Course notes: Week 7, Sheaves (horrors continue)

1. Proof of the existence of $Loc(M)$

1.1. The construction of $Loc(M)$ is forced on us. let $U \subset Spec(R)$ be an open subset, and let $U_i, i \in I$ be its cover, where each U_i is of the form U_{f_i}. Note that U is not quasi-compact, so we cannot assume that the set I is finite.

By (Sh 1) and (Sh 2) we must have:

$$\Gamma(U, Loc(M)) = \{s_i \in M_{f_i} \,|\, res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j), \}$$

where the restriction maps are the natural morphisms

$$\Gamma(U_i, Loc(M)) = M_{f_i} \to M_{f_i \cdot f_j} \simeq \Gamma(U_i \cap U_j, Loc(M)).$$

What we must show is the correctness of this definition. I.e., that $\Gamma(U, Loc(M))$ thus defined is independent of the choice of the cover by basic open subsets.

1.2. **Step 1.** Assume first that U is itself a basic open subset U_f. If we denote $R' = R_f$ and $M' = M_f$, then $U_f \simeq Spec(R')$. So effectively, this case does not differ from U_f being the entire $Spec(R) = X$, which we will assume, to simplify the notation. In particular, it is quasi-compact.

A priori, we must have $\Gamma(X, Loc(M)) = M$. The natural maps $M \to M_{f_i}$ define us a map

$$M \to \{s_i \in M_{f_i} \,|\, res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j), \}$$

and we must show that it is an isomorphism. Observe, first, that it is injective, by Serre's lemma. Hence, we must prove the surjectivity.

Lemma 1. *Suppose that $J \subset I$ is a subset, such that $U_j, j \in J$ cover $Spec(R)$, and suppose that for a collection $\{s_i \in M_{f_i}\}$ as above, there exists a section $s \in M$, such that $res^X_{U_j}(s) = s_j$ for $j \in J$. Then $res^X_{U_i}(s) = s_i$ for all $i \in I$.*

1.3. **Exercise 1.** Prove the lemma. Hint: use Serre's lemma and the fact that for every i, $U_j \cap U_i, j \in J$ cover U_i.)

Using the lemma, we can assume that the set I is finite.

1.4. **Exercise 2.** Figure out where in the argument we use the finiteness of I. (Hint: localization with respect to a multiplicative subset respects arbitrary direct sums ((a) prove it), but not infinite direct products ((b) give a counterexample).)

Let us now rewrite the condition on $\{s_i\}$ as a kernel of some map. We can do it for any sheaf $\mathcal{F}$. Consider two maps

$$\prod_{i \in I} \Gamma(U_i, \mathcal{F}) \rightrightarrows \prod_{(i_1, i_2) \in I \times I} \Gamma(U_{i_1} \cap U_{i_2}, \mathcal{F}).$$

Recall that to define a map into a direct product we must specify the composition of this map with the projection on each of the factors.

The first map, projected onto the (i_1, i_2) factor sends $\{s_i\}$ to $res^{U_{i_1}}_{U_{i_1} \cap U_{i_2}}(s_{i_1})$. The second map, projected onto the same factor sends $\{s_i\}$ to $res^{U_{i_2}}_{U_{i_1} \cap U_{i_2}}(s_{i_2})$.

Observe that collections $\{s_i\} \in \prod_{i \in I}, \Gamma(U_i, \mathcal{F})$, that agree on the overlaps are the same as the *equalizer* (i.e., the kernel of the difference) if these two maps.

Thus, we have to show that

$$0 \to M \to \prod_{i \in I} M_{f_i} \to \prod_{(i_1,i_2) \in I \times I} M_{f_i \cdot f_j} \tag{1}$$

is exact.

Lemma 2. *Let* $0 \to N_1 \to N_2 \to N_3$ *be a complex of* R*-modules. Then it is exact if and only if the corresponding localized complexes* $0 \to (N_1)_{f_i} \to (N_2)_{f_i} \to (N_3)_{f_i}$ *are exact for every* i*.*

The lemma follows immediately from the exactness of localization and Serre's lemma.

Hence, we have to study the localization of the complex appearing in (1) with respect to every choice of $i' \in I$. Set $R' := R_{f_{i'}}$, $M' := M_{f_{i'}}$, $U' := U_{f_{i'}}$, $U'_i := U' \cap U_i = U_{f_i \cdot f_{i'}}$. The complex of interest looks as follows:
This complex looks as follows:

$$0 \to M' \to \prod_{i \in I} M'_{f_i} \to \prod_{(i_1,i_2) \in I \times I} M'_{f_i \cdot f_j}$$

We must show that every collection $\{s'_i \in M'_{f_i}\}$, that agrees on the overlaps comes from an element $m' \in M'$. But for $i = i'$, $U'_i = U'$. I.e., we are in the situation of Lemma 1 for R replaced by R'! Hence the existence of such m' follows.

