Course notes: Week 1, categories and functors

1. Categories

1.1. A category $\mathcal{C}$ consists by definition of a *class* of objects, denoted $Ob(\mathcal{C})$, and for any two objects $X, Y \in Ob(\mathcal{C})$ a set of morphisms from X to Y, denoted $Hom(X,Y)$, plus some additional data. Before defining this data, let us give some examples.

1.2. **Example 1.** $\mathcal{C} = Sets$. Objects of $Sets$ is the class of *all* sets. Morphisms between a set X and a set Y are by definition all maps of sets $X \to Y$.

1.3. **Example 2.** $\mathcal{C} = Top$. Objects of Top are topological spaces. By definition, $Hom(X,Y)$ is the set of all continuous maps $X \to Y$.

1.4. **Example 3.** Let A be a ring. We define the category of left A-modules, denoted $A - mod$, to have as objects all left A-modules, and $Hom(X,Y)$ to be the set of all A-module maps $X \to Y$.

When A is a field k, we will often denote $A - mod$ by $Vect_k$, and call it the category of k-vector spaces.

1.5. **Exercise 1.** Give 5 more examples of categories that you encounter in real life.

1.6. Now we specify the additional data appearing in the definition of categories:

- For every $X \in Ob(\mathcal{C})$ there exists a distinguished element $1_X \in Hom(X,X)$, called the identity map.
- For $X, Y, Z \in Ob(\mathcal{C})$ we are given a map of sets $\circ : Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$, called the composition.

These data must satisfy the following conditions:

- For $\alpha \in Hom(X,Y)$, the compositions $1_Y \circ \alpha \in Hom(X,Y)$ and $\alpha \circ 1_X \in Hom(X,Y)$ are both equal to α.
- For $X, Y, Z, W \in Ob(\mathcal{C})$, and α, β, γ in $Hom(X,Y), Hom(Y,Z), Hom(Z,W)$, respectively, the compositions $\gamma \circ (\beta \circ \alpha)$ and $(\gamma \circ \beta) \circ \alpha$ coincide as elements of $Hom(X,W)$

Sometimes we will write instead of $\alpha \in Hom(X,Y)$ just $\alpha : X \to Y$.

1.7. **Examples.** In examples 1-3 above, set 1_X to be the identity map $X \to X$, and the composition the natural composition of maps.

1.8. **Exercise 2.** Show that the two properties above hold in examples 1-3, as well as in your own 5 examples.

1.9. **Notion of isomorphism.** Let X and Y be two sets. In practically makes no sense to ask the question when they coincide. Instead, there is a much more reasonable notion, applicable for any category.

We say that two objects $X, Y \in Ob(\mathcal{C})$ are isomorphic if there exist elements (maps) $\alpha \in Hom(X,Y)$ and $\beta \in Hom(Y,X)$, such that $\alpha \circ \beta = 1_Y$ and $\beta \circ \alpha = 1_X$.

A morphism $\alpha : X \to Y$ is called an isomorphism if there exists $\beta : Y \to X$, such that $\alpha \circ \beta = 1_Y$ and $\beta \circ \alpha = 1_X$.

1.10. **Exercise 3.** Show that if β exists, it is unique.

1.11. **Opposite category.** Let $\mathcal{C}$ be a category. We introduce another category, denoted $\mathcal{C}^{op}$, called the opposite of $\mathcal{C}$, be setting $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$, and for $X, Y \in Ob(\mathcal{C})$, $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$.

We set $1_X \in Hom_{\mathcal{C}^{op}}(X,X)$ to coincide with $1_X \in Hom_{\mathcal{C}}(X,X) \simeq Hom_{\mathcal{C}^{op}}(X,X)$. The composition is defined as follows. For $\alpha \in Hom_{\mathcal{C}^{op}}(X,Y)$, $\beta \in Hom_{\mathcal{C}^{op}}(Y,Z)$, let us denote by α', β' the corresponding elements of $Hom_{\mathcal{C}}(Y,X)$ and $Hom_{\mathcal{C}}(Z,Y)$, respectively. Then $\beta \circ \alpha$ is the element of $Hom_{\mathcal{C}^{op}}(X,Z)$ equal to the element $\alpha' \circ \beta' \in Hom_{\mathcal{C}}(Z,X)$.

