Course notes: Week 2, Finite generation and Noetherian rings

1. Finitely generated modules

1.1. Unless specified otherwise, in this section we will work with an arbitrary (not necessarily commutative) ring A. By an A-module we will mean a left A-module.

An A-module M is called finitely generated (f.g.) if it can be represented as the quotient of a free module A^n, $n \in \mathbb{N}$.

1.2. **Exercise 1.** Prove that a module M is finitely generated if and only if any increasing sequence of submodules $M_1 \subset M_2 \subset ... \subset M_i \subset M$ with $\underset{i}{\cup} M_i = M$ stabilizes. I.e., there exists an index i, such that $M_i = M$, and hence $M_j = M_i$ for $j \geq i$.

1.3. **Exercise 2.** Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of A-modules. This means that M_1 is a submodule of M, and $M_2 \simeq M/M_1$.
(a) Assume that M_1 and M_2 are finitely generated. Show that M is then also finitely generated.
(b) Assume that M is f.g. Show that M_2 is also f.g. Is it true that M_1 is f.g.?

2. Noetherian rings

2.1. In this section the algebra A continues to be arbitrary. Recall that a left ideal $I \subset A$ is the same thing as a submodule of A, regarded as a left module over itself.

A ring A is called left Noetherian if every every ideal $I \subset A$ is finitely generated as a left A-module.

2.2. **Exercise 3.** Show that a field, regarded as a ring, is always Noetherian.

Theorem 1. *Let A be a Noetherian ring, M is f.g. A-module, and $M' \subset M$ a submodule. Then M' is also f.g.*

Proof. let $N := A^n \twoheadrightarrow M$ be a surjection, and N' the preimage of M' in N. Since M' is a quotient of N', it is enough to show that N' is finitely generated.

Let $N_i = A^i$, $i = 0, ..., n$ be the standard increasing sequence of A-submodules. Set $N'_i = N' \cap N_i$. Using Exercise 2 and induction, it is enough to show that all N'_i/N'_{i-1} are f.g. A-modules.

However, each N'_i/N'_{i-1} for $i = 1, ..., n$ is a submodule of N_i/N_{i-1}. But each N_i/N_{i-1} is isomorphic to A, as an A-module. By assumption, every left A-submodule of A is finitely generated.

□

2.3. **Exercise 4.** Let $A \to B$ be a surjection of algebras. Show that if A is Noetherian, then so is B.

2.4. **Exercise 5.** Let $A \to B$ be a homomorphism of algebras, and assume that B is f.g. *as a left A-module.* Assume that A is Noetherian. Show that B is then also Noetherian.

2.5. **Polynomial algebras over a ring.** For a ring A we define another ring $A[x_1, ..., x_n]$ as follows. Its elements are finite sums

$$\sum_{i_1,...,i_n \leq N} a_{i_1,...,i_n} \cdot x_1^{i_1} \cdot ... \cdot x_n^{i_n},$$

where $a_{i_1,...,i_n} \in A$. Addition on this set is obvious. Multiplication is defined also "naturally":

$$(a \cdot x_1^{i_1} \cdot ... \cdot x_n^{i_n}) \cdot (b \cdot x_1^{j_1} \cdot ... \cdot x_n^{j_n}) = (a \cdot b) \cdot x_1^{i_1+j_1} \cdot ... \cdot x_n^{i_n+j_n}.$$

Note that in this ring $x_i \cdot x_j = x_j \cdot x_i$ and $a \cdot x_i = x_i \cdot a$ for $a \in A$.

2.6. **Exercise 6.** Let B be another ring. Show that a homomorphism

$$\Phi : A[x_1, ..., x_n] \to B$$

is the same as a data of $\phi : A \to B$ and n elements $b_1, ..., b_n \in B$ with the property that $b_i \cdot b_j = b_j \cdot b_i$, and $\phi(a) \cdot b_i = b_i \cdot \phi(a)$, $a \in A$.

2.7. **Exercise 7.** Let for a moment A be commutative. Introduce the category $A-alg$, called the category of A-algebras, whose objects are pairs (B, ϕ), consisting of a rings B and a homomorphism $\phi : A \to B$, such that $b \cdot \phi(a) = \phi(a) \cdot b$ for all $a \in A, b \in B$. Morphisms in this category between (B, ϕ) and (B', ϕ') are ring homomorphisms $\psi' : B \to B'$, such that $\psi \circ \phi = \phi'$. Consider the functor $F : A - alg \to Sets$, that assigns to (B, ϕ) the set of all $n - tuples$ of elements $b_1, ..., b_n \in B$ that pairwise commute. Show that this functor is *represented* by the object $A[x_1, ..., x_n] \in A - alg$.

2.8. **Hilbert's basis theorem.**

Theorem 2. *Let A be a Noetherian ring, and let $B = A[x]$ be the ring of polynomials over A in 1 variable. Then B is also Noetherian.*

Proof. Let $J \subset B$ be an ideal. We define a subset $I \subset A$ by letting $a \in I$ if there exists an element $\Sigma\, a_i \cdot x^i \in I$, whose leading coefficient is a. Clearly, I is a left ideal in A. Let $i^1, ..., i^n \in I$ be a finite subset that generates I; it exists by the Noetherianness assumption. For each k Let $f^k \in B$ be an element , such that i^k appears as the leading coefficient if f^k. Let d be the maximum of the degrees of f^k.

