Course notes: Week 9a, Kähler differentials

1. MODULES OF DIFFERENTIALS

1.1. Let x be a point on a differentiable manifold M. Let us recall that the tangent space $T_x(M)$ can be defined as the space of all linear maps $\psi : C^\infty(M) \to \mathbb{C}$, such that $\psi(f \cdot g) = f(x) \cdot \psi(g) + \psi(f) \cdot g(x)$.

1.2. Let k be a field, and A a commutative k-algebra. Let M be an A-module. A k-derivation $A \to M$ is a k-linear (but not A-linear !) map $\psi : A \to M$, such that

$$\psi(a_1 \cdot a_2) = a_1 \cdot \psi(a_2) + a_2 \cdot \psi(a_1).$$

1.3. **Exercise 0.** Let $f(t)$be a polynomial with coefficients in k. Show that $\psi(f(a)) = f'(a) \cdot \psi(a)$, where f' is the derivative.

Let $Der_k(A, M)$ be the set of such derivations. This is abelian group, and thus we get a functor

$$A - mod \to Ab : M \mapsto Der_k(A, M).$$

Proposition 1. *The above functor is representable.*

Proof. We construct an A-module, denoted $\Omega_k(A)$, and called the module of Kähler differentials of A over k, such that, functorially:

$$Hom(\Omega_k(A), M) \simeq Der_k(A, M).$$

Consider the tensor product $algebra A \underset{k}{\otimes} A$; it maps to A by the multiplication map on A. Let I_A be the kernel. This is a module over $A \underset{k}{\otimes} A$. Consider the quotient $I_A/(I_A)^2$.

1.4. **Exercise 1.** (a) Show that $I_A/(I_A)^2 \simeq I_A \underset{A \underset{k}{\otimes} A}{\otimes} A$. (b) Deduce that the action of $A \underset{k}{\otimes} A$ on $I_A/(I_A)^2$ factors through A.

Thus, $I_A/(I_A)^2$ is an A-module. We set it to be the sought-for $\Omega_k(A)$. Let us now verify the defining property. Let M be an A-module.

For a map $\phi : I_A/(I_A)^2 \to M$ define a derivation $\psi : A \to M$ by $\psi(a) = \phi(a \otimes 1 - 1 \otimes a)$. (Note that $a \otimes 1 - 1 \otimes a \in I_A$.)

1.5. **Exercise 2.** Show that ψ is indeed a derivation.

Vice versa, for a derivation $\psi : A \to M$ we define the map $\phi : I_A/(I_A)^2 \to M$ as follows. For an element $c = \Sigma\, a_i \otimes b_i \in I_A$ set

$$\phi(c) = \Sigma\, b_i \cdot \psi(a_i).$$

1.6. **Exercise 3.** (a) Show that ϕ is a well-defined map $\phi : I_A/(I_A)^2 \to M$. (b) Show that the above two constructions are inverses of one-another.

□

1.7. **Exercise 4.** Let $A = k[x_1, ..., x_n]$. (a) Show that to specify a derivation $\psi : A \to M$ is the same at to specify n elements $m_i \in M$ such that $\psi(x_i) = m_i$. (b) Deduce that $\Omega_k(A)$ is a free A-module on n generators.

1.8. **Generaliztions.** Let more generally, A be a commutative ring over another commutative ring A_0. Define the functor $Der_{A_0} : A - mod \to Ab$ by setting $Der_{A_0}(A, M)$ to be the set of A_0-linear maps $A \to M$, that satisfy the Leibniz rule:

$$\psi(a_1 \cdot a_2) = a_1 \cdot \psi(a_2) + a_2 \cdot \psi(a_1)$$

for $a_1, a_2 \in A$. For $A_0 = k$ we recover $Der_k(A, M)$.

1.9. **Exercise 5.** Show that the above functor is representable, by explicitly constructing the corresponding object $\Omega_{A_0}(A)$.

Let now $A \to B \to C$ be 3 commutative rings. Let M be a C-module, which can also be regarded as a B-module.