1.5. **Step 2.** Let now U be arbitrary. We must show that if $U_i, i \in I$ and $U_{i'}, i' \in I'$ are two coverings of U, then the two sets

$$\{s_i \in M_{f_i} \,|\, res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j), \} \text{ and } \{s_{i'} \in M_{f_{i'}} \,|\, res^{U_{i'}}_{U_{i'} \cap U_{j'}}(s_{i'}) = res^{U_{j'}}_{U_{i'} \cap U_{j'}}(s_{j'}), \}$$

are canonically isomorphic. Set $I'' = I \times I'$, and for $i'' = (i, i')$ define $U_{i''} := U_i \cap U_{i'}$. We claim that each of the sets above identifies with the corresponding set for I''. More precisely, that the natural map

$$\{s_i \in M_{f_i} \,|\, res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j), \} \to \{s_{i''} \in M_{f_{i''}} \,|\, res^{U_{i''}}_{U_{i''} \cap U_{j'}}(s_{i''}) = res^{U_{j''}}_{U_{i''} \cap U_{j''}}(s_{j''}), \} \tag{2}$$

is an isomorphism, and similarly, for I'.

Given a collection $\{s_{i''}\}$ we reconstruct the corresponding collection $\{s_i\}$ as follows. For every fixed i, the open subsets $U_i \cap U_{i'}, i' \in I'$ cover U_i, and the elements $s_{(i,i'')} \in M_{f_i \cdot f_{i'}}$ are compatible on the overlaps. Hence, by Step 1, there exists $s_i \in M_{f_i}$, such that its image in $M_{f_i \cdot f_{i'}}$ equals $s_{(i,i'')}$.

1.6. **Exercise 3.** (a) Prove that the resulting collection $\{s_i\}$ agrees on the overlaps.
(b) Show that the map constructed just now is indeed the inverse to the map of (2).

1.7. **Step 3.** Thus, we have a well-defined $\Gamma(U, Loc(M))$ for every M. It remains to construct the restriction maps $\Gamma(U_2, Loc(M)) \to \Gamma(U_1, Loc(M))$ for $U_1 \subset U_2$.

1.8. **Exercise 4.** Carry out the construction of restriction maps.

2. Construction of a sheaf associated to a presheaf

2.1. We want to show the existence of the functor $PreSh(X) \to Sh(X)$, which is the left adjoint to the tautological functor $Sh(X) \to PreSh(X)$. I.e., given a presheaf $\mathcal{F}$, we would like to produce a sheaf $\underline{\mathcal{F}}$, such that

$$Hom_{Sh(X)}(\underline{\mathcal{F}}, \mathcal{F}') = Hom_{PreSh(X)}(\mathcal{F}, \mathcal{F}'),$$

whenever $\mathcal{F}'$ is a sheaf. The idea is to "kill" those sections of $\mathcal{F}$ that fail to satisfy (Sh 1), and add more sections in order to have (Sh 2).

The scheme of the construction is as follows. We will construct a functor $PreSh(X) \to PreSh(X) : \mathcal{F} \mapsto \mathcal{F}^+$ with the following properties:

- (i) $Hom_{PreSh(X)}(\mathcal{F}^+, \mathcal{F}') = Hom_{PreSh(X)}(\mathcal{F}, \mathcal{F}')$, whenever $\mathcal{F}'$ is a sheaf.
- (ii) For any $\mathcal{F}$, $\mathcal{F}^+$ satisfies (Sh 1).
- (iii) If $\mathcal{F}$ satisfies (Sh 1), then $\mathcal{F}^+$ satisfies (Sh 2).

2.2. **Exercise 5.** Show that the above three properties imply that the functor $\mathcal{F} \mapsto \mathcal{F}^{++}$ maps $PreSh(X)$ to $Sh(X)$ and is the desired left adjoint.

2.3. Here is the construction of $\mathcal{F}^+$. Let U be an open subset of X. Consider the set A, whose elements are coverings of U. I.e., for every $a \in A$ we have a subset I_a in the set of all open subsets of U, such that $\underset{i \in I_a}{\cup} U_i = U$.