2. Functors

2.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. A (covariant) functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an assignment for each $X \in Ob(\mathcal{C}_1$ an object $F(X) \in Ob(\mathcal{C}_2)$ and a map of sets $F : Hom_{\mathcal{C}_1}(X,Y) \to Hom_{\mathcal{C}_2}(F(X), F(Y))$, such that the following two properties hold:

- $F(1_X) = 1_{F(X)}$ and
- $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$.

2.2. **Example 4.** Set $\mathcal{C}_1 = A - mod$ and $\mathcal{C}_2 = Sets$. We define the functor F (called the forgetful functor) as follows. To each A-module N it assigns the *set* underlying N. For a map $\alpha : N_1 \to N_2$, $F(\alpha)$ is the map of underlying sets.

2.3. **Example 5.** Set $\mathcal{C}_1 = Sets$ and $\mathcal{C}_2 = Top$. We define the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ that assigns to every set X the topological space, which is X as a set with the discrete topology. To map $\alpha : X \to Y$ we assign the corresponding map $X \to Y$ of discrete topological spaces.

2.4. **Example 6.** Let $\mathcal{C}_1 = A - mod$ and $\mathcal{C}_2 = Ab$ (the category of abelian groups). Let M be a right A-module. We define the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ by $F(N) = M \underset{A}{\otimes} N$. For $\alpha : N_1 \to N_2$, let $F(\alpha) : M \underset{A}{\otimes} N_1 \to M \underset{A}{\otimes} N_2$ be the natural map.

2.5. **Example 7.** Let $\mathcal{C}_1, \mathcal{C}_2 = Top$, and let Z be a topological space. We define the functor F by $F(X) = X \times Z$, where $X \times Z$ has the product topology. For $\alpha : X \to Y$, we let $F(\alpha)$ be the natural map $X \times Z \to Y \times Z$.

2.6. **Contravariant functors.** A contravariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is by definition the same as a covariant functor $F : \mathcal{C}_1^{op} \to \mathcal{C}_2$.

2.7. **Exercise 4.** Write down what a contravariant functor is in concrete terms. Show that the notion of contravariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is equivalent to that of covariant functor $\mathcal{C}_1 \to \mathcal{C}_2^{op}$.

Henceforth, we will primarily speak about covariant functors, it will be up to you to work out the contravariant case.

2.8. **Yoneda's functors.** Let X be an object of $\mathcal{C}$. We define the covariant functor $h^X : \mathcal{C} \to Sets$ as follows. For an object $Y \in Ob(\mathcal{C})$ we set $h^X(Y)$ to be the set $Hom(X,Y)$. For a morphism $\alpha : Y_1 \to Y_2$, we set $h^X(\alpha) : Hom(X, Y_1) \to Hom(X, Y_2)$ to be given by composition.

We define a contravariant functor $h_X : \mathcal{C} \to Sets$ by setting $h_X(Y) = Hom(Y,X)$, and for a morphism $\alpha : Y_1 \to Y_2$, we set $h_X(\alpha) : Hom(X, Y_2) \to Hom(X, Y_1)$ to be given by composition with α.

2.9. **Exercise 5.** Show that $h_X : \mathcal{C}^{op} \to Sets$ and $h^X : \mathcal{C} \to Sets$ satisfy the axioms of functors.

3. Morphisms of functors

3.1. Let $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ be two functors. A morphism of functors $T : F \Rightarrow G$ is an assignment to every $X \in Ob(\mathcal{C}_1)$ a map $T(X) \in Hom_{\mathcal{C}_2}(F(X), G(X))$, such that for $\alpha \in Hom_{\mathcal{C}_1}(X, Y)$, the compositions $G(\alpha) \circ T(X)$ and $T(Y) \circ F(\alpha)$ coincide as elements of $Hom_{\mathcal{C}_2}(F(X), G(Y))$.

Morphisms of functors are also called "natural transformations".

3.2. **Example 8.** Let $\mathcal{C}_1 = A - mod$, $\mathcal{C}_2 = Ab$, and the functors $F, F' : \mathcal{C}_1 \to \mathcal{C}_2$ be given by tensor products with right modules M and M', respectively. Let $M \to M'$ be a morphism of right modules.

