Course notes: Week 8, filtrations and Hilbert polynomials

1. Filtrations

1.1. Let M be an abelian group. An (increasing) filtration on it is a sequence of subgroups $... \subset M_{-j} \subset ... \subset M_0 \subset M_1 \subset ...$, such that their union is all of M. In a similar way we define filtered R-modules, where R is a ring.

Unless specified otherwise, we'll assume that $M_{-j} = 0$ for j large enough.

If $M' \subset M$ and M is filtered, M' acquires an induced filtration by setting $M'_i = M' \cap M_i$. If M'' is a quotient module of M, it also acquires a filtration by setting M''_i to be the image of M_i.

1.2. For two filtered modules M and N a map $\phi : M \to N$ is said to be filtered if $\phi(M_i) \subset N_i$. In this way, filtered modules form a category, denoted $R - mod^{fil}$.

1.3. **Exercise 1.** (a) Show that the category $R - mod^{fil}$ is additive, admits kernels and co-kernels. (b*) Show that this category is *not* abelian.

1.4. **Associated graded.** If M is a filtered module, its associated graded, denoted $gr(M)$ is defined as $\underset{i\in\mathbb{Z}}{\oplus} M_i/M_{i-1}$. We denote M_i/M_{i-1} by $gr^i(M)$.

1.5. **Exercise 2.** Show that $gr(\cdot)$ is a functor $R - mod^{fil} \to R - mod$.

Note that if M is a filtered module, and M' (resp., M'') is a submodule (resp., quotient module) with the induced filtration, then $gr(M')$ (resp., $gr(M'')$) is a submodule (resp., quotient module) of $gr(M)$.

1.6. **Exercise 3.** (a) Show that a filtered map $M \to N$ can be injective (resp., surjective), without the map $gr(M) \to gr(N)$ having this property. (b) Show that if a filtered map $\phi : M \to N$ is such that $gr(\phi) : gr(M) \to gr(N)$ is surjective (resp., injective), then ϕ is surjective (resp., injective).

1.7. **Graded and filtered rings.**

We say that a ring R is (positively) graded, if as an abelian group it's written as $\underset{i\in\mathbb{N}}{\oplus} R^i$, such that $1 \in R^0$, and the multiplication maps $R^i \otimes R^j \to R^{i+j}$. Evidently, R^0 is a sub-ring of R, and each R^i is an R^0-module.

A graded module M over a graded ring R is an R-module, written as $\underset{i\in\mathbb{Z}}{\oplus} M^i$, such that the multiplication maps $R^i \otimes M^j \to M^{i+j}$. In particular, every M^j is an R^0-module. Unless specified otherwise, we'll assume that $M^{-i} = 0$ for i large enough.

We say that a ring R is (positively) filtered, if as an abelian group it's written as $\underset{i\in\mathbb{N}}{\cup} R_i$, $1 \in R_0$, and the multiplication maps $R_i \otimes R_j \to R_{i+j}$.

A filtered module M over a filtered ring R is an R-module, which is filtered as an abelian group, such that the multiplication maps $R_i \otimes M_j \to M_{i+j}$.

Evidently, R is a filtered module over itself, and $R_0 \subset R$ is a sub-ring. For any M as above, each M_i is a module over R_0.

1.8. **Exercise 4.** Show that any graded ring can be regarded as a filtered ring by setting R^i to be $\underset{j\le i}{\oplus} R_j$, and similarly for modules.

1.9. **Exercise 5.** (a) Show that if R is a filtered ring, then $gr(R)$ is a graded ring. (b) If M is a filtered R-module, then $gr(M)$ is a graded $gr(R)$-module. (c) Show that if R is a graded ring, regarded as a filtered ring, then its $gr(\cdot)$ is isomorphic to R as a graded algebra; and similarly for modules.