For every $a \in A$ set

$$M_a = \{s_i \in \Gamma(\mathcal{F}, U_i), i \in I_a \,|\, res^{U_i}_{U_i \cap U_j}(s_i) = res^{U_j}_{U_i \cap U_j}(s_j).$$

The set A is partially ordered. Namely, we say that $a \leq b$ if for every $j \in I_b$ there exists $i \in I_a$, such that $U_j \subset U_i$.

Proposition-Construction 1. *For $a \leq b$ we have a natural map $M_a \to M_b$. If $a \leq b \leq c$, the composition $M_a \to M_b \to M_c$ equals the map $M_a \to M_c$.*

Proof. Let a and b be as above. For a collection $\{s_i\} \in M_a$ and $j \in I_b$, choose $i \in I_a$, such that $U_j \subset U_i$. Define s_j as $res^{U_i}_{U_j}(s_i)$. If i' is another choice of an element of I_a, such that $U_j \subset U_i$, then we claim that

$$res^{U_{i'}}_{U_j}(s_{i'}) = res^{U_i}_{U_j}(s_i).$$

2.4. **Exercise 6.** Prove it!

Hence, the collection $\{s_j, j \in I_b\}$ is well-defined.

2.5. **Exercise 7.** (a) Show that $\{s_j, j \in I_b\}$ constructed above is indeed an element of M_b, i.e., that s_j's agree on the overlaps. (b) Prove the second assertion of the proposition (about $a \leq b \leq c$).

□

2.6. **Exercise 8.** Show that A is directed. (Hint: any two open covers admit a common refinement.)

Set

$$\Gamma(U, \mathcal{F}^+) = \varinjlim_{A} M_a.$$

2.7. Let now U' be a subset of U. We need to construct a map

$$\Gamma(U, \mathcal{F}^+) \to \Gamma(U', \mathcal{F}^+).$$

Recall that to construct a map *from* an inductive limit somewhere is the same as to have a compatible system of maps from our inductive system to this "somewhere".

Hence, we must construct a compatible family of maps $M_a \to \Gamma(U', \mathcal{F}^+)$. For $a \in A$ consisting of a covering U_i of U, define a covering of U' by $U' \cap U_i$. We thus obtain an element $a' \in A'$, where A' is the corresponding indexing set for U'. Let $M'_{a'}$ be the corresponding set.

The restriction map defines us a map $M_a \to M'_{a'}$. By the construction of inductive limits, we have a map $M'_{a'} \to \Gamma(U', \mathcal{F}^+)$. By composing, we obtain the desired map

$$M_a \to \Gamma(U', \mathcal{F}^+).$$

The fact that for $a \leq b$ the map composed map $M_a \to M_b \to \Gamma(U', \mathcal{F}^+)$ equals the map $M_a \to \Gamma(U', \mathcal{F}^+)$ is a straightforward verification. Thus, we obtain a map $res^U_{U'} : \Gamma(U, \mathcal{F}^+) \to \Gamma(U', \mathcal{F}^+)$, as required. It is easy to see that the system of maps $res^U_{U'}$ for $\mathcal{F}^+$ is compatible with double inclusions. So, $\mathcal{F}^+$ is indeed a presheaf.

2.8. **Exercise 9.** Prove properties (i)-(iii) of $\mathcal{F}^+$.

2.9. **Example: the constant sheaf.** Consider the naive constant presheaf $\underline{R}'_X$. As you proved in the previous problem set, in the category of presheaves of R-modules, we have:

$$Hom_{PreSh(X)}(\underline{R}'_X, \mathcal{F}') \simeq \Gamma(X, \mathcal{F}')$$

for every presheaf $\mathcal{F}'$.

Define the constant sheaf $\underline{R}_X$ as the sheaf associated to $\underline{R}'_X$. By definition, we have:

$$Hom_{Sh(X)}(\underline{R}_X, \mathcal{F}) \simeq Hom_{PreSh(X)}(\underline{R}'_X, \mathcal{F}).$$

Hence, for a sheaf $\mathcal{F}$,

$$Hom_{Sh(X)}(\underline{R}_X, \mathcal{F}) \simeq \Gamma(X, \mathcal{F}').$$

2.10. **Exercise 10.** Show that

$$\Gamma(U, \underline{R}_X) = \underset{\underset{B}{\longrightarrow}}{lim} \prod_{i \in I_b} R,$$

where B is a set, whose elements, denoted b, are partitions of U into a disjoint union $U = \bigcup_{i \in I_b} U_i$ of open subsets.