Then $T(N) : N \underset{A}{\otimes} M \to N \underset{A}{\otimes} M'$ is a morphism of functors.

3.3. **Example 9.** Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor. The identity morphism $F \Rightarrow F$ assigns to every $X \in Ob(\mathcal{C}_1)$ the identity element $1_{F(X)} \in Hom(F(X), F(X))$.

3.4. **Example 10.** Let $\mathcal{C}$ be a category and α be an element of $Hom_{\mathcal{C}}(X_2, X_1)$. Set $F_1, F_2 : \mathcal{C} \to Sets$ be h^{X_1} and h^{X_2}, respectively. We define the morphism of functors $h^\alpha : h^{X_1} \Rightarrow h^{X_2}$ as follows. For $Y \in \mathcal{C}$, the map $Hom(X_1, Y) \to Hom(X_2, Y)$ is given by composition with α.

Theorem 1. *Any morphism of functors $T : h^{X_1} \Rightarrow h^{X_2}$ has the form h^α for a unique element $\alpha \in Hom_{\mathcal{C}}(X_2, X_1)$*

This theorem is called Yoneda's lemma.

Proof. Let us first construct the element α. Consider the morphism

$$T(X_1) : Hom(X_1, X_1) \simeq h^{X_1}(X_1) \to h^{X_2}(X_1) \simeq Hom(X_2, X_1),$$

and apply it to the element $1_{X_1} \in Hom(X_1, X_1)$. We set $\alpha = T(X_1)(1_{X_1}) \in Hom(X_2, X_1)$.

Let us show that T equals h^α. For $Y \in \mathcal{C}$ and $\beta \in h^{X_1}(Y) \simeq Hom(X_1, Y)$ we have to show that $T(\beta) = \beta \circ \alpha \in h^{X_2}(Y) \simeq Hom(X_2, Y)$.

By the axiom of morphisms of functors, we have a commutative diagram:

$$\begin{array}{ccc} Hom(X_1, Y) & \xrightarrow{T(Y)} & Hom(X_2, Y) \\ {\scriptstyle h^{X_1}(\beta)}\uparrow & & {\scriptstyle h^{X_2}(\beta)}\uparrow \\ Hom(X_1, X_1) & \xrightarrow{T(X_1)} & Hom(X_2, X_1). \end{array}$$

Applying it to $1_{X_1} \in Hom(X_1, X_1)$ we arrive to the required conclusion.

□

3.5. **Exercise 6.** Let $\alpha : X_1 \to X_2$ be an element of $Hom_{\mathcal{C}}(X_1, X_2)$. Construct a map of contravariant functors $h_\alpha : h_{X_1} \Rightarrow h_{X_2}$. Show that any morphism of functors $h_{X_1} \Rightarrow h_{X_2}$ equals h_α for a unique $\alpha \in Hom_{\mathcal{C}}(X_1, X_2)$.

3.6. **Exercise 7.** Let F_1, F_2, F_3 be three functors $\mathcal{C}_1 \to \mathcal{C}_2$, and let $T : F_1 \Rightarrow F_2$ and $S : F_2 \Rightarrow F_3$ be morphisms of functors. Define the composition $S \circ T : F_1 \Rightarrow F_3$.

3.7. We say that a map of functors $T : F \Rightarrow G$ is an isomorphism if for every $X \in Ob(\mathcal{C}_1)$, the map $T(X) : F(X) \to G(X)$ is an isomorphism.

3.8. **Exercise 8.** Show that : $F \Rightarrow G$ is an isomorphism if and only if there exists a morphism of functors $S : G \Rightarrow F$ such that $T \circ S$ and $S \circ T$ are both identity morphisms. Show that the morphism of functors h^α (resp., h_α) is an isomorphism if and only α is.

3.9. **Representable functors.** Altough the assertion of Exercise 8 is a trivialilty, it is one of the most powerful tools in category theory. The upshot is that in order to "know" an object $X \in Ob(\mathcal{C})$ is it sufficient to "know" $Hom(Y, X)$ or $Hom(X, Y)$ for all $Y \in Ob(\mathcal{C})$.