1.10. **Exercise 6.** (a) Show that we have an exact sequence

$$0 \to Der_B(C, M) \to Der_A(C, M) \to Der_A(B, M).$$

(b) Deduce that we have an exact sequence of C-modules

$$\Omega_A(B) \underset{B}{\otimes} C \to \Omega_A(C) \to \Omega_B(C) \to 0.$$

(c) Deduce that if B is a f.g. A-algebra, then $\Omega_A(B)$ is a f.g. B-module.

1.11. **Exercise 7.** Show that if $S \subset A$ is a multiplicative subset, $\Omega_{A_0}(A_S) \simeq \Big(\Omega_{A_0}(A)\Big)_S$.

1.12. **Tangent and cotangent spaces.** Let A be again a k-algebra, and let $x = \mathfrak{m} \in Specm(A) =: X$ be a maximal ideal corresponding to a homomorphism $\chi : A \to k$. We will regard k as an A-module via χ.

We define $T_x(X)$ as $Der_k(A, k)$. Let $\Omega_k(A)_x$ be the fiber of $\Omega_k(A)$ at x, i.e., $\Omega_k(A)_x \simeq \Omega_k(A) \underset{A}{\otimes} k$.

1.13. **Exercise 8.** Show that $T_x(X)$ is the dual k-vector space to $\Omega_k(A)_x$.

By the above exercise, it makes sense to denote $\Omega_k(A)_x$ also by $T^*_x(X)$.

1.14. **Exercise 9.** Show that $\Omega_k(A)_x$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$.

2. Differentials and dimension

2.1. Let A be a f.g. domain over an *algebraically closed* field k. We saw that $\Omega_k(A)$ is a f.g. A-module. We will show:

Theorem 1. *Over a dense open subset of $X = Spec(A)$, $\Omega_k(A)$ is locally free of rank n, where $n = \dim(A)$.*

2.2. **Exercise 10.** Let A be a Noetherian domain, and M a f.g. A-module, such that

$$\dim_{Frac(A)}(M \underset{A}{\otimes} Frac(M)) = n.$$

(a) Show that for some $f \in A$, M_f is a locally free A_f-module of rank n. (b) Show that the set of prime ideals $\mathfrak{p}$, such that the *fiber* of M at $\mathfrak{p}$ is of dimension n over $Frac(A/\mathfrak{p})$, is open in $Spec(M)$. (c) Show that if f is such that $Spec(A_f)$ is contained in the above open, then M_f is locally free over A_f. (d) Show that at all other primes the dimension of the fiber is $> n$.

The set of prime ideals of A, for which the fiber of $\Omega_k(A)$ is of dimension n is called to smoothness locus of A. We say that A is smooth if the smoothness locus is the entire $Spec(A)$. By Exercise 4, $k[x_1, ..., x_n]$ is smooth. In general, smooth algebras are those, whose properties are most close to those of $k[x_1, ..., x_n]$.

Let $K = Frac(A)$. This is a f.g. field over k of transcendency degree n. By Exercise 9, to prove the theorem, we have to show that $\dim_K(\Omega_k(K)) = n$. This will be very easy when $char(k) = 0$, but some additional complications arise in the $char(k) = p$ case.

2.3. **Separable field extensions.** Let L be a field, and $f(t)$ be a non-constant polynomial with coefficients in L. We say that $f(t)$ is separable if $f'(t) \neq 0$. In other words, if $f(t) = \Sigma\, a_i \cdot t^i$, we need that not all i, for which $a_i \neq 0$ are divisible by p, where $p = char(k)$. When $char(k) = 0$ every polynomial is separable.

Let $L' \supset L$ be a finite field extension.

Theorem 2. *The following conditions are equivalent:*

(a) *For every element $a \in L'$ its minimal polynomial over L is separable.*

(b) *L' is generated over L by elements, for each of which the minimal polynomial is separable.*

(c) $\Omega_L(L') = 0$.

(d) *If $\overline{L}$ is the algebraic closure of L, then the Artinian algebra $L' \underset{L}{\otimes} \overline{L}$ has no nilpotents.*

Extensions satisfying the equivalent conditions of the theprem are called separable. If $char(L) = 0$, then every finite extension is separable.