Lemma 1. *Let R be a filtered ring, and M a filtered R-module. Assume that $gr(M)$ is f.g. as a module over $gr(R)$. Show that M is f.g. as an R-module.*

Proof. Let $m'_1, ..., m'_k$ be elements of $gr(M)$ (of degrees $d_1, ..., d_n$, respectively), such that they generate $gr(M)$ as a $gr(R)$-module. Choose elements $m_j \in M_{d_j}$, which project onto m'_j under $M_{d_j} \to gr^{d_j}(M)$. We claim that these elements generate M as an R-module.

We argue by induction on i that every element of M_i can be written as $\Sigma\, r_j \cdot m_j$. By assumption, if $i << 0$, then $M_i = 0$ and there is nothing to prove. Assume the validity for some i, and let m be an element of M_{i+1}. Let m' be its image in $gr^i(M)$. Then there exist elements $r'_j \in gr^{j-d_j}(R)$, such that $m' = \Sigma r'_j \cdot m'_j$. Let $r_j \in R_{j-d_j}$ be some liftings. Consider $m_1 = m - \Sigma\, r_j \cdot m_j$. This is an element of M_i, but its projection to M_{i-1} is 0. Hence, $m_1 \in M_{i-1}$, and we are done by the induction hypothesis.

□

If k is a field, we'll say that A is a filtered k-algebra, if A is filtered as a ring, and the homomorphism $k \to A$ maps to A_0.

2. Good filtrations

2.1. Let R be a filtered ring, and let M be a non-filtered f.g. A-module.

Proposition 1. *We can choose a filtration on M so that $gr(M)$ is f.g. as a $gr(R)$-module. If $Fil_i(M), i \in \mathbb{Z}$ and $Fil'_i(M), i \in \mathbb{Z}$ are two such filtrations, there exists an integers k such that*

$$Fil'_{i-k}(M) \subset Fil_i(M) \subset Fil'_{i+k}(M).$$

A filtration as in the lemma is called a "good" filtration.

Proof. Let $\overline{m} = \{m_1, ..., m_k\} \subset M$ be a finite subset of elements that generated M over R. Set M_i to be the image of $R_i \times \overline{m}$ in M under the multiplication map.

We obtain a surjective map of filtered R-modules $R^k \to M$, such that the filtration on M is the induced one. Hence,

$$gr(R^n) \simeq gr(R)^n \twoheadrightarrow gr(M),$$

and $gr(M)$ is f.g.

Let us rename this filtration and call $M_i = Fil_i(M)$. For the second assertion of the lemma, it is enough to compare this filtration with any other one, call it $Fil'i$.

By assumption, there exists k_1, such that $\overline{m} \subset Fil'_{k_1}(M)$. Then $Fil_i(M) = R_i \cdot \overline{m} \subset Fil'_{i+k_1}$. (Note that for this direction we haven't used the assumption that Fil'_i is "good".)

Lemma 2. *If Fil'_i is a "good" filtration, then for some n, the maps $R_i \otimes Fil'_n \to Fil'_{i+n}$ are surjective for all i.*

Let us finish the proof of the proposition modulo the lemma. Take k_2 be such that $Fil'_n(M) \subset M_{k_2}$, where n is from the lemma. Then

$$Fil'_i(M) \subset Fil'_{i+n} \subset R_i \cdot Fil'_n(M) \subset R_i \cdot M_{k_2} \subset M_{k_2+i}.$$

Finally, set k to be $max(k_1, k_2)$

□

Let us now prove the lemma:

Proof. Let $gr'(M)$ be the associated graded of M with respect to the filtration Fil'_i. Let $\overline{m}' \subset gr'(M)$ be a finite collection of elements that generates it over $gr(R)$. There exists an integer n, such that $\overline{m}' \subset \bigoplus_{i \leq n} gr^i(M)$.

2.2. **Exercise 6.** Show that n satisfies the condition of the lemma. (Hint: use induction.)

