

1. THE NOTION OF DIMENSION

1.1. We will work with finitely generated algebras over a field k (for the peace of mind let us assume that k is algebraically closed, but everything is valid in the general case as well).

Let A be such an algebra, and assume that A is a domain, in order for $X = \text{Spec}(A)$ to be irreducible. We want to say what $\dim(X) := \dim(A)$ is. Here are some basic requirements:

- (i) $\dim(\mathbb{A}^n) = n$, where $\mathbb{A}^n = \text{Spec}(k[x_1, \dots, x_n])$.
- (ii) If $Y := \text{Spec}(B) \rightarrow X := \text{Spec}(A)$ for a homomorphism $\phi : A \rightarrow B$ is dominant (i.e., the image is dense, which is equivalent to ϕ being injective), then $\dim(Y) \geq \dim(X)$. Moreover, if $B = A_f$, we want that $\dim(A) = \dim(B)$.
- (iii) If $f : Y \rightarrow X$ is finite (i.e., $\phi : A \rightarrow B$ is such that B is f.g. as an A -module), then $\dim(Y) \leq \dim(X)$.
- (iv) If ϕ is surjective but not an isomorphism (i.e., Y is a proper closed subset in X), then $\dim(Y) < \dim(X)$.

1.2. **Exercise 1.** Show that if $f : Y \rightarrow X$ is finite in the above sense, then the preimage of every point $\mathfrak{p} \in X$ is a finite set. (Hint: prove first that the preimage $f^{-1}(\mathfrak{p})$ is isomorphic to the set of prime ideals in $B \otimes_A \text{Frac}(A/\mathfrak{p})$.)

2. TRANSCENDENCY DEGREE

2.1. Let $k \subset K$ be an extension of field. An element $x \in K$ is said to be algebraic, if it is integral, i.e., if it satisfies a non-zero polynomial equation

$$a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_1 \cdot x + a_0 = 0$$

where $a_i \in k$. Since k is a field, it does not matter whether we require that $a_n = 1$ or not, as long as it's non-zero.

2.2. **Exercise 2.** Show that x is algebraic if and only if the extension $k' := k(x)_K \subset K$, that it generates, is finite (i.e., is finite-dimensional as a k -vector space). Deduce that if x is algebraic, then x^{-1} is also algebraic over k .

We say that K is algebraic over k if every element $x \in K$ is.

2.3. **Exercise 3.** (a) Show that if $x_1, \dots, x_n \in K$ are elements, which are algebraic over k , then the subfield in K generated by them is an algebraic extension of k . (b) Show that an extension K is finite if and only if it's algebraic and finitely generated as a field over k . (c) Show that if $k' \supset k$ is algebraic, and $k'' \supset k'$ is algebraic, then $k'' \supset k$ is algebraic.

Note that for a field extension $k \subset K$ there is a difference between being generated as a field and as an algebra by a collection of elements $x_1, \dots, x_n \in K$:

The latter means that every element in K is the sum of products of the x_i 's. I.e., K is the minimal k -subalgebra that contains x_1, \dots, x_n .

On the other hand, being generated as a field means that every element in K can be represented as a ratio of two elements, each of which is the sum of products of the x_i 's. I.e., K is the minimal subfield that contains x_1, \dots, x_n .

Note that if A is f.g. as a k -algebra, then $\text{Frac}(A)$ is f.g. as a field extension.

In the sequel, when we speak about fields, "generation" will be understood in the field sense, unless specified otherwise.

2.4. Elements x_1, \dots, x_n of K are said to be algebraically independent over k if they do not satisfy a (multy)-polynomial equation with coefficients in k .

2.5. **Exercise 4.** Show that the condition is equivalent to the fact that the homomorphism of k -algebras $k[x_1, \dots, x_n] \rightarrow K$ is injective.

If x_1, \dots, x_n are algebraically independent, we obtain a map $k(x_1, \dots, x_n) := \text{Frac}(k[x_1, \dots, x_n]) \rightarrow K$.

