

1. BASIC PROPERTIES

1.1. The ring  $R$  will be assumed commutative throughout, and  $S \subset R$  a multiplicative subset. We have seen that the ring  $R_S$  is flat as an  $R$ -module, i.e., the functor  $M \mapsto M_S := R_S \otimes_R M$  is exact.

Let  $M$  be a module and  $N \subset M$  a submodule. By the above procedure we obtain the  $R_S$ -submodule  $N_S$  inside  $M_S$ . Vice versa, let  $N'$  be an  $R_S$ -submodule of  $M_S$ . We define  $\text{sat}(N') \subset M$  to be the  $R$ -submodule equal to the preimage of  $N'$  under the natural map  $M \rightarrow M_S$ .

**Lemma 1.** *For  $N$  and  $N'$  as above, we have the inclusion  $N \subset \text{sat}(N_S)$  and  $(\text{sat}(N'))_S = N'$ .*

1.2. **Exercise 1.** Prove the lemma.

**Corollary 1.** *Suppose  $R$  is Noetherian. Then  $R_S$  is Noetherian as well.*

*Proof.* Let  $\{M'_i\}$  be a growing sequence of ideals in  $R_S$ . Define  $M_i \subset R$  as  $\text{sat}(M'_i)$ . By the Noetherian assumption on  $R$ , starting from some  $i$  we have  $M_i = R$ . Hence, for the same  $i$  we have  $M'_i = (M_i)_S = R_S$ .  $\square$

2. MODULES AND ZARISKI TOPOLOGY

2.1. Let  $M$  be an  $R$ -module. We define  $\text{supp}(M) \subset \text{Spec}(R)$  to be the set of all prime ideals  $\mathfrak{p} \subset R$ , such that the localization  $M_{\mathfrak{p}} \neq 0$ .

For an  $R$ -module  $M$ , set  $\text{Ann}(M)$  to be ideal of  $R$  consisting of all elements  $a \in R$  such that for  $a \cdot m = 0$  for all  $m \in M$ .

**Proposition 1.** *Suppose that  $M$  is finitely generated. Then  $\text{supp}(M)$  is closed; in fact,  $\text{supp}(M) = V(\text{Ann}(M))$ .*

*Proof.* We claim more generally that if  $M$  is finitely generated,  $M_S = 0$  if and only if  $\text{Ann}(M) \cap S \neq \emptyset$ . This clearly implies the proposition.

Suppose that  $\text{Ann}(M) \cap S \neq \emptyset$ , and let  $a$  be an element of the intersection. Then for any element  $\frac{m}{s} \in M_S$ , we have  $\frac{m}{s} = \frac{a \cdot m}{a \cdot s} = 0$ . (Note that this direction does not use the finite generation assumption.)

In the opposite direction, assume that  $M_S = 0$ . Let  $m_1, \dots, m_n$  be a finite (!) set of generators of  $M$ . Since  $\frac{m_i}{1} \in M_S$  are all zero, there exist  $s_i \in S$ , such that  $s_i \cdot m_i = 0 \in M$ . Let  $s$  be the product of the  $s_i$ 's. Then  $s \cdot m_i = 0$ , and hence  $s \cdot m = 0 \forall m \in M$ . Hence,  $s \in \text{Ann}(M) \cap S$ .  $\square$

2.2. **Exercise 2.** Take  $R = k[x]$ . Give an example of an *infinitely* generated module  $M$ , such that  $\text{supp}(M)$  isn't a Zariski closed subset of  $\text{Spec}(R)$ .

2.3. **Exercise 3.** (a) Show that if  $M \neq 0$ , then  $\text{supp}(M) \neq \emptyset$ . (b) Show that  $\text{supp}(M)$  always contains a maximal ideal.