□

2.3. **Exercise 7.** Assume that $gr(R)$ is Noetherian. (a) Let M be an R-module with a "good" filtration, and let $N \subset M$ be a submodule. Show that the induced filtration on N is also "good". (b) Show that R is also Noetherian.

3. Hilbert polynomials

3.1. The main example of graded algebras is the polynomial algebra $A = k[x_1, ..., x_n]$. In the sequel we'll mostly regard it as a filtered algebra.

Let M be a finitely generated module over A (without any filtration). We want to assign to M an integer, called its dimension (it basically reflects the dimension of the support of M).

Choose a good filtration M_i on M. Consider the function

$$\mathbb{Z} \to \mathbb{N} : \phi_M(i) = \dim_k(M_i).$$

Theorem 1. *There exists a polynomial p_M with rational coefficients of degree $\leq n$, such that $\phi_M(i) = p_M(i)$ for $i >> 0$.*

This polynomial is called the Hilbert polynomial of M. The proof relies on the following basic

Lemma 3. *Let ψ be a function $\mathbb{Z} \to \mathbb{N}$, such that for all $i >> 0$, $\psi(i+1) - \psi(i) = q(i)$, where q is a polynomial of degree m. Then there exists a polynomial p of degree $m+1$, such that $\psi(i) = q(i)$ for $i >> 0$.*

Proof. We proceed by induction on m. If $m = 0$, i.e. $\psi(i+1) = \psi(i) + n$, then $\psi(i) = i \cdot n + c$, where c is a constant.

Suppose the assertion is true for $m-1$. Let $c \cdot t^m$ be the leading term of q. Note that

$$t^m = (m+1)^{-1} \cdot ((t+1)^{m+1} - t^{m+1}) + \text{l.o.t.}.$$

Hence, if we put

$$\psi'(t) = \psi(t) - (c \cdot (m+1)^{-1}) \cdot t^{m+1},$$

we would obtain that $\psi'(i+1) - \psi'(i)$ is a polynomial of degree $\leq m-1$.

□

Now let us prove the theorem:

Proof. We proceed by induction on n. Consider the map $M \to M$, given by the multiplication by t_n. Let N' and N'' be its kernel and cokernel, respectively. Let us endow them with the induced filtrations; both are "good". We have an exact sequence:

$$0 \to N'_i \to M_i \xrightarrow{t_n} M_{i+1} \to N''_{i+1} \to 0.$$

Hence,

$$\phi_M(i+1) - \phi_M(i) = \phi_{N''}(i+1) - \phi_{N'}(i).$$

By the induction hypothesis, the RHS is a polynomial of degree $\leq n-1$. Applying the lemma, we conclude that ϕ_M is a polynomial of degree $\leq n$.

□

Proposition 2. *The degree of the polynomial ϕ_M does not depend on the choice of the good filtration.*

3.2. **Exercise 8.** Prove the proposition using Prop. 1.

Thus, $\deg(\phi_M)$ depends only on M. It is called the dimension of M.

3.3. **Exercise 9.** Show that if M' is a sub- (or quotient) module of M, then $\dim(M') \leq \dim(M)$.

4. Dimensions of modules over f.g. algebras

4.1. Our present goal is to define the notion of dimension for a f.g. module M over a f.g. k-algebra A.

If A is such an algebra, choose a surjective homomorphism $k[x_1, ..., x_n] \to A$, and set $\dim_A(M)$ to be the dimension of M as a module over the polynomial algebra. However, we have to insure that this definition does not depend on the presentation of A as a quotient of a polynomial algebra.