Set M be the A-submodule of $B = A[x]$ equal to the intersection of J and the submodule (isomorphic to A^d) of polynomials of degrees $< d$. By Theorem 1, M is finitely generated over A. Let $j^1, ..., j^m$ be a finite set of generators. We claim that these elements together with the f^k's generate J as a left B-module.

Suppose the contrary. Let $g \in J$ be an element it of the minimal degree, which is not in the ideal generated by the j^l's and f^k's. By assumption, the degree of g is $\geq d$. Let a be the leading coefficient of g. By assumption, a can be written as $\Sigma_k\, a_k \cdot i^k$. Consider the element

$$g' = g - \Sigma_k\, x^{\deg(g)-\deg(f^k)} \cdot a_k \cdot f^k.$$

It also belongs to J, and it belongs to the ideal (f^k, j^l) if and only if g does.

However, $\deg(g') < \deg(g)$, by construction. This is a contradiction.

□

Corollary 1. *Let k be a field. Then the ring (k-algebra) $k[x_1, ..., x_n]$ is Noetherian.*

2.9. **Exercise 8.** Show that the algebra $k[x_1, x_2...]$ (the polynomial algebra on infinitely many variables) is not Noetherian.

2.10. **Exercise 9*.** Modify the proof of Hilbert's basis theorem to show that if A is a Noetherian ring, then the ring of formal power series $A[[x]]$ (consisting of infinite expressions $\Sigma\, a_n \cdot x^n$) is also Noetherian.

3. Integral dependence

3.1. In this section all algebras will be commutative. Let $\phi : A \to B$ be a homomorphism of rings. (In this case we say that B is an A-algebra.)

Let $x \in B$ be an element, and let $A[x]_B \subset B$ be the subring of B generated by A and x, i.e., $A[x]_B$ is the image of the homomorphism $A[x] \to B$ (see Exercise 6).

Proposition 1. *The following conditions are equivalent:*
(a) There exists an integer n, such that $x^n = \Sigma_{k<n}\, \phi(a_k) \cdot x^k$.
(b) The ring $A[x]_B$ is finitely generated as an A-module.

Proof. If (a) holds, then x^k, $1 \leq k \leq n-1$ generate $A[x]_B$ as an A-module.

If (b) holds, consider the increasing sequence M_i of A-submodules of $A[x]_B$, where M_i is generated by x^k, $1 \leq k \leq i$. By Exercise 1, this sequence stabilizes. Hence, for some n, $M_n = M_{n-1}$. Hence, $x^n \in M_{n-1}$.

□

If the conditions of the above proposition hold, we say that x is integral over A.

Proposition 2. *If x is integral over A and $x' \in A[x]_B$, then x' is also integral over A.*

We'll need this assertion only under the assumption that A is Noetherian. The proof in the general case is a little tricky.

3.2. **Exercise 10.** Prove the proposition for A Noetherian.

In the exercises below you may use the above proposition, although we didn't give a proof in the general case. Also, feel free to assume that A is Noetherian.

3.3. **Exercise 11.** Let the homomorphism $\phi : A \to B$ factor as $A \to A' \xrightarrow{\phi'} B$, and assume that A' is finitely generated *as an A-module.* Let $x \in B$ be integral over A'. Show that then x is integral over A as well.

3.4. We say that the ring B is integral over A if every $x \in B$ is integral over A.

3.5. **Exercise 12.** Show that if $x_1, ..., x_n \in B$ are integral over A, then the ring $A[x_1, ..., x_n]_B$ is integral over A.

3.6. **Exercise 13.** Show that the subset of elements in B, which are integral over A forms a subring of B.

3.7. We say that B is finitely generated as an A-algebra, if there exist elements $x_1, ..., x_n \in B$, such that the resulting homomorphism $A[x_1, ..., x_n] \to B$ (see Exercise 6) is surjective.

In this case we say that $x_1, ..., x_n$ generate B as an A-algebra. This condition is equivalent, of course, to the fact that the subalgebra $A[x_1, ..., x_n]_B$ equals the entire B.

3.8. **Exercise 14.** Let A be Noetherian, and let B be a f.g. A-algebra. Show that B is also Noetherian.

3.9. **Exercise 15.** Let B be an A-algebra. Show that B is finitely generated *as an A-module* if and only it it is finitely generated *as an A-algebra* **and** *is integral*.

3.10. **Exercise 16*.** Assume that $\phi : A \to B$ is injective and that both A and B have no zero-divisors. Assume that B is integral over A. Show that A is a field if and only if B is.