2.4. **Exercsie 11.** Deduce formally that if $L' \supset L$ and $L'' \supset L'$ are separable, then $L'' \supset L$ is separable. (Hint: use Exercise 6 and condition (d)).

The complete proof of Theorem 2 will be given later. Here we will discuss just one step used in (b) $\Rightarrow$ (c), which explains the relevance of separability to differentials.

Proposition 2. *Let L' be generated over L by a, which satisfies an in irreducible separable polynomial $f(t)$ over L. Then for any field $k \subset K$, the restriction map*

$$Der_k(L', V) \to Der_k(L, V)$$

is an isomorphism for any L'-vector space V.

2.5. **Exercise 12.** Show that the fact that $Der_k(L', V) \simeq Der_k(L, V)$ for some non-zero L'-vector space V is equivalent to the fact that $\Omega_k(L) \underset{L}{\otimes} L' \simeq \Omega_k(L')$.

Proof. The assumption implies that $L' = L[t]/f(t)$, where the image of t in L' is a.

Let us first prove the injectivity of the above map. Let ψ be a k-derivation $L' \to V$, such that $\psi|_L = 0$. Then ψ is an L-derivation. Hence, by Exercise 0,

$$0 = \psi(f(a)) = f'(a) \cdot \psi(a).$$

However, $f'(a) \neq 0$, which implies $\psi(a) = 0$. Hence, $\psi = 0$.

Let now ψ be a k-derivation $L \to V$. If $f(t) = \Sigma\, b_i \cdot t^i$, we define $\psi(a)$ by the condition

$$\psi(a) \cdot f'(a) + \Sigma\, \psi(b_i) \cdot a^i = 0 \in K.$$

This is well-defined since $f'(a) \neq 0$.

We extend ψ to the entire L' by linearity and the Leibniz rule:

$$\psi(b \cdot a^k) = \psi(b) \cdot a^k + k \cdot a^{k-1} \cdot b \cdot \psi(a).$$

2.6. **Exercise 13.** Show that ψ is well-defined.

$\square$

Using Exercise 12, Theorem 1 follows from the next lemma:

Lemma 1. *Let K be a field of transcendency degree n over an algebraically closed field k. Then K contains a subfield of the form $K' = k(x_1, ..., x_n)$, over which it is finite and separable.*

Note that in *char* $= 0$ the lemma is trivial: the assertion follows follows from the definition of transcendency degree.

Proof. By assumption, K contains a subfield $K_0 = k(x_1, ..., x_n)$, over which it is finite. We will construct a growing collection of subfields $K_i'' \subset K$, each of which is separable over some subfield of the form $K_i' = k(y_1, ..., y_n)$, and such that $K_i'' \subset K_0$ for all i.

Set $K_0' = K_0'' = K_0$. If K is separable over K_0, we are done.

If not, let $a \in K$ be an element, whose minimal polynomial $f(t)$ over $k(x_1, ..., x_n)$ isn't separable. By eliminating extra variables, we can assume that a does not satisfy a polynomial equation over any $k(x_1, ..., \hat{x_i}, ..., x_n)$.

Multiplying $f(t)$ by a *polynomial* in the x_i's, we eliminate the denominators. I.e., we can assume that there exists an *irreducible* polynomial in $n+1$ variables $g(t_1, ..., t_n, t)$ over k, such that $f(t) = g(x_1, ... x_n, t)$.

2.7. **Exercise 14.** Define the polynomial $f_i(t_i)$ over the field $k(x_1, ..., \hat{x_i}, ..., x_n, a)$ by $f_i(t_i) = g(x_1, ..., t_i, ..., x_n, a)$. Show that f_i is irreducible. (Hint: use the fact that if a ring has no divisors, then the same is true for every localization.)

Let us write the $g(t)$ as

$$\Sigma\, c_{j_1, ..., j_n, m} \cdot x_1^{j_1} \cdot ... \cdot x_n^{j_n} \cdot t^m,$$

$c_{j_1, ..., j_n, m} \in k$, where all m's that appear with non-zero coefficients are divisible by p.

We claim that not all the powers j_i that appear are divisible by p. Indeed, if this were so, since k is algebraically closed, we could extract the p-th root of every coefficient $c_{j_1,...,j_n}$, and hence, we could extract the p-th root of the entire polynomial $f(t)$, which would be a contradiction.