2.4. **Exercise 4.** (a) Show that an  $R$ -module  $M$  is flat if and only if each  $M_{\mathfrak{p}}$  is flat as a  $R_{\mathfrak{p}}$ -module for all  $\mathfrak{p} \in \text{Spec}(R)$ . (b) Show for  $M$  to be flat it is sufficient that  $M_{\mathfrak{m}}$  be flat for all *maximal* ideals  $\mathfrak{m} \subset R$ .

**2.5. Serre's lemma.** Let  $f_1, \dots, f_n, \dots$  be elements of  $R$ , such that they generate the unit ideal. Then the map

$$M \rightarrow \prod_i M_{f_i}$$

is injective.

The meaning of this assertion is that the localizations  $M_{f_i}$  essentially capture all the information about the module  $M$ .

*Proof.* Let  $m$  be an element in the kernel of the above map. This means that for every  $i$  there exists an integer  $n_i$ , such that  $f_i^{n_i} \cdot m = 0$ .

However, the elements  $f_i^{n_i}$  also generate the unit ideal. Indeed, this condition is equivalent to  $\bigcup_i U_{f_i^{n_i}} = \text{Spec}(R)$ , and  $U_{f_i^{n_i}} = U_{f_i}$ .

Hence, there exist elements  $a_i \in R$  (all but finitely many being zero) such that  $\sum a_i \cdot f_i^{n_i} = 1$ . Hence, we obtain that  $m = 1 \cdot m = 0$ . □

**2.6. Exercise 5.** Let  $M$  and  $f_1, \dots, f_n, \dots$  be as above. Show that  $M$  is flat if and only if each  $M_{f_i}$  is flat as a  $R_{f_i}$ -module.

### 3. NAKAYAMA'S LEMMA

**3.1.** There is a nice criterion to determine whether  $\mathfrak{p} \in \text{supp}(M)$  for a finitely-generated module  $M$ :

**Theorem 1.** *Suppose that  $M_{\mathfrak{p}} \neq 0$ . Then  $(M/\mathfrak{p} \cdot M) \otimes_{R/\mathfrak{p}} \text{Frac}(R/\mathfrak{p}) \neq 0$ .*

**3.2. Exercise 6.** Show that

$$M \otimes_R \text{Frac}(R/\mathfrak{p}) \simeq (M/\mathfrak{p} \cdot M) \otimes_{R/\mathfrak{p}} \text{Frac}(R/\mathfrak{p}) \simeq (M_{\mathfrak{p}})/(\mathfrak{p}_{\mathfrak{p}} \cdot M_{\mathfrak{p}}).$$

The localization  $M_{\mathfrak{p}}$  is commonly referred to as "the stalk of  $M$  at  $\mathfrak{p}$ "—this is a sheaf-theoretic terminology. The  $\text{Frac}(R/\mathfrak{p})$ -vector space  $M \otimes_R \text{Frac}(R/\mathfrak{p})$  is usually called "the fiber of  $M$  at  $\mathfrak{p}$ ". So, Nakayama's lemma is the assertion that for a finitely generated module, if its fiber is 0, then so is the stalk.

Note that taking the fiber is a particular case of the general construction of taking  $M \otimes_R R'$  for a homomorphism  $R \rightarrow R'$ ; in the case at hand  $R'$  is a field.

Using the second isomorphism of Exercise 6, we obtain that the statement of Nakayama's lemma is equivalent to the following one:

**Theorem 2.** *Let  $R$  be a local ring with the maximal ideal  $\mathfrak{m}$ , and  $M$  a finitely-generated  $R$ -module. Suppose that  $M/\mathfrak{m} \cdot M = 0$ . Then  $M = 0$ .*

*Proof.* This is John's argument. Assume that  $M/\mathfrak{m} \cdot M = 0$ , i.e.,  $M = \mathfrak{m} \cdot M$ . Let  $m_1, \dots, m_n$  be generators of  $M$ . We can assume that this set is minimal, i.e., that no subset generates  $M$ .