3. Construction of cokernels

3.1. Let us now prove the existence of cokernels in the category if sheaves. Thus, let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be a map of sheaves. We need to show that the functor $Sh(X) \to Sets$ given by

$$\mathcal{F} \mapsto \{\beta : \mathcal{F}_2 \to \mathcal{F} \,|\, \beta \circ \alpha = 0\}$$

is representable by some $\mathcal{F}_3 \in Sh(X)$.

Let $\mathcal{F}'_3$ be the cokernel of α is the category of presheaves.

3.2. **Exercise 11.** Show that $\underline{\mathcal{F}_3}$ is the desired cokernel.

3.3. Let us now describe explicitly when a map of sheaves is a surjection *in the category of sheaves.*

Proposition 1. *A map* $\alpha : \mathcal{F}' \to \mathcal{F}$ *is a surjection of sheaves if and only if for every* $s \in \Gamma(U, \mathcal{F})$ *there exists a covering* $U = \underset{i \in I}{\cup} U_i$*, such that* $res^U_{U_i}(s) \in \Gamma(U_i, \mathcal{F})$ *is the image of some* $s'_i \in \Gamma(U_i, \mathcal{F}')$.

I.e., a map of sheaves is surjective *not when every section of* $\mathcal{F}$ *is the image of a section of* $\mathcal{F}'$, which would be too strong, but when this holds locally.

3.4. **Exercise 12.** (a) Show that the above condition is sufficient for surjectivity directly (without resorting to the construction of the associated sheaf). (b) Show the "only if part" using Exercise 10, and properties (ii) and (iii) of the +-construction.

3.5. **Exercise 13.** Let $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ be a map of sheaves, such that $ker(\alpha) = 0$ and $coker(\alpha) = 0$. Show that α is an isomorphism.

Note that the implication $\{ker(\alpha) = 0 \text{ and } coker(\alpha) = 0\} \Rightarrow \{\alpha \text{ is an isomorphism}\}$ is not true in every category. Consider the category, whose objects are Banach spaces, and morphisms are continuous linear maps. Then a morphism is an injection if it's an injective in the naive sense (disregard the topology). A morphism is a surjection if its image is dense. (Show it!) Then, evidently, not every dense embedding is an isomorphism.

3.6. In what follows we'll say that a complex of sheaves $... \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to ...$ is exact, if $\mathcal{F}_1$ maps surjectively onto the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By the above discussion, this does *not* imply that for every open subset U, the complex

$$... \to \Gamma(U, \mathcal{F}_1) \to \Gamma(U, \mathcal{F}_2) \to \Gamma(U, \mathcal{F}_3) \to ...$$

is exact.

3.7. **Example.** Let X be a differentiable manifold. Let $\mathcal{F}^i$ be the sheaf Ω^i_X of i-forms. We have a complex

$$\underline{\mathbb{R}}_X \to C^\infty_X \to \Omega^1_X \to ...$$

3.8. **Exercise 14.** Show that the above complex of sheaves is exact.

Of course, for the above complex of sheaves, the corresponding complex of global sections is not always exact: it gives the De Rham cohomology of X.

3.9. **Exercise 15.** Show that the functor $\Gamma(U, \cdot) : Sh(X) \to Ab$ is left-exact.

4. More on localization

4.1. Let R be a commutative ring, and $X = Spec(R)$. Consider the sheaf of abelian groups $Loc(R)$. It is easy to see that it has a natural structure of a sheaf of rings; it's usually denoted by $\mathcal{O}_X$. Note that every $Loc(M)$ is naturally a sheaf of $\mathcal{O}_X$-modules, i.e. each $\Gamma(U, Loc(M))$ has a structure of a module over the ring $\Gamma(U, \mathcal{O}_X)$. Let $\mathcal{O}_X - mod$ denote the category of *sheaves of* $\mathcal{O}_X$*-modules.* Thus, we obtain a functor

$$Loc : R - mod \to \mathcal{O}_X \quad \text{mod} .$$

Note that the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) : \mathcal{O}_X - mod \to Ab$ naturally factors through a functor $\mathcal{O}_X - mod \to R - mod$, because $\Gamma(X, \mathcal{O}_X) \simeq R$.

4.2. **Exercise 16.** Show that *Loc* is the left adjoint of the above functor $\Gamma : \mathcal{O}_X - mod \to R - mod$.

4.3. **Exercise 17.** Show that *Loc* is exact (i.e., it maps short exact sequences to short exact sequences).

4.4. Let us have one more example of how the exactness of the functor $\Gamma(U, \cdot)$ fails even for sheaves of the form $Loc(M)$.

