula for left Kan extensions, the left adjoint takes an arrow $\alpha \in \mathcal{M}^{\mathcal{C}_0}$ to the disjoint union $\sqcup_{c \in \mathcal{C}_0} \mathcal{C}(c, -) \cdot \alpha_c$. Objects in $\mathcal{J}_{\mathcal{C}_0}$ are natural transformations $\mathcal{C}_0(a, -) \cdot i$ for some $a \in \mathcal{C}$ and $i \in \mathcal{J}$. As \mathcal{C}_0 is discrete, this natural transformation consists of the arrow i at the component for a and the identity arrow at the initial object of \mathcal{M} at all other objects of \mathcal{C} . The image of this object under Lan_i is $\mathcal{C}(a, -) \cdot i$, by the above formula. From this point, it is easy to see that

$$\text{Lan}_i \mathcal{J}_{\mathcal{C}_0} = \mathcal{J}_{\text{proj}} \quad \text{and} \quad \text{Lan}_i \mathcal{J}_{\mathcal{C}_0} = \mathcal{J}_{\text{proj}}.$$

In order to apply Theorem 3.5 and conclude that $\mathcal{M}^{\mathcal{C}}$ has a natural model structure cofibrantly generated by $\mathcal{J}_{\text{proj}}$ and $\mathcal{J}_{\text{proj}}$, we must show that the right adjoint i^* takes the underlying maps of the coalgebras for the comonad generated by $\mathcal{J}_{\text{proj}}$ to weak equivalences in $\mathcal{M}^{\mathcal{C}_0}$. In other words, we must show that the coalgebras for the comonad generated by $\mathcal{J}_{\text{proj}}$ are levelwise weak equivalences.

Coalgebras for the comonad generated by $\mathcal{J}_{\text{proj}}$ are in the left class of the underlying wfs $(\square(\mathcal{J}_{\text{proj}}^{\square}), \mathcal{J}_{\text{proj}}^{\square})$ that this category generates. From the adjunction, we know that $\mathcal{J}_{\text{proj}}^{\square}$ is the class of levelwise algebras for the original monad generated by \mathcal{J} on \mathcal{M}^2 .

Let α be an element of $\square(\mathcal{J}_{\text{proj}}^{\square})$ and factor α using the induced nwfs $(\mathbb{L}_t^{\mathcal{C}}, \mathbb{R}^{\mathcal{C}})$ on $\mathcal{M}^{\mathcal{C}}$, not that nwfs cofibrantly generated by $\mathcal{J}_{\text{proj}}$. The components of the right factor $R^{\mathcal{C}}\alpha$ are algebras for the monad \mathbb{R} generated by \mathcal{J} because $R^{\mathcal{C}}\alpha$ is an algebra for the monad $\mathbb{R}^{\mathcal{C}}$. So α lifts against $R^{\mathcal{C}}\alpha$, which means that α is a retract of $L_t^{\mathcal{C}}\alpha$. The constituent maps $(L_t^{\mathcal{C}}\alpha)_c = L_t(\alpha_c)$ are coalgebras for the comonad on \mathcal{M}^2 generated by \mathcal{J} ; in particular they are weak equivalences, since \mathcal{J} is the generating category of trivial cofibrations. So levelwise the arrows of α are retracts of weak equivalences; hence α consists of levelwise weak equivalences. Theorem 3.5 may now be used to establish the projective natural model structure. \square

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