Then  $m_1$  can be written as

$$\sum_i a_i \cdot m_i, \quad a_i \in \mathfrak{m}.$$

Then

$$(1 - a_1) \cdot m_1 = \sum_{i \neq 1} a_i \cdot m_i.$$

However, as  $a_1 \in \mathfrak{m}$ ,  $1 - a_1$  is invertible. Hence, we can express  $m_1$  as a sum  $\sum_{i \neq 1} b_i \cdot m_i$ . This contradicts the minimality assumption.  $\square$

**3.3. Exercise 7.** Give an example of how the assertion of Nakayama's lemma fails for infinitely generated modules for both  $R = \mathbb{Z}$  and  $R = k[t]$ .

#### 4. APPLICATIONS OF NAKAYAMA'S LEMMA

4.1. We will now give the following criterion of flatness for finitely generated modules over a Noetherian ring.

**Theorem 3.** *Let  $M$  be a f.g. module over a Noetherian ring  $R$ . Then it is flat if and only if it is locally free. I.e., that there exist elements  $f_1, \dots, f_n$  generating the unit ideal in  $R$ , such that the localizations  $M_{f_i}$  are free.*

Locally free modules are algebro-geometric analogs of vector bundles in differential geometry.

The "if" direction follows from Exercise 5. Note that the statement is equivalent to the fact that for any prime ideal  $\mathfrak{p}$  there exists an element  $f \notin \mathfrak{p}$ , such that  $M_f$  is free as a  $R_f$ -module.

4.2. **Exercise 8.** Show that the latter condition is equivalent to the fact  $M_{\mathfrak{p}}$  is free as a  $R_{\mathfrak{p}}$ -module for every prime ideal  $\mathfrak{p}$ . (Hint: use the finite generation assumption.)

Thus, it is sufficient to prove the following theorem:

**Theorem 4.** *Let  $M$  be a finitely generated flat module over a local Noetherian ring  $R$ . Assume that  $M$  is flat. Then it is free.*

4.3. **Exercise 9.** Take  $R = \mathbb{Z}$ . Give an example of an infinitely generated module, which is flat, but not free.

*Proof.* Set  $k = R/\mathfrak{m}$ , and consider the  $k$ -vector space  $V = M/\mathfrak{m} \cdot M$ . Let  $m'_1, \dots, m'_n$  be a basis in  $V$ . Let  $m_i$  be an element of  $M$ , projecting to  $m'_i$ . Consider the map

$$R^n \rightarrow M,$$

corresponding to  $m_1, \dots, m_n$ . We claim that this map is an isomorphism. Indeed, the surjectivity follows from the following:

**Lemma 2.** *Let  $\alpha : M_1 \rightarrow M_2$  be a map of modules over a local Noetherian ring, with  $M_2$  finitely generated. Suppose that the map of vector spaces  $M_1/\mathfrak{m} \cdot M_1 \rightarrow M_2/\mathfrak{m} \cdot M_2$  is surjective. Then  $\alpha$  is surjective.*

4.4. **Exercise 10.** Deduce the lemma from Theorem 2.

Let  $M'$  be the kernel of  $R^n \rightarrow M$ . By the flatness assumption,

$$M'/\mathfrak{m} \cdot M' \simeq M' \otimes_R R/\mathfrak{m} \simeq \ker(R^n/\mathfrak{m} \cdot R^n \rightarrow M/\mathfrak{m} \cdot M).$$

However, the latter map is an isomorphism, by construction. Hence,  $M'/\mathfrak{m} \cdot M' = 0$ , and we are done by Nakayama's lemma.  $\square$

4.5. We will now give another characterization of locally free modules finitely generated modules over a ring  $R$ .

**Proposition 2.** *Assume that  $R$  contains no non-zero nilpotent elements. Let  $M$  be a f.g. module, such that the dimension of fibers of  $M$  for all prime ideals is the same. (I.e., we are looking at  $\dim_{\text{Frac}(R/\mathfrak{p})}(M \otimes_R \text{Frac}(R/\mathfrak{p}))$ .) Then  $M$  is locally free.*

4.6. **Exercise 11.** Give an example how the assertion of the proposition fails when  $R$  has nilpotents.