Proposition 3. *Let M be a f.g. module over an algebra $k[x_1, ..., x_n, y_1, ..., y_m]$ of dimension d'. Assume that M is f.g. as a module over the subalgebra $k[x_1, ..., x_n]$ of dimension d. Then $d = d'$.*

Proof. By induction we can assume that $m = 1$. Let $m_1, ..., m_k$ be generators of M as a module over $A = k[x_1, ..., x_n]$, and let $Fil_i(M)$ be a "good" filtration defined as in the proof of Prop. 1, i.e., $Fil_i(M) = A_i \cdot \{m_1, ..., m_k\}$. Let $Fil'_i(M)$ be the good filtration with respect to $A[y]$, defined in the same way, i.e.,

$$Fil'_i(M) = \sum_{0 \leq j \leq i} y^j \cdot Fil_{i-j}(M).$$

Evidently, $Fil_i(M) \subset Fil'_i(M)$, so the corresponding Hilbert polynomials satisfy $\phi_M(i) \leq \phi'_M(i)$, which implies the inequality on the degrees: $d \leq d'$.

Let k be the integer such that for each $m \in \{m_1, ..., m_k\}$,

$$y \cdot m \in Fil_k(M).$$

Then we obtain:

$$Fil'_i(M) \subset Fil_{k \cdot i}(M).$$

Hence, $\phi'_M(i) \leq \phi_M(i \cdot k)$. This implies $d \geq d'$.

□

4.2. **Exercise 10.** (a) Deduce that $\dim_A(M)$ is independent of the presentation of A as a quotient of a polynomial algebra. (b) Let M be a f.g. module over B, and let $A \to B$ be a homomorphism, such that M is f.g. over A as well. Show that $\dim_A(M) = \dim_B(M)$.

For a f.g. k-algebra A, we define $\dim(A)$ to be $\dim_A(A)$.

Theorem 2. *If A is integral, the two notions of dimension coincide.*

Proof. By the Noether normalization lemma, we can find a polynomial algebra $k[x_1, ..., x_n]$, that injects into A, and such that A is finite as a $k[x_1, ..., x_n]$-module. By what we saw earlier, the transcendental dimension of A over k equals n.

By the previous exercise, $\dim_A(A) = \dim_{k[x_1,...,x_n]}(A)$. However, since $k[x_1, ..., x_n]$ injects into A,

$$n \geq \dim_{k[x_1,...,x_n]}(A) \geq \dim_{k[x_1,...,x_n]}(k[x_1, ..., x_n]) = n.$$

□

4.3. Finally, we are ready to prove property (iv) of the dimension function. For that we will use the second definition of dimension.

Proof. Thus, let A be a domain, and B be proper quotient of A. We have to show that

$$\dim(A) = \dim_A(A) > \dim_A(B) = \dim_B(B).$$

Let x be any element in $ker(A \to B)$. By the assumption, the multiplication by x is an injective map $A \to A$ and the cokernel of this map surjects onto B.

Thus, it is sufficient to prove the following lemma:

Lemma 4. *Let M be a f.g. module over A, and $x \in A$ be such that $x : M \to M$ is injective. Then $\dim_A(M) \geq 1 + \dim_A(M/x \cdot M)$.*

□

Proof. We can assume that A is a polynomial algebra, let $deg(x) \leq k$. Choose a good filtration on M. We have

$$0 \to M_i \to M_{i+k} \to (M/x \cdot M)_{i+k} \to 0.$$

Hence,

$$\phi_M(i + k) - \phi_M(i) = \phi_{M/x\cdot M}(i + k).$$

This implies that $\deg(\phi_{M/x\cdot M}) < \deg(\phi_M)$.

□

4.4. **Exercise 11.** Let A be a k-domain, and let $A \supset \mathfrak{p}_1 \supset ... \supset \mathfrak{p}_n = 0$ be a chain of prime ideals. Show that $n \leq \dim(A)$.

The maximum of lengths of chains of prime ideals in a commutative ring R is called its Krull dimension (if the maximum is finite). We have seen that $\dim_{Krull}(A) \leq \dim(A)$.

4.5. **Exercise 12.** Show that $\dim_{Krull}(A) = \dim(A)$. (Use the Noether normalization lemma and the going up theorem.)