Course notes: Week 10, Projective and injective objects

1. PROJECTIVE OBJECTS

1.1. Unless specified otherwise, R will be an arbitrary (not necessarily commutative) ring.

Let $\alpha : M_1 \twoheadrightarrow M_2$ be a surjection of R-modules, and let $\beta : N \to M_2$ be a map. Evidently, we can't always find a lifting $\beta' : N \to M_1$, whose composition with α is β.

For example, take $M_1 = \mathbb{Z}$, $M_2 = \mathbb{Z}/2\mathbb{Z}$, and β be the identity map.

1.2. We say that a module N is projective if in the above situation a lifting β' always exist.

Lemma 1. *The following conditions are equivalent:*

(1) *N is projective.*

(2) *Every surjection $N' \to N$ admits a right inverse.*

Proof. Implication (1) ⇒ (2) is tautological. To prove (2) ⇒ (1), let α and β be as above, and conider the Cartesian product of M_1 and N over M_2:

$$N' := M_1 \underset{M_2}{\times} N := \{m_1 \in M_1, n \in N \mid \alpha(m_1) = \beta(n)\}.$$

Let α' denote the evident map $N' \to N$.

1.3. **Exercise 1.** Show that α' is surjective.

By assumption, there exists a splitting $N \to N'$.

1.4. **Exercise 2.** Show that the composition $N \to N' \to M_1$ is the desired β'. □

Proposition 1. *Every projective R-module is a direct summand of a free R-module.*

Proof. If N is any module, we can always find a surjection $R^I \to N$ for some index set I. If N is projective, this map admits a splitting, i.e., N is a direct summand of R^I.

1.5. **Exercise 3.** (a) Show that if $N = R$, then it is projective. (b) Show that a direct sum (finite or infinite) of projective R-modules is projective. (c) Show that a direct summand of a projective R-module is projective.

Thus, Exercise 3 shows that a direct summand of a free R-module is projective. □

1.6. **Exercise 4.** (a) Show that every projective R-module is flat. (b) Show that $\mathbb{Q}$ is flat, but not projective as a $\mathbb{Z}$-module.

2. INJECTIVE OBJECTS

2.1. Let $\alpha : M_1 \hookrightarrow M_2$ be an injective map of R-modules, and let $\beta : M_1 \to N$ be a map. It is *not* always true that we can extend β to a map $\beta' : M_2 \to N$.

If a module N is such that an extension is always possible, it is called injective.

2.2. **Exercise 5.** Formulate and prove the analog of Lemma 1 for injective modules.

2.3. **Exercise 6.** (a) Show that $k(t)$ as a module over $k[t]$ is injective. (b) Show that $\mathbb{Q}$ as a module over $\mathbb{Z}$ is injective.

Proposition 2. *Let R be a k-algebra, where k is a field. Then R-module can be embedded into an injective R-module.*

We will later prove the same result for any R.

Proof. For a k-vector space V, consider the left R-module $Hom_k(R, V)$, where R on itself acts via right multiplication. We have:

$$Hom_R(M, Hom_k(R, V)) \simeq Hom_k(M, k).$$

Hence, the functor $Hom(\cdot, Hom_k(R, V))$ is exact, which means that $Hom_k(R, V)$ is injective.

Apply this to $V = M$, regarded as k-vector space. We have a natural embedding of R-modules

$$M \simeq Hom_R(R, M) \to Hom_k(R, M).$$

□

3. Ext^1

3.1. Let M and N be two R-modules. Consider the set, whose elements are isomorphism classes of sghort exact sequences

$$0 \to N \to E \to M \to 0,$$

where we say that E and E' are isomorphic if there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "id"'] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

This set is called the set of extensions of M by means of N, and denoted $Ext^1_R(M, N)$ (or simply $Ext^1(M, N)$ if R is fixed).

3.2. We claim that $Ext^1(M, N)$ has a natural structure of an abelian group. Namely, if E and E' are two extensions as above, we define their Baer sum E'' as follows:

$$E'' = coker(N \to E \underset{M}{\times} E'),$$

where the map $N \to E \oplus E'$ is the anti-diagonal map.

We define the negative of an extension E to be E as an R-module, but the morphism to M to be the negative of the initial one.

We call an extension E trivial if it splits, i.e., if $E \simeq N \oplus M$.

3.3. **Exercise 7.** Show that the operation defined above is associative and commutative; the trivial extension is the zero element; the operation of passing to the negative extension is the negative.

3.4. Let $N_1 \to N_2$ be a map of R-modules, and M another R-module. We define the push-out map $Ext^1(M, N_1) \to Ext^1(M, N_2)$ as follows: to an extension

$$0 \to N_1 \to E_1 \to M \to 0$$

we attach $E_2 := coker(N_1 \to (E_1 \oplus N_2))$.

3.5. **Exercise 8.** Show that E_2 is an extension of M by N_2, and that the resulting map $Ext^1(M, N_1) \to Ext^1(M, N_2)$ is a map of abelian groups.

3.6. Let now $M_1 \to M_2$ be a map of R-modules, and N another R-module. We define the map $Ext^1(M_2, N) \to Ext^1(M_1, N)$ as follows. To an extension

$$0 \to N \to E_2 \to M_2 \to 0$$

we attach $E_1 := E_2 \underset{M_2}{\times} M_1$.