*Proof.* Using Exercise 8, we can assume that  $R$  is local. Consider a surjection  $R^n \rightarrow M$ , constructed as in the previous proof. (Here  $n$  is the dimension of fibers of  $M$ , which is assumed constant.)

Let  $\underline{a} = (a_1, \dots, a_n) \in R^n$  be an element in the kernel of the above map. For a prime ideal  $\mathfrak{p}$  consider the map

$$(R^n) \otimes_R \text{Frac}(R/\mathfrak{p}) \rightarrow M \otimes_R \text{Frac}(R/\mathfrak{p}).$$

By construction, this map is surjective, and it is also injective, because the dimensions of the two vector spaces are the same. Hence, the image of  $\underline{a}$  in every  $(R^n) \otimes_R \text{Frac}(R/\mathfrak{p})$  is 0. I.e., each  $a_i$  belongs to all  $\mathfrak{p}$ . This means that  $a_i$  is nilpotent, and hence, 0. □

## 5. THE "GOING UP" THEOREM

**Theorem 5.** *Let  $\phi : A \rightarrow B$  be a homomorphism of rings. Assume that  $B$  is finitely generated as an  $A$ -module. Then the map  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is closed (i.e., the image of a closed subset is closed).*

5.1. **Exercise 12.** Show that the assertion of the theorem is equivalent to the following one:

**Theorem 6.** *For  $\phi : A \rightarrow B$  as above, let  $I \subset A$  and  $J \subset B$  be ideals such that  $\phi^{-1}(J) = I$ . Let  $\mathfrak{p} \supset I$  be a prime ideal in  $A$ . Then there exists a prime ideal  $\mathfrak{q} \supset J$  such that  $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ .*

*Proof.* We can replace  $A$  by  $A/I$  and  $B$  by  $B/J$ , so that we can assume that  $I = J = 0$ , and  $\phi$  is an embedding.

Since the functor of localization is exact, the homomorphism  $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$  is injective. In particular,  $B_{\mathfrak{p}} \neq 0$ . By Nakayama's lemma,  $B' := (B/B \cdot \mathfrak{p}) \otimes_{A/\mathfrak{p}} \text{Frac}(A/\mathfrak{p})$

is non-zero. In particular, the homomorphism  $\text{Frac}(A/\mathfrak{p}) \rightarrow B'$  is injective. Let  $\mathfrak{q}'$  be any point in  $\text{Spec}(B')$ , and let  $\mathfrak{q}$  be its image in  $\text{Spec}(B)$  with respect to the morphism, corresponding to the tautological homomorphism  $B \rightarrow B'$ .

We claim that  $\Phi(\mathfrak{q}) = \mathfrak{p}$ . Indeed,  $\Phi(\mathfrak{q})$  is the image of  $\mathfrak{q}'$  under  $\text{Spec}(B') \rightarrow \text{Spec}(B) \rightarrow \text{Spec}(A)$ , but the latter composition equals  $\text{Spec}(B') \rightarrow \text{Spec}(A/\mathfrak{p}) \rightarrow \text{Spec}(A)$ , and the image of the second arrow consists of the unique point  $\mathfrak{p}$ . □

## 6. ARTINIAN RINGS

6.1. A ring  $R$  is called Artinian if any decreasing sequence of ideals  $I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$  stabilizes.

**Proposition 3.** (a) Any prime ideal in an Artinian ring is maximal. (b) An Artinian ring has finitely many maximal ideals.