Take $R = k[x, y]$ (where k is assumed algebraically closed for simplicity). Let $Y \subset X$ be the closed subset $\{0\}$, i.e. $Y = V(\mathfrak{m}_0)$, where $\mathfrak{m}_0$ is the maximal ideal corresponding to the evaluation at $0 \in k^2$. Set $U = X - Y$; evidently $U = U_x \cup U_y$.

4.5. **Exercise 18.** Show that $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is an isomorphism. (Hint: you have to recall how $\Gamma(U, Loc(M))$ is defined, when U is not a basic open.)

Consider the following short exact sequence of R-modules:

$$0 \to k[x, y] \to k[x, y]_x \to k[x, y]_x/k[x, y] \to 0.$$

4.6. **Exercise 19*.** Show that the complex

$$0 \to \Gamma(U, Loc(k[x, y])) \to \Gamma(U, Loc(k[x, y]_x)) \to \Gamma(U, Loc(k[x, y]_x/k[x, y])) \to 0$$

is *not* exact.

5. Direct and inverse images

5.1. Let $f : Y \to X$ me a continuous map of topological spaces. We will define a functor $f_* : Sh(Y) \to Sh(X)$, called the direct image of sheaves.

For $\mathcal{F} \in Sh(Y)$ we define $f_*(\mathcal{F}) \in Sh(X)$ as follows:

$$\Gamma(U, f_*(\mathcal{F})) \simeq \Gamma(f^{-1}(U), \mathcal{F}).$$

5.2. **Exercise 20.** (a) Show that $f_*(\mathcal{F})$ satisfies the sheaf axioms. (b) For $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ define a map $f_*(\alpha) : f_*(\mathcal{F}_1) \to f_*(\mathcal{F}_2)$.

Let us consider some examples.

5.3. **Exercise 21.** Let $X = pt$ (the topological space consisting of one point) and Y any topological space; f is then the tautological map. (a) Show that the category $Sh(X)$ is then equivalent to the category of abelian groups. (b) Show that *under this equivalence* the functor f_* is isomorphic to $\Gamma(Y, \cdot)$.

5.4. **Exercise 22.** Let $\phi : A \to B$ be a homomorphism of commutative rings, $X = Spec(A)$, $Y = Spec(B)$, and $f = \Phi-$ the corresponding map of spectra. Show that for a B-module M, $f_*(Loc_B(M))$ is naturally isomorphic to $Loc_A(M)$, where the superscript indicates where we are taking the corresponding sheaf, and in the latter case M is regarded as an A-module via ϕ.

5.5. **Inverse images.** The rest is optional (but also very important for mathematical survival).

Theorem 1. *The functor f_* admits a left adjoint.*

The left adjoint, which will be constructed below is called "the inverse image" or "pull-back" functor, and denoted f^*.

Proof. First, we construct a functor $f^{'*} : Sh(X) \to PreSh(Y)$. Let $\mathcal{F}$ be a sheaf on X. For an open $U \subset Y$ we set

$$\Gamma(U, f^{'*}(\mathcal{F})) = \underset{\underset{U_X}{\longrightarrow}}{lim}\, \Gamma(U_X, \mathcal{F}),$$

where U_X runs over the set ordered set of open subsets of X, which contain the image of U. (The ordering is defined so that $U'_X \geq U''_X$ is $U'_X \subset U''_X$, i.e., "the smaller the subset, the bigger the element".)

It's easy to see that $f^{'*}(\mathcal{F})$ is indeed a presheaf. We claim that for a sheaf $\mathcal{F}_Y$ on Y, we have:

$$Hom_{PreSh(Y)}(f^{'*}(\mathcal{F}), \mathcal{F}_Y) \simeq Hom_{Sh(X)}(\mathcal{F}, f_*(\mathcal{F}_Y)),$$

functorially in $\mathcal{F}_Y$. Finally, we define

$$f^*(\mathcal{F}) := \underline{f^{'*}(\mathcal{F})}.$$

From the above, $f^*(\cdot)$ is indeed the left adjoint to $f_*(\cdot)$.

□

5.6. If $Y = pt$, and $f : Y \to X$ corresponds to a point $x \in X$, the sheaf $f^*(\mathcal{F})$ on Y, thought of as an abelian group, is called "the stalk of $\mathcal{F}$ at x".

5.7. **Exercise 23*.** Let $X = Spec(R)$, $\mathcal{F} = Loc(M)$, and $x \in X$ correspond to some prime ideal $\mathfrak{p}$. Show that the stalk of $\mathcal{F}$ at x is isomorphic to $M_{\mathfrak{p}}$.

5.8. Those who want to know how to interpret fibers, are welcome to schedule an appointment.