

Course notes: Week 6, Sheaves (a horror story)

1. BASIC DEFINITIONS

1.1. Let  $X$  be a topological space. A presheaf  $\mathcal{F}$  (of sets, abelian groups,  $R$ -modules) on  $X$  is an assignment for every open subset

$$(U \subset X) \mapsto \Gamma(U, \mathcal{F}),$$

where  $\Gamma(U, \mathcal{F})$  is a set (resp., abelian group,  $R$ -module), and for every inclusion  $U_1 \subset U_2$  of open subsets a map

$$res_{U_1}^{U_2} : \Gamma(U_2, \mathcal{F}) \rightarrow \Gamma(U_1, \mathcal{F})$$

of sets (resp., abelian groups,  $R$ -modules), such that for a double inclusion  $U_1 \subset U_2 \subset U_3$ ,

$$res_{U_1}^{U_2} \circ res_{U_2}^{U_3} = res_{U_1}^{U_3},$$

and for  $U = U_1 = U_2$ ,  $res_U^U = Id$ . The set  $\Gamma(U, \mathcal{F})$  is called "the set of sections of  $\mathcal{F}$  over  $U$ ".

1.2. **Exercise 1.** Define the category  $Open(X)$  to have as objects open subsets of  $X$ , and  $Hom(U_1, U_2)$  be empty unless  $U_1$  is contained in  $U_2$ , and if  $U_1 \subset U_2$  be a 1-element set. The composition is defined naturally. Show that a presheaf of sets (resp., abelian groups,  $R$ -modules) is the same as a functor  $Open(X) \rightarrow Sets^{op}$  (resp.,  $Ab^{op}$ ,  $R-mod^{op}$ ).

1.3. A presheaf is called a sheaf if the following two axioms hold:

(Sh 1) If  $U$  is an open subset equal to  $\bigcup_{i \in I} U_i$  (where  $I$  is some, possibly infinite, indexing set), the map

$$\Gamma(U, \mathcal{F}) \rightarrow \prod_{i \in I} \Gamma(U_i, \mathcal{F})$$

is injective. (I.e., if two sections over  $U$  are such that their restrictions on every  $U_i$  coincide, then they are equal.)

(Sh 2) If for  $U$  and  $U_i, i \in I$  as above there are sections  $s_i \in \Gamma(U_i, \mathcal{F})$ , such that for every  $i, j \in I$ ,

$$res_{U_i \cap U_j}^{U_i}(s_i) = res_{U_i \cap U_j}^{U_j}(s_j),$$

then there exists a section  $s \in \Gamma(U, \mathcal{F})$ , such that  $s_i = res_{U_i}^U(s)$ .

1.4. **Exercise 2.** Show that given (Sh 1), the section  $s$ , whose existence is required in (Sh 2) is automatically unique.

2. EXAMPLES

2.1. Let  $X$  be a topological space. Define the sheaf  $Cont_X$  as follows:  $\Gamma(U, Cont_X)$  is the  $\mathbb{C}$ -vector space of  $\mathbb{C}$ -valued continuous functions on  $U$ . The restriction maps are defined naturally.

Variant: let  $X$  be a  $C_X^\infty$ -manifold. Define the sheaf  $C^\infty$  by setting  $\Gamma(U, C_X^\infty)$  to be the  $\mathbb{C}$ -vector space of smooth functions on  $U$ . Similarly, we can define the sheaves  $\Omega_X^i$  of  $i$ -differential forms.

2.2. **Exercise 3.** Check the sheaf axioms in the above examples.

**2.3. The constant sheaf.** Assume for simplicity that  $X$  is such that every open subset of  $U$  is the union of non-intersecting connected sets (its connected components).

For a ring  $R$ , let  $\underline{R}_X$  be the sheaf of  $R$ -modules, such that

$$\Gamma(U, \underline{R}_X) = \prod_i R,$$

where the product is taken over the set of connected components of  $U$ . For  $U_1 \subset U_2$  the restriction map is defined as follows. If  $U_1 = \bigcup_{i \in I} (U_1)_i$  and  $U_2 = \bigcup_{j \in J} (U_2)_j$  are the decompositions into connected components, for every  $i$  there exists a unique  $j$ , such that  $(U_1)_i \subset (U_2)_j$ . This defines a map

$$\prod_{j \in J} R \rightarrow \prod_{i \in I} R.$$

**2.4. Exercise 4.** (a) Show that  $\underline{R}_X$  indeed satisfies the sheaf axioms. (b) Show that the presheaf, whose set of sections over every  $U$  is simply  $R$  *does not* satisfy the sheaf axioms.

### 3. THE CATEGORY OF SHEAVES

**3.1.** In what follows we will mostly speak about (pre)sheaves of abelian groups. Modifications for sheaves of sets or sheaves of  $R$ -modules are straightforward.

Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two presheaves. A morphism  $\alpha : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is an assignment for every open subset  $U \subset X$  a map

$$\alpha_U : \Gamma(U, \mathcal{F}_1) \rightarrow \Gamma(U, \mathcal{F}_2),$$

such that for  $U_1 \subset U_2$  the diagram

$$\begin{array}{ccc} \Gamma(U_2, \mathcal{F}_1) & \longrightarrow & \Gamma(U_2, \mathcal{F}_2) \\ \text{res}_{U_1}^{U_2} \downarrow & & \text{res}_{U_1}^{U_2} \downarrow \\ \Gamma(U_1, \mathcal{F}_1) & \longrightarrow & \Gamma(U_1, \mathcal{F}_2) \end{array}$$

commutes.

A morphism between two sheaves is by definition the same thing as a morphism between them as presheaves.

If  $\alpha : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  and  $\beta : \mathcal{F}_2 \rightarrow \mathcal{F}_3$  are morphisms of (pre)sheaves, we can compose them and obtain a morphism  $\beta \circ \alpha : \mathcal{F}_1 \rightarrow \mathcal{F}_3$ . Thus, (pre)sheaves form a category, denoted  $Sh(X)$  (resp.,  $PreSh(X)$ ). We have an obvious functor  $Sh(X) \rightarrow PreSh(X)$ . One of the main points, as we shall see, is that this functor admits a left adjoint.

**3.2. Example.** Let  $Cont_X$  be the sheaf defined as above, and let  $Cont_X^*$  be the sheaf, such that  $\Gamma(U, Cont_X^*)$  is the abelian group *under multiplication* that consists of invertible continuous functions on  $U$ . We have a natural map  $Cont_X \rightarrow \Gamma(U, Cont_X^*)$  given by  $f \mapsto \exp(2 \cdot \pi \cdot i \cdot f)$ .

**3.3. Exercise 5.** Show that we have an isomorphism between the following two functors  $Sh(X) \rightarrow Ab$ :

$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F}), \text{ and } \mathcal{F} \mapsto Hom_{Sh(X)}(\underline{\mathbb{Z}}_X, \mathcal{F}).$$

**3.4. Kernels.** Let  $\alpha : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  be a map of (pre)sheaves. Define the functor  $\ker(\alpha) : Sh(X) \rightarrow Sets^{op}$  by

$$\ker(\alpha)(\mathcal{F}) = \{\beta \in Hom(\mathcal{F}, \mathcal{F}_1) \mid \alpha \circ \beta = 0\}.$$

**Lemma 1.** *The above functor is representable.*

**3.5. Exercise 6.** Prove the lemma by showing that the (pre)sheaf  $U \mapsto \ker(\alpha_U)$  represents the above functor. In particular, it is a sheaf, if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are.

**3.6. Exercise 7.** Show that the kernel of the exponential map (see example above) is isomorphic to the sheaf  $\underline{\mathbb{Z}}_X$ .

We say that  $\alpha$  is injective if  $\ker(\alpha)$  is the zero sheaf.

**3.7. Co-kernels.** For  $\alpha$  as above, define the functor  $\operatorname{coker}(\alpha) : Sh(X) \rightarrow Sets$  by

$$\operatorname{coker}(\alpha)(\mathcal{F}) = \{\beta \in Hom(\mathcal{F}_2, \mathcal{F}) \mid \beta \circ \alpha = 0\}.$$

We say that  $\alpha$  is a surjection if  $\operatorname{coker}(\alpha)(\mathcal{F}) = 0$  for all (pre)sheaves  $\mathcal{F}$ .

**3.8. Exercise 8.** Show that the exponential map is a surjection of sheaves.

**3.9. Exercise 9.** (a) Show that the above functor on the category of it presheaves is representable by  $U \mapsto \operatorname{coker}(\alpha_U)$ . (b\*) Give an example that even if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are sheaves, the result of the construction of point (a) does not satisfy the sheaf axioms. (Hint: show that the exponential map is *not always* a surjection of pre-sheaves.)

Nonetheless, we will prove the following theorem:

**Theorem 1.** *The functor  $\operatorname{coker}(\alpha)$  on the category  $Sh(X)$  is representable.*

#### 4. FROM MODULES TO SHEAVES

**4.1.** In this section  $R$  will be a commutative ring, and the topological space  $X = Spec(R)$  endowed with the Zariski topology. Let  $M$  be an  $R$ -module. We will prove the following theorem:

**Theorem 2.** *There exists a unique sheaf of abelian groups on  $X$ , denoted  $Loc(M)$ , characterized by the following property:*

$$\Gamma(U_f, Loc(M)) = M_f,$$

and the map  $res_{U_f.g}^{U_f} : \Gamma(U_f, Loc(M)) \rightarrow \Gamma(U_{f.g}, Loc(M))$  identifies with the natural map  $M_f \rightarrow M_{f.g}$ .

**4.2. Exercise 10\*.** Prove the theorem.