Course notes: Week 1, Recap of tensor products

1. Definition of tensor product

1.1. Recall the notion of an abelian group, given by generators and relations.

1.2. Let A be a ring, M a right A-module and N a left A-module. The tensor product $M \underset{A}{\otimes} N$ is the abelian group, generated by symbols $m \otimes n$, $m \in M, n \in N$ and subject to the following relations:

- $0 \otimes n = 0$, $m \otimes 0 = 0$;
- $m_1 \otimes n + m_2 \otimes n = (m_1 + m_2) \otimes n$, $m \otimes n_1 + m \otimes n_2 = m \otimes (n_1 + n_2)$;
- $m \cdot a \otimes n = m \otimes a \cdot n$, $a \in A$.

1.3. **Exercise 1.** Let $A = \mathbb{Z}$, $M = \mathbb{Z}/2$, $N = \mathbb{Z}/3$. Show that $M \underset{A}{\otimes} N = 0$.

1.4. **Exercise 2.** Suppose that M is a left module *also* over a ring B, such that the actions of A and B on M commute. Then $M \underset{A}{\otimes} N$ has a structure of left B-module.

1.5. **Exercise 3.** Let $M = A$, regarded as a right A-bimodule. Show that $A \underset{A}{\otimes} N \simeq N$, as a left A-module.

1.6. **Exercise 4.** Suppose that $M = M_1 \oplus M_2$. Show that $M \underset{A}{\otimes} N \simeq M_1 \underset{A}{\otimes} N \oplus M_2 \underset{A}{\otimes} N$.

1.7. **Exercise 5.** Suppose that A is commutative, and note that in this case any left A-module is also a right A-module, and vice versa. Observe that from the previous exercise $M \underset{A}{\otimes} N$ has *a priori two* structures of A-module. Show that these two structures coincide.

1.8. **Exercise 6.** Suppose again that A is commutative. Show that there are natural isomorphisms $M \underset{A}{\otimes} N \simeq N \underset{A}{\otimes} M$ and $M_1 \underset{A}{\otimes} (M_2 \underset{A}{\otimes} M_3) \simeq (M_1 \underset{A}{\otimes} M_2) \underset{A}{\otimes} M_3$.

2. Tensor product of vector spaces

2.1. Suppose now that $A = k$ is a (commutative) field. Then A-modules are the same as k-vector spaces. If no confusion is likely to occur, we will write $V \otimes W$ instead of $V \underset{k}{\otimes} W$.

2.2. **Exercise 7.** Let $\{e_i\}$ be a basis of a vector space V and $\{f_j\}$ is a basis of a vector space W, where the indices i and j run over (not necessarily finite)sets I and J, respectively. Show that the vectors $e_i \otimes f_j$ form a basis of $V \otimes W$.

2.3. For a vector space V, we will denote by V^* the dual vector space, consisting of all linear functionals $V \to k$.

Note that we are in algebra, so that no inner form (scalar product) on V can be used, and it is important to distinguish V from V^*.

2.4. **Exercise 8.** Construct a natural linear map $V \otimes V^* \to k$.

2.5. For two vector spaces V_1, V_2, we will denote by $Hom(V_1, V_2)$ the vector space of all linear maps $V_1 \to V_2$. In particular, $V^* = Hom(V, k)$.

2.6. **Exercise 9.** Show that for 3 vector spaces U, V and W, we have a natural isomorphism $Hom(U \otimes V, W) \simeq Hom(U, Hom(V, W))$.

2.7. **Exercise 10.** Assume that V is finite-dimensional. Show that there are natural isomorphisms $Hom(V, W) \simeq V^* \otimes W$, and $Hom(W, V) \simeq W^* \otimes V$.

2.8. **Exercise 11.** In the above exercise take $V = W$. Show that the resulting map $Hom(V, V) \simeq V^* \otimes V \to k$ equals the trace of a matrix.

3. Tensor products and bilinear maps

3.1. We return to the general framework of modules over a non-commutative ring A. let M be a right A-module and N a left A-module. Let Λ be an abelian group.

An A-bilinear pairing $M \times N \to \Lambda$ is by definition a map of sets $\phi : M \times N \to \Lambda$, such that

- $\phi(0, n) = \phi(m, 0) = 0$.
- $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$, $\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$.
- $\phi(m \cdot a, n) = \phi(m, a \cdot n)$.

3.2. **Exercise 12.** Show that A-bilinear pairings $\phi : M \times N \to \Lambda$ are in bijection with maps of abelian groups $M \underset{A}{\otimes} N \to \Lambda$.