4. A step toward Nullstellensatz

4.1. Let k be a (commutative) field. In this section all rings will be commutative k-algebras.

Consider the polynomial algebra $k[x_1, ..., x_n]$ in n variables. Let $f^1, ..., f^k \in k[x_1, ..., x_n]$ be a bunch of elements. We can look for the set of their common zeroes in k^n:

$$V(f_1, ..., f_k) = \{(c_1, ..., c_n) \in k^n \mid f^i(c_1, ..., c_n) = 0, \forall i = 1, ..., k\}.$$

Let f be any element in the ideal $(f^1, ..., f^k)$, generated by the f^i's. Then, evidently, $f(c_1, ..., c_n) = 0$ for any $(c_1, ..., c_n) \in V(f_1, ..., f_k)$.

Moreover, since any ideal $I \subset k[x_1, ..., x_n]$ has the form $(f^1, ..., f^k)$ for a finite collection of elements $f^1, ..., f^k \in k[x_1, ..., x_n]$ (due to Hilbert's basis theorem), we can speak of the set of zeroes $V(I)$ of an ideal I.

4.2. Now, we'd like to ask the inverse question. Can we reconstruct the ideal I, knowing $V(I)$? Naively, one might guess that if $f(c_1, ..., c_n) = 0$ for any $(c_1, ..., c_n) \in V(I)$, then $f \in I$. But this is completely false. E.g., unless k is algebraically closed, $V(I)$ might be empty.

So, we assume that k is algebraically closed. But the naive guess is still false:

4.3. **Exercise 17.** Show that if $f^m \in I$ for some power $m \in \mathbb{N}$, then $f(c_1, ..., c_n) = 0$ for any $(c_1, ..., c_n) \in V(I)$.

Hilbert's Nullstellensatz asserts that the above exercise describes the range of failure of the naive guess:

Theorem 3. *Assume that k is algebraically closed. If $f(c_1, ..., c_n) = 0$ for any $(c_1, ..., c_n) \in V(I)$, then $f^m \in I$ for some power $m \in \mathbb{N}$.*

4.4. **Weak Nullstellensatz.** Today we will perform a step in the direction of this theorem. We will prove the following:

Theorem 4. *Let k be an algebraically closed field. Then every maximal ideal of $k[x_1, ..., x_n]$ is the kernel of a homomorphism $k[x_1, ..., x_n] \to k$.*

Before giving the proof, note any homomorphism $\phi : k[x_1, ..., x_n] \to k$ is an evaluation map at some point of k^n. Indeed, the i-coordinate of the point is recovered as $\phi(x_i)$.

Proof. Let $\mathfrak{m} \subset k[x_1, ..., x_n]$ be a maximal ideal, and let E be the quotient. By assumption, E is a field, and is finitely generated as a k-algebra. Hence, Weak Nullstellensatz results from the following assertion:

Proposition 3. *Let k be a field and let E be a finitely generated k-algebra, which is a field. Then E is a finite field extension of k.*

Indeed, k being algebraically closed means that any finite field extension $E \supset k$ must coincides with k itself. Thus, it remains to prove the above proposition.

□

4.5. **Exercise 18.** Assuming the above proposition, describe all maximal ideals of the algebra $k[x_1, ..., x_n]$ for k not necessarily algebraically closed.

We will deduce the proposition from the following lemma:

Lemma 1. *Let $A \subset B \subset C$ be algebras. Assume that C is finitely generated as an A-algebra, and that C is finitely generated as a B-module. Assume that A is Noetherian. Then B is also finitely generated as an A-algebra.*

4.6. **Proof of the lemma.** Let $x_1, ..., x_m$ generate C as an A-algebra, and let $y_1, ..., y_n$ generate C as a B-module. Then

$$x_i = \Sigma_j \, b_{ij} \cdot y_j \tag{1}$$

$$y_k \cdot y_l = \Sigma_j \, b_{klj} \; y_j \tag{2}$$

for some $b_{ij}, b_{klj} \in B$.

Let B_0 be the A-subalgebra of B, generated by all the b_{ij}, b_{klj}. By the Hilbert Basis Theorem, B_0 is Noetherian, because A is.

Using (1) and (2) we obtain that the y_j's generate C as a B_0-module. Since $B \subset C$, by Noetherianness, we obtain that B is also finitely generated as a B_0-module.

In particular, B is finitely generated as a B_0-algebra, and hence as an A-algebra.

4.7. **Proof of the proposition.** Let E be generated as an algebra over k by elements $x_1, ..., x_n$. We can assume that for some $0 \leq r \leq n$, the elements x_i, $i \leq r$ are algebraically independent, and each of the elements x_j, $j > r$ is algebraic (=integral) over $F = k(x_1, ..., x_r)$. We claim that $r = 0$, which is exactly the assertion of the proposition.

By construction and Exercise 12, E is f.g. as an F-module. Applying the lemma, we obtain that F is finitely generated as k-algebra. If $r > 0$, we obtain that the field $F = k'(x_n) := k(x_1, ..., x_{r-1})(x_r)$ is finitely generated as an algebra over the field $k' := k(x_1, ..., x_{r-1})$. But this is a contradiction:

4.8. **Exercise 19.** Show that for a field k', the field of rational functions $k'(x)$ is never finitely generated as a k'-algebra.