*Proof.* For point (a), let  $\mathfrak{p}$  be a prime ideal; then  $A/\mathfrak{p}$  is a domain and an Artinian ring. We claim that it is a field, i.e., that every non-zero element is invertible. For  $f \in A/\mathfrak{p}$  consider the ideals  $I_n \subset A/\mathfrak{p}$ , generated by  $f^n$ . This chain stabilizes, hence for some  $n$ ,  $f^n \in I_{n+1}$ , i.e.,  $f^n = f^{n+1} \cdot a$ . Since  $A/\mathfrak{p}$  is a domain,  $f \cdot a = 1$ .

For point (b), let  $\mathfrak{m}_1, \mathfrak{m}_2, \dots$  be a sequence of different maximal ideals. Suppose, by contradiction, that this sequence is infinite. Consider  $I_n = \bigcap_{i=1, \dots, n} \mathfrak{m}_i$ . By assumption, this chain stabilizes. Hence, for some  $n$ ,  $\bigcap_{i=1, \dots, n} \mathfrak{m}_i \subset \mathfrak{m}_{n+1}$ . We claim that for some  $i = 1, \dots, n$   $\mathfrak{m}_i \subset \mathfrak{m}_{n+1}$  (which would be a contradiction). Suppose not; then there exist elements  $f_i \in \mathfrak{m}_i - \mathfrak{m}_i \cap \mathfrak{m}_{n+1}$ . Consider  $\prod_i f_i$ . It belongs to the above intersection, and hence to  $\mathfrak{m}_{n+1}$ . But this contradicts the fact that  $\mathfrak{m}_{n+1}$  is prime. □

6.2. Next we will study Artinian local rings.

6.3. **Exercise 13.** Show that in Artinian local ring every element in the maximal ideal is nilpotent.

6.4. **Exercise 14.** Show that if  $R$  is an Artinian and Noetherian local ring with maximal ideal  $\mathfrak{m}$ , then  $\mathfrak{m}^n = 0$  for some  $n$ .

6.5. **Exercise 15\*.** (a) Show that the conclusion of Exercise 11 is true *without* the Noetherian assumption. (b) Deduce that an Artinian local ring is automatically Noetherian.

**Theorem 7.** Let  $R$  be an Artinian ring. Then  $R \simeq \bigoplus_{i=1, \dots, n} R_{\mathfrak{m}_i}$ , where  $\mathfrak{m}_i$  is the (finite) set of its maximal ideal.

This theorem shows that an Artinian ring is a finite direct sum of Artinian local rings. Combined with Exercise 15, we conclude that every Artinian local ring is Noetherian.

*Proof.* We have the natural map  $R \rightarrow \bigoplus R_{\mathfrak{m}_i}$ . To show that it is an isomorphism, it suffices to show that for every prime ideal of  $R$ , the map

$$R_{\mathfrak{p}} \rightarrow \bigoplus (R_{\mathfrak{m}_i})_{\mathfrak{p}}$$

is an isomorphism. But any such  $\mathfrak{p}$  is one of the  $\mathfrak{m}_i$ 's. Hence, it suffices to see that for  $i \neq j$ , the localization  $(R_{\mathfrak{m}_i})_{\mathfrak{m}_j}$  is 0.

6.6. **Exercise 16.** Deduce that  $(R_{\mathfrak{m}_i})_{\mathfrak{m}_j} = 0$  from Exercise 13. □

## 7. DIRECT LIMITS

7.1. This section is not related to the previous material. Let  $I$  be a partially ordered set. We say that  $I$  is directed (or filtered) if for any  $i_1, i_2 \in I$ , there exists  $j \in I$  with  $j \geq i_1$  and  $j \geq i_2$ . From now on posets will be assumed directed.

Let  $\mathcal{C}$  be a category. An  $I$ -family of objects of  $\mathcal{C}$  is an assignment  $i \in I \mapsto X_i \in \text{Ob}(\mathcal{C})$ , and for each pair  $i, j \in I, i \leq j$  a morphism  $\alpha_{i,j} : X_i \rightarrow X_j$ , so that for  $i \leq j \leq k$ , we have  $\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k}$ .