3.7. **Exercise 8.** Show that E_1 is an extension of M_1 by N, and that the resulting map $Ext^1(M_2, N) \to Ext^1(M_1, N)$ is a map of abelian groups.

4. Boundary homomorphisms

4.1. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of R-modules, and M another R-module. We define the map

$$Hom(M, N_3) \to Ext^1(M, N_2)$$

by sending $\alpha : M \to N_3$ to the corresponding induced extension of M by means of N_1.

Lemma 2. *This is a map of abelian groups.*

Proof. Let α_1 and α_2 be two maps $M \to N_3$. Consider the two extensions E_i of M by means of N_1, $i = 1, 2$.

Consider also the map $M \oplus M \overset{\alpha_1 \oplus \alpha_2}{\longrightarrow} N_3$; let E be the corresponding induced extension:

$$0 \to N_1 \to E \to M \oplus M \to 0.$$

It is easy to see that both the Baer sum of E_1 and E_2, and the extension corresponding to $\alpha_1 + \alpha_2$ identify with the extension induced by the diagonal map $M \to M \oplus M$ (check it!).

□

Proposition 3. *We have an exact sequence of abelian groups*

$$\begin{aligned} 0 \to Hom(M, N_1) \to Hom(M, N_2) \to Hom(M, N_3) \to \\ Ext^1(M, N_1) \to Ext^1(M, N_2) \to Ext^1(M, N_3). \end{aligned}$$

Proof. We have to check the exactness at three terms $Hom(M, N_3)$, $Ext^1(M, N_1)$, $Ext^1(M, N_2)$.

Consider $\alpha : M \to N_3$. The vanishing of its image in $Ext^1(M, N_1)$ means that the projection $N_2 \underset{N_3}{\times} M \to M$ admits a splitting, i.e., a map $M \to N_2$, whose composition with $N_2 \to N_1$ is α. But this exactly means that α comes from an element in $Hom(M, N_2)$.

Let now $0 \to N_1 \to E_1 \to M \to 0$ be an extension, giving a class in $Ext^1(M, N_1)$. The vanishing of the push-out $E_1 \oplus N_2/N_1$ means that we have a splitting $M \oplus N_2 \simeq (E_1 \oplus N_2)/N_1$. In particular, by composing with $(E_1 \oplus N_2)/N_1 \to N_2/N_1 \simeq N_3$ we produce a map $\alpha : M \to N_3$, and *pre-composing* with $E_1 \to (E_1 \oplus N_2)/N_1$ a map $E_1 \to N_2$.

It is easy to see that the resulting map $E_1 \to N_2 \oplus M$ is in fact an isomorphism onto $N_2 \underset{N_3}{\times} M$, i.e. our E_1 was induced by means of α.

Finally, let

$$0 \to N_2 \to E_2 \to M \to 0$$

be an extension. The vanishing of the push-out $(E_2 \oplus N_3)/N_2$ means that the map $N_2 \to N_3$ can be extended to a map $\gamma : E_2 \to N_3$. Consider $ker(\gamma)$. We have the maps

$$N_1 \to ker(\gamma) \text{ and } ker(\gamma) \to M,$$

and it is easy to see that they form a short exact sequence

$$0 \to N_1 \to ker(\gamma) \to M \to 0.$$

Hence, we have obtained a class in $Ext^1(M, N_1)$. It is easy to see that its image in $Ext^1(M, N_2)$ corresponds to the original extension E_2.

□

4.2. Let now $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules, and N another R-module. We define a map

$$Hom(M_1, N) \to Ext^1(M_3, N)$$

by associating to $\alpha : M_1 \to N$ the push-out of the extension M_2.

4.3. **Exercise 9.** Show that this is a map of abelian groups.

Proposition 4. *In the above situation we have an exact sequence*

$$0 \to Hom(M_3, N) \to Hom(M_2, N) \to Hom(M_1, N) \to$$
$$Ext^1(M_3, N) \to Ext^1(M_2, N) \to Ext^1(M_1, N).$$

4.4. **Exercise 10.** Prove the proposition.

5. INJECTIVE MODULES, REVISITED

5.1. Let us note that a module N over a ring R is injective if and only if for all M, $Ext^1(M, N) = 0$. (Similarly, a module M is projective if and only if $Ext^1(M, N) = 0$ for all N.)

5.2. **Exercise 10*.** (a) Show that for a module N to be injective it is sufficient to check that $Ext^1(M, N) = 0$ for all f.g. modules M. (b) Use Prop. 4 to show that for a module N to be injective it is enough to show check that $Ext^1(R/I, N) = 0$ for all ideals $I \subset R$. (c) Deduce that N is injective if and only if for every ideal $I \subset R$ and a map $I \to N$ there exists an extension $R \to N$.

5.3. **Exercise 11*.** (a) Show that for $R = \mathbb{Z}$ a module N is injective if and only if it is divisible, i.e., for every $n \in N$ and $k \in \mathbb{Z}$ there exists $n' \in N$ such that $k \cdot n' = n$. (b) Show that every abelian group (i.e., $\mathbb{Z}$-module) can be embedded into an injective abelian group. (c) Prove the analog of Prop. 2 for any ring R.