Course notes: Week 4, a study of tensor products

1. Exactness properties of tensor products

1.1. Until specified otherwise, a ring R will be arbitrary (not necessarily commutative).

Let N be a fixed right R-module, and $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of left R-modules.

Proposition 1. *We have an exact sequence*

$$N \underset{R}{\otimes} M_1 \to N \underset{R}{\otimes} M_2 \to N \underset{R}{\otimes} M_3 \to 0.$$

The meaning of the proposition is that the map $N \underset{R}{\otimes} M_2 \to N \underset{R}{\otimes} M_3$ is surjective, and its kernel is the image of $N \underset{R}{\otimes} M_1$. However, the map of the latter into $N \underset{R}{\otimes} M_2$ is **not** necessarily surjective.

1.2. **Example.** Take $R = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z}$, and the map $M_1 \to M_2$ being the multiplication by 2. Then $M_3 \simeq \mathbb{Z}/2\mathbb{Z}$, and the map $N \underset{R}{\otimes} M_2 \to N \underset{R}{\otimes} M_3$ is an isomorphism. The map $N \underset{R}{\otimes} M_1 \to N \underset{R}{\otimes} M_2$ is on the contrary the zero map. This is example is rather typical.

1.3. **Exercise 1.** Produce a similar example for $R = k[x]$, where k is a field.

Lemma 1. *Let $\Lambda_1 \to \Lambda_2 \to \Lambda_3 \to 0$ be a complex of abelian groups. Then it is exact if and only if for any abelian group Λ, the complex*

$$0 \to Hom(\Lambda_3, \Lambda) \to Hom(\Lambda_2, \Lambda) \to Hom(\Lambda_1, \Lambda)$$

is exact.

1.4. **Exercise 2.** (a) Prove the lemma. (b) Deduce the proposition from the lemma, using the description of tensor products as bilinear maps.

1.5. **Exercise 3.** Give a direct proof of the proposition.

A functor $R - mod \to Ab$ is called right (resp., left) *exact* if it transforms short exact sequences to sequences which are exact except for the left (resp., right) place. Thus, the functor of tensor product with a given module is right-exact. If a functor transforms short exact sequences to short exact sequences it is called *exact.*

2. Flatness

2.1. **Definition.** A right (resp., left) R-module is called flat if the functor of tensor product with it is exact. I.e., a right module N is flat if whenever $M_1 \to M_2$ is an injection of left modules, then the map of abelian groups $N \underset{R}{\otimes} M_1 \to N \underset{R}{\otimes} M_2$ is injective.

2.2. If N is isomorphic to R itself, then it is flat. Indeed, in this case the functor $M \mapsto N \underset{R}{\otimes} M$ is isomorphic to the composition of the identity functor $R - mod \rightarrow R - mod$, followed by the forgetful functor $R - mod \rightarrow Ab$. Hence, it is exact.

Since tensor products commute with arbitrary (not necessarily finite) direct sums, any *free* R-module (i.e., one isomorphic to a direct sum of the module R) is also flat.

Theorem 1. *Let $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N \rightarrow 0$ be a short exact sequence of right R-modules. Assume that N is flat. Then for any left R-module M, we have a short exact sequence*

$$0 \rightarrow N_1 \underset{R}{\otimes} M \rightarrow N_2 \underset{R}{\otimes} M \rightarrow N \underset{R}{\otimes} M \rightarrow 0.$$

I.e., the non-exactness of the sequence appearing in Proposition 1 has to do with non-flatness of the *left* module M_3. Whether or not M_1 and M_2 are flat does not matter.

Proof. Choose generators for M as an R-module. Then we have a map $P \twoheadrightarrow M$, where P is a free left R-module. (I.e, $P \simeq R^I$, where I is the above set of generators.) Let M' be the kernel of $P \rightarrow M$.