7.2. To an  $I$ -family  $\{X_i, \alpha_{i,j}\}$  we attach a functor  $F : \mathcal{C} \rightarrow \text{Sets}$  by setting for  $X \in \mathcal{C}$ ,  $F(X)$  to be the set of collections of maps  $\beta_i : X_i \rightarrow X, \forall i \in I$ , such that whenever  $i \leq j$ ,  $\beta_j \circ \alpha_{i,j} = \beta_i$ .

We say that an  $I$ -family  $\{X_i, \alpha_{i,j}\}$  admits an inductive limit (aka direct limit, co-limit) if this functor is representable. In this case the representing object is called "the inductive limit of this family", and denoted

$$\lim_{\rightarrow} X_i.$$

It isn't true that every system admits an inductive limit.

7.3. **Example.** Let  $\mathcal{C}$  be the category  $\text{Vect}_k^{fd}$  of *finite-dimensional* vector spaces (over some field  $k$ ). Let  $I = \mathbb{N}$  (natural numbers, and the  $I$ -family be  $X_n = k^n$ , such that for  $n \leq m$ ,  $k^n \rightarrow k^m$  be the standard embedding, corresponding to the first  $n$  basis vectors.

7.4. **Exercise 17.** (a) Show that the above system does not admit an inductive limit in  $\text{Vect}_k^{fd}$ . (b) Show that this system does have a limit, when viewed in the category  $\text{Vect}_k$  of all vector spaces.

7.5. From now on we will specify to the case when  $\mathcal{C} = R\text{-mod}$ , where  $R$  is some ring (not necessarily commutative). Most of the discussion equally applies to the category  $\text{Sets}$ .

**Proposition-Construction 1.** *Any system  $\{M_i, \alpha_{i,j}\}$  in  $R\text{-mod}$  admits an inductive limit.*

*Proof.* Define an  $R$ -module  $N$  as follows. Its elements are pairs  $(i \in I, m_i \in M_i)$ , where we identify  $(i_1 \in I_1, m_{i_1} \in M_{i_1})$  and  $(i_2 \in I_2, m_{i_2} \in M_{i_2})$  if there exists  $j \in I$  with  $j \geq i_1, i_2$  such that  $\alpha_{i_1,j}(m_1) = \alpha_{i_2,j}(m_2)$ .

The addition on  $N$  is defined as follows: the sum of  $(i_1, m_1)$  and  $(i_2, m_2)$  is  $(j, m)$  if  $j \geq i_1, i_2$  and  $m = \alpha_{i_1,j}(m_1) + \alpha_{i_2,j}(m_2)$ .

7.6. **Exercise 18.** Show that the module  $N$  constructed above indeed represents the functor  $F$ . □

7.7. **Exercise 19.** Show that  $N$  constructed above is isomorphic to

$$\text{coker} \left( \bigoplus_{i,j \in I, i \leq j} M_i \rightarrow \bigoplus_k M_k \right),$$

where the map in the formula sends

$$\sum_{i,j \in I, i \leq j} m_i \mapsto \sum_j (m_j - \alpha_{i,j}(m_i)).$$

7.8. **Exercise 20.** Let  $M \simeq \varinjlim M_i$ . Show that for a right  $R$ -module  $N$ ,

$$N \otimes_R M \simeq \varinjlim N \otimes_R M_i.$$

7.9. **Exercise 21.** Let  $R$  be commutative, and  $f \in R$  a non-nilpotent element. Show that as an  $R$ -module, the localization  $M_f$  can be described as the inductive limit of the  $\mathbb{N}$ -system of modules, where each  $M_m \simeq M$ , but  $\alpha_{m,n} : M \rightarrow M$  for  $n \geq m$  is the map of multiplication by  $f^{n-m}$ .