The transcendency degree of K over k is the maximal number of algebraically independent elements (if such a number is finite). If there are ∞ -many algebraically independent elements, we say that the transcendency degree is infinite.

2.6. **Exercise 5.** Let $\{x_1, \dots, x_n, \dots\}$ be a (possibly infinite) collection of elements of K . Show that the following conditions are equivalent: (a) for any $y \in K$, the collection $\{y, x_1, \dots, x_n\}$ is algebraically dependent. (b) K is an algebraic extension of the subfield $k(x_1, \dots, x_n, \dots)_K$ generated by the x_i 's.

We will call a collection of elements $\{x_1, \dots, x_n, \dots\} \subset K$ "maximal" if it satisfies the equivalent conditions of the above exercise. Note that we are not requiring for $\{x_1, \dots, x_n, \dots\}$ to be algebraically independent.

Proposition 1. Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ be two collections of elements of K . Assume that the x_i 's are algebraically independent, and $\{y_1, \dots, y_m\}$ is maximal. Then $n \leq m$.

Proof. Consider the collection of elements $\{x_1, y_1, y_2, \dots, y_m\}$. By assumption, they are not algebraically independent. Hence, there exists a polynomial on $m + 1$ variables F_1 with coefficients in k , such that $F_1(x_1, y_1, \dots, y_m) = 0$. This polynomial involves at least one y_j (for, otherwise, x_1 would be algebraic over k). With no loss of generality, we can assume that $j = 1$.

We obtain that y_1 is algebraic over $k(x_1, y_2, \dots, y_m)_K$. Hence, $k(x_1, y_1, y_2, \dots, y_m)_K$ is an algebraic extension of $k(x_1, y_2, \dots, y_m)_K$. Hence, by Exercise 3, K is algebraic over $k(x_1, y_2, \dots, y_m)_K$. By Exercise 5, we conclude that the collection $\{x_1, y_2, \dots, y_m\}$ is maximal.

Continuing the process, we can replace all the y_j 's by the x_i 's. Hence, we obtain that the collection $\{x_1, \dots, x_m\}$ is maximal. Hence, $\{x_1, \dots, x_m, x_{m+1}\}$ cannot be algebraically independent. □

Corollary 1. If $K \supset k$ is finitely generated, then it has a finite transcendency degree.

2.7. **Exercise 6.** Prove the corollary.

2.8. For a domain A as above, we set $\dim(A)$ to be the transcendency degree of $\text{Frac}(A)$ over k . By Exercise 6, $\dim(A)$ is finite. Now we have to verify the properties. Property (i) is obvious. Property (ii) follows from the next

Lemma 1. For $k \subset K' \subset K$, $\text{tr.deg.}(K'/k) \leq \text{tr.deg.}(K/k)$.

2.9. **Exercise 7.** Prove the lemma.

To establish property (iii) we'll use a trick, called the Noether normalization lemma. (Here we'll need the assumption that k is infinite, which is satisfied if it is algebraically closed.)

Theorem 1. *Let $A \rightarrow B$ be a map of f.g. algebras over k , such that B is f.g. as an A -module. We can produce a commutative diagram:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ k[x_1, \dots, x_n] & \longleftarrow & k[y_1, \dots, y_m], \end{array}$$

where $m \leq n$ and y_j 's are linear combinations of x_i 's, and A (resp., B) is finite over $k[x_1, \dots, x_n]$ (resp., $k[y_1, \dots, y_m]$), and the vertical maps are injective.

Usually, the Noether normalization theorem is stated just for A : that we can find an injective homomorphism $k[x_1, \dots, x_n] \rightarrow A$, such that A is f.g. as a module over $k[x_1, \dots, x_n]$.

The above commutative diagram can be geometrically translated as

$$\begin{array}{ccc} X & \longleftarrow & Y \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{A}^m, \end{array}$$

where $\mathbb{A}^m \simeq k^m$ is a linear space quotient of $\mathbb{A}^n = k^n$, not necessarily respecting the basis, and the vertical maps are dominant and finite.