We shall say that a covariant (resp., contravariant) functor $F : \mathcal{C} \to Sets$ is representable if it is *isomorphic* to a functor of the form h^X (resp., h_X). In this case we will call X "the representing object". Exercise 8 implies that the representing object is defined *up to a canonical isomorphism*. I.e, any isomorphism of functors $h^{X_1} \simeq h^{X_2}$ (resp., $h_{X_1} \simeq h_{X_2}$) comes from a unique isomorphism $X_1 \simeq X_2$.

3.10. **Example 11.** Let A be a ring, and M and N be a right and a lef A-modules respectively. Define a covariant functor from the category Ab of abelian groups to $Sets$, that assigns to every Λ the set of A-bilinear pairings $M \times N \to \Lambda$. We have seen that this functor is representable by $M \underset{A}{\otimes} N$.

3.11. **Example 12.** Let $\alpha : M_1 \to M_2$ be a map of left A-modules. Define a contravariant functor $F : A - mod \to Sets$ as follows: $F(M)$ is the set of all maps $\beta : M \to M_1$, such that the composition $\alpha \circ \beta : M \to M_2$ is 0. This functor is representable by $ker(\alpha)$.

3.12. **Exercise 9.** Give a similar interpretation to $coker(\alpha)$.

3.13. **Compositions of functors.** Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_3$ be functors. We define their composition $G \circ F : \mathcal{C}_1 \to \mathcal{C}_2$ as follows. For $X \in Ob(\mathcal{C}_1)$, set $G \circ F(X) = G(F(X))$. For $\alpha \in Hom_{\mathcal{C}_1}(X, Y)$, we set $G \circ F(\alpha) \in Hom_{\mathcal{C}_3}(G \circ F(X), G \circ F(Y))$ to be $G(F(\alpha))$.

3.14. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. How to express the idea that $\mathcal{C}_1$ and $\mathcal{C}_2$ are "the same"? The appropriate notion is called an equivalence of categories:

A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be an equivalence of categories *if there exists* a functor $G : \mathcal{C}_2 \to \mathcal{C}_1$, such that $F \circ G : \mathcal{C}_2 \to \mathcal{C}_2$ *is isomorphic* to the identity functor on $\mathcal{C}_2$ (cf. Example 9), and $G \circ F : \mathcal{C}_1 \to \mathcal{C}_1$ is isomorphic to the identity functor on $\mathcal{C}_1$. In this case G is called "a quasi-inverse" of F.

3.15. **Exercise 10.** Show that if G_1 and G_2 are both quasi-inverse functors of F, then G_1 and G_2 are isomorphic as functors $\mathcal{C}_2 \to \mathcal{C}_1$.

3.16. **Exercise 11.** Show that $F : \mathcal{C}_1 \to \mathcal{C}_2$ is an equivalence of categories if and only if the following properties hold:

- For $X, Y \in Ob(\mathcal{C}_1)$, the map $F : Hom_{\mathcal{C}_1}(X, Y) \to Hom_{\mathcal{C}_2}(F(X), F(Y))$ is an isomorphism.
- For every $X' \in Ob(\mathcal{C}_2)$, there exists some $X \in Ob(\mathcal{C}_1)$, such that $F(X)$ is isomorphic to X'.

In general we will say that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is faithful (resp., full. fully-faithful) if for $X, Y \in Ob(\mathcal{C}_1)$, the map $F : Hom_{\mathcal{C}_1}(X, Y) \to Hom_{\mathcal{C}_2}(F(X), F(Y))$ is injective (resp., surjective, bijective).

We will say that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is essentially surjective (or surjective on objects) if for every $X' \in Ob(\mathcal{C}_2)$, there exists some $X \in Ob(\mathcal{C}_1)$, such that $F(X)$ is isomorphic to X'

Thus, the exercise says: equivalence=fully-faithfulness + essential surjectivity.

3.17. **Exercise 12.** Give a non-trivial example of an equivalence of categories from the previous quarter.