We have a big commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & N_1 \underset{R}{\otimes} M' & \xrightarrow{\alpha''} & N_2 \underset{R}{\otimes} M' & \xrightarrow{\delta} & N \underset{R}{\otimes} M' & \longrightarrow & 0 \\
 & & \gamma\downarrow & & \gamma'\downarrow & & \downarrow & & \\
0 & \longrightarrow & N_1 \underset{R}{\otimes} P & \xrightarrow{\alpha'} & N_2 \underset{R}{\otimes} P & \longrightarrow & N \underset{R}{\otimes} P & \longrightarrow & 0 \\
 & & \beta\downarrow & & \beta'\downarrow & & \downarrow & & \\
 & & N_1 \underset{R}{\otimes} M & \xrightarrow{\alpha} & N_2 \underset{R}{\otimes} M & \longrightarrow & N \underset{R}{\otimes} M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We will no perform a "diagram chase". Let x be an element in $ker(\alpha)$. Choose y so that $x = \beta(y)$. Since $\beta'(\alpha'(y)) = 0$, there exists z such that $\gamma'(z) = \alpha'(y)$. Then $\delta(z) = 0$. Hence, there exists w, such that $z = \alpha''(w)$. Consider $\gamma(w)$. We have: $\alpha'(\gamma(w)) = \alpha'(y)$. Hence, $y = \gamma(w)$. But then $x = \beta(\gamma(w)) = 0$.

□

3. Change of rings

Proposition 2. *Let $R_1 \rightarrow R_2$ be a homomorphism of rings. Let M be a left R_1-module and N a right R_2-module. We can regard N is a right R_1-module and $R_2 \underset{R_1}{\otimes} M$ as a left R_2-module. Then*

$$N \underset{R_1}{\otimes} M \simeq N \underset{R_2}{\otimes} (R_2 \underset{R_1}{\otimes} M).$$

3.1. **Exercise 4.** Prove the proposition.

3.2. **Exercise 5.** Deduce that if N is a flat right R_1-module, then $N \underset{R_1}{\otimes} R_2$ is a flat right R_2-module.

3.3. We will say that R_2 is (right) flat over R_1 if it is flat as a right R_1-module. (Of course, if both R_1 and R_2 are commutative, there is no distinction between left and right.)

3.4. **Exercise 6.** (a) Show that in the situation of the previous proposition, if R_2 is right flat over R_1 and N is flat as a R_2-module, then it is flat also as a R_1-module. (b) Show that the assertion would be false without the assumption that R_2 is right flat over R_1.

4. Exactness of localization

4.1. From now on R will be a commutative ring and $S \subset R$ a multiplicative subset.

Theorem 2. *The ring R_S is flat over R.*

Before proving the theorem, let us describe differently the functor $M \mapsto M_S := M \underset{R}{\otimes} R_S$. We claim that M_S is formed by expressions $\frac{m}{s}$, $m \in M$, $s \in S$, such that $\frac{m_1}{s_1} = \frac{m_2}{s_2}$ if and only if there exist $s', s'' \in S$, such that $m_1 \cdot s' = m_2 \cdot s''$ and $s_1 \cdot s' = s_2 \cdot s''$. The addition and action of R_S are defined in the evident manner.

4.2. **Exercise 7.** Prove that the module described above is indeed isomorphic to $M \underset{R}{\otimes} R_S$.

4.3. **Exercise 8.** (a) Show that the category $R_S - mod$ is equivalent to the full subcategory of $R - mod$, formed by those modules M, on which the action of every $s \in S$ is invertable. (b) Show that $M \mapsto M_S$ is the right adjoint to the functor of embedding the above subcategory into the ambient category of R-modules.

Now let us prove the theorem:

Proof. Let $\alpha : M_1 \to M_2$ be an injection of R-modules. We must show that $(M_1)_S \to (M_2)_S$ is also injective. Let $\frac{m}{s}$ be an element of $(M_1)_S$. If its image in $(M_2)_S$ is zero, there exists an element $s' \in S$, such that $s' \cdot \alpha(m) \in M_2$ is zero. By assumption, this means that $s' \cdot m \in M_1$ is zero. But this in turn implies that $\frac{m}{s} = 0$ as an element of $(M_1)_S$.

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