The proof of the theorem relies on the following key lemma:

Lemma 2. *Let $\phi : k[x_1, \dots, x_n] \rightarrow A$ be such that A is a finite as a module over $k[x_1, \dots, x_n]$. If ϕ is not injective, we can find y_1, \dots, y_{n-1} , which are linear combinations of the x_i 's so that A is finite as a module over $k[y_1, \dots, y_{n-1}]$.*

Let's prove the lemma first:

Proof. Since ϕ isn't injective, there exists a polynomial in n variables F , such that

$$F(\phi(x_1), \dots, \phi(x_n)) = 0.$$

Let d be the degree of this polynomial, and let f be its homogeneous component of highest degree. Without loss of generality, we can assume that x_n enters f . Since k was assumed infinite, f is not identically zero on k^n . Hence, there exist $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}, \lambda'_n \in k$, such that

$$f(\lambda'_1, \dots, \lambda'_n) \neq 0.$$

By homogeneity, if we set $\lambda_i = \lambda'_i \cdot \lambda_n^{-1}$,

$$f(\lambda_1, \dots, \lambda_{n-1}, 1) = f(\lambda'_1, \dots, \lambda'_{n-1}, \lambda'_n) \cdot \lambda_n^{-d} = c \neq 0.$$

Set $y_j = x_j - \lambda_j \cdot x_n$ for $j = 1, \dots, n-1$. Let $A' \subset A$ be the subalgebra, generated by the $\phi(x_i)$'s, $i = 1, \dots, n$ and $A'' \subset A'$ be the subalgebra of A' , generated by the $\phi(y_j)$'s, $j = 1, \dots, n-1$.

Claim: $\phi(x_n) \in A'$ is integral over A'' .

Proof: We'll produce a monic polynomial $G(t)$ of degree d with coefficients in A'' , such that $G(x_n) = 0$. Indeed, set

$$G(t) = c^{-1} \cdot F(y_1 + \lambda_1 \cdot t, \dots, y_{n-1} + \lambda_{n-1} \cdot t, t).$$

2.10. **Exercise 8.** Show that $G(t)$ has the required properties.

2.11. **Exercise 9.** Deduce that A is finitely generated as a module over A'' . □

2.12. **Exercise 10.** Deduce the theorem from the lemma.

Let us now prove that our definition satisfies property (iii).

Lemma 3. *Let A be a f.g. k -domain, and let $k[x_1, \dots, x_n] \rightarrow A$ be an injective homomorphism, such that A is f.g. as a $k[x_1, \dots, x_n]$ -module. Then $\dim(A) = n$.*

Proof. The assumption implies that $\text{Frac}(A)$ is algebraic over $k(x_1, \dots, x_n)$. □

Thus, Theorem 1 gives us an "effective" way to calculate the dimension.

Let now $\phi : A \rightarrow B$ be a homomorphism of domains, such that B is finite as an A -module. Consider the diagram as in the theorem. We obtain that $\dim(B) = m \leq n = \dim(A)$. This proves property (iii).

Finally, it remains to prove point (iv). This is more serious, and we'll have to work a different approach to the notion dimension, based on generating functions.

2.13. **Digression: another proof of weak Nullstellensatz.** We'll use the above Noether normalization lemma to prove the following Theorem:

Theorem 2. *Let k be an algebraically closed field, and $K \supset k$ a field extension, such that K is f.g. as a k -algebra. Then $K = k$.*

Proof. Let $k[x_1, \dots, x_n] \rightarrow K$ be an injective homomorphism, such that K is f.g. as module over $k[x_1, \dots, x_n]$. Assume that $n > 0$. Then there exists a non-zero prime ideal \mathfrak{m} in $k[x_1, \dots, x_n]$.

By the going-up theorem (which relied only on Nakayama's lemma), there exists a prime ideal \mathfrak{m}' in K , such that $\mathfrak{m}' \cap k = \mathfrak{m}$. This is a contradiction, since K is a field. □