4. Adjoint functors

4.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories, and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ be (covariant) functors. We say that F is the left adjoint functor of G, or that G is the right adjoint of F is for every $X \in Ob(\mathcal{C}_1)$ and $Y \in Ob(\mathcal{C}_2)$ we are given an isomorphism of sets

$$Hom_{\mathcal{C}_2}(F(X), Y) \simeq Hom_{\mathcal{C}_1}(X, G(Y)),$$

such that the following axioms hold:

- For a morphism $\alpha : X \to X'$ in $\mathcal{C}_1$, the diagram
$$\begin{array}{ccc} Hom_{\mathcal{C}_2}(F(X), Y) & \longrightarrow & Hom_{\mathcal{C}_1}(X, G(Y)) \\ \uparrow & & \uparrow \\ Hom_{\mathcal{C}_2}(F(X'), Y) & \longrightarrow & Hom_{\mathcal{C}_1}(X', G(Y)) \end{array}$$
commutes, and
- For a morphism $\beta : Y \to Y'$ in $\mathcal{C}_2$, the diagram
$$\begin{array}{ccc} Hom_{\mathcal{C}_2}(F(X), Y) & \longrightarrow & Hom_{\mathcal{C}_1}(X, G(Y)) \\ \downarrow & & \downarrow \\ Hom_{\mathcal{C}_2}(F(X), Y') & \longrightarrow & Hom_{\mathcal{C}_1}(X, G(Y')) \end{array}$$
commutes.

4.2. **Exercise 13.** Let $\mathcal{C}_1 = A - mod$ and $\mathcal{C}_2 = B - mod$, where A and B are two rings. Let $\phi : A \to B$ be a homomorphism. Let $G : B - mod \to A - mod$ be the forgetful functor, i.e., for an B-module N, $G(N)$ is regarded as an A-module via ϕ. Define the functor $F : A - mod \to B - mod$ by $G(M) = B \underset{A}{\otimes} M$. Show that F is the right adjoint of G.

4.3. We will say that a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ admits a right (resp., left) adjoint, if there exists a functor $G : \mathcal{C}_2 \to \mathcal{C}_1$ which is a right (resp., left) adjoint of F.

4.4. **Exercise 14.** Show that if G_1, G_2 are both right (resp., left) adjoint functors of F, then $G_1 \simeq G_2$ as functors $\mathcal{C}_2 \to \mathcal{C}_1$.

4.5. **Exercise 15.** Show that F admits a right adjoint if and only if for every $Y \in Ob(\mathcal{C}_2)$, the functor $\mathcal{C}_1^{op} \to Sets$, given by $X \mapsto Hom_{\mathcal{C}_2}(F(X), Y)$ is representable. Formulate a similar condition for the existence of a left adjoint.

4.6. **Exercise 16.** Show that F is an equivalence of categories and G is its quasi-inverse, then F and G are mutually (both left and right) adjoint.

4.7. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be categories. We define their product $\mathcal{C}_1 \times \mathcal{C}_2$ to be another categories, whose objects are pairs $X \times Y$, $X \in Ob(\mathcal{C}_1), Y \in Ob(\mathcal{C}_2)$, and

$$Hom_{\mathcal{C}_1 \times \mathcal{C}_2}(X' \times Y', X'' \times Y'') = Hom_{\mathcal{C}_1}(X', X'') \times Hom_{\mathcal{C}_2}(Y', Y'').$$

4.8. **Exercise 17.** Show that two functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ are mutually adjoint if and only if we are given and isomorphism between two functors $H_1, H_2 : \mathcal{C}_1^{op} \times \mathcal{C}_2 \to Sets$, where

$$H_1(X \times Y) = Hom_{\mathcal{C}_2}(F(X), Y),\ H_2(X \times Y) = Hom_{\mathcal{C}_1}(X, G(Y)).$$

4.9. Let Γ be a (discrete) group. Define the category $\Gamma - mod$ to consist of abelian groups, acted on by Γ, and morphisms being morphisms of abelian groups, that respect the Γ-action.

Let $\Gamma' \subset \Gamma$ be a subgroup. We have a batural forgetful functor $F : \Gamma{-}mod \to \Gamma'-$ mod .

4.10. **Exercise 18.** Construct two functors $G, G' : \Gamma' - mod \to \Gamma -$ mod , where G is a *left* adjoint of F, and G' is a *right* adjoint of F.