2.8. **Exercsie 15*.** (a) Show that we are not using the full algebraic closedness of k, but rather that the map $x \mapsto x^p$ is surjective. (Such fields are called perfect.) (b) Show that a field is perfect if and only if every finite extension of it is separable.

Hence, there exists $i \in \{1, ..., n\}$, such that the polynomial $f_(t_i)$ (cf. Exercise 14) is separable. Hence, if we set $K_1'' = k(x_1, ..., x_i, ..., x_n, a)$, it would be separable as an extension of $K_1' = k(x_1, ..., \hat{x_i}, ..., x_n, a)$. By constriction $K_1'' \supset K_0'' = K_0$, and the containment is strict.

If K is separable over K_1'' we are done by Exercise 12. In not, proceeding as above we can find a field $K_2' \subset K$, isomorphic to the field of rational functions on n variables, and another field $\tilde{K}_2'' \supset K_2'$, which contains K_1', and which is separable over K_2'. Finally, we define K_2'' to be the subfield of K, generated by $\tilde{K}_2''$ and K_1''.

Since K_1'' over K_1' is separable, by Theorem 2, it is generated by elements that satisfy separable irreducible polynomials. Hence, the same is true for K_2'' over $\tilde{K}_2''$. Exercise 12 implies that K_2'' is separable over K_2'. Evidently, K_2'' strictly contains K_1''.

If K is separable over K_2'' we are done by Exercise 12. If not we continue the process. Since K was finite over K_0, this process will terminate.

□

3. Separability

3.1. Let us now prove Theorem 1. The implication (a) ⇒ (b) is trivial.

3.2. **Exercise 16.** Deduce the implication (b) ⇒ (c) from Proposition 2.

3.3. Let us assume that condition (c) holds, and let us deduce (d).

Lemma 2. *Let $A \to B$ and $A \to A'$ be homomorphisms of commutative algebras. Set $B' = B \underset{A}{\otimes} A'$. Then $\Omega_{A'}(B') \simeq \Omega_A(B) \underset{B}{\otimes} B'$.*

3.4. **Exercise 17.** Prove the lemma.

By the lemma, we obtain that $\Omega_{\overline{L}}(L' \underset{L}{\otimes} \overline{L}) = 0$.

Set $A = L' \underset{L}{\otimes} \overline{L}$; this is an Artinian algebra over $\overline{L}$. Hence, it has the form $\oplus A_i$, where A_i are local Artinian algebras, such that $\Omega_{\overline{L}}(A_i) = 0$.

We need to show that $\overline{L} \to A_i$ is an isomorphism for every i. I.e., that the maximal ideal $\mathfrak{m}_i$ of A_i is 0. By Nakayama's lemma, it enough to show that $\mathfrak{m}_i/\mathfrak{m}_i^2 = 0$.

Since A_i is f.g. (in fact f.d.) over $\overline{L}$,

$$A_i/\mathfrak{m}_i \simeq \overline{L}.$$

Now, our assertion follows from Exercise 9.

3.5. Finally, let us assume that $L' \supset L$ satisfies (d). Note that if L'' is a sub-extension of L', then it also satisfies (d), since

$$L'' \underset{L}{\otimes} \overline{L} \subset L' \underset{L}{\otimes} \overline{L}.$$

Let $a \in L'$ be an element with minimal polynomial $f(t)$ over L, i.e., $L'' \simeq L[t]/f(t)$ is a sub-extension. We have:

$$L'' \underset{L}{\otimes} \overline{L} \simeq \overline{L}[t]/f(t).$$

Suppose $f(t)$ wasn't separable. Let us write

$$f(t) = \Sigma\, a_i \cdot t^{p \cdot i},$$

and define a polynomial $g(t)$ over $\overline{L}$ by

$$f(t) = \Sigma\, b_i \cdot t^{p \cdot i},$$

where $(b_i)^p = a_i$.

Then $f(t) = (g(t))^p$. Hence, the image of $g(t)$ in $\overline{L}[t]/f(t)$ is nilpotent, which contradicts the assumption.