ENRICHED MODEL CATEGORIES AND DIAGRAM CATEGORIES

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Abstract. We collect in one place a variety of known and folklore results in enriched model category theory and add a few new twists. One twist is a new perspective on equivariant model categories. A central theme is a general procedure for constructing a Quillen adjunction, often a Quillen equivalence, between a given $\mathscr{V}$-model category and a category of diagrams in $\mathscr{V}$, where $\mathscr{V}$ is any good enriching category. From this perspective, we rederive the result of Schwede and Shipley that reasonable stable model categories are Quillen equivalent to diagram categories of spectra (alias categories of module spectra). The general theory will be applied to G-spectra in a sequel [10], and for that we need quite a few technical improvements and modifications of general model categorical results. We collect those here. They are bound to have applications in a variety of other contexts.

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Introduction

The categories, $\mathscr{M}$ say, that occur in nature have both hom sets $\mathscr{M}(X,Y)$ and enriched hom objects $\underline{\mathscr{M}}(X,Y)$ in some related category, $\mathscr{V}$ say. Technically $\mathscr{M}$ is enriched over $\mathscr{V}$. In topology, the enrichment is often given simply as a topology on the set of maps between a pair of objects, and its use is second nature. In algebra, enrichment in abelian groups is similarly familiar in the context of additive and Abelian categories. In homological algebra, this becomes enrichment in chain complexes, and the enriched categories go under the name of DG-categories.

Quillen's model category theory encodes homotopical algebra in general categories. In and of itself, it concerns just the underlying category, but the relationship with the enrichment is of fundamental importance in nearly all of the applications.

The literature of model category theory largely focuses on enrichment in the category of simplicial sets and related categories with a simplicial flavor. Although there are significant technical advantages to working simplicially, the reason for this is nevertheless probably more historical than mathematical. Simplicial enrichment often occurs naturally, but it is also often arranged artificially, replacing a different naturally occurring enrichment with a simplicial one. This gives a certain simplicial flavor to the literature that perhaps impedes the wider dissemination of model theoretic techniques. For example, it can hardly be expected that those in representation theory and other areas that deal naturally with DG-categories will read much of the simplicially oriented model category literature, even though it is directly relevant to their work. Even in topology, it usually serves no mathematical purpose to enrich simplicially in situations in equivariant, parametrized, and classical homotopy theory that arise in nature with topological enrichments.

While we also have explicit questions in mind, one of our goals is to summarize and explain some of how model category theory works in general in enriched contexts, adding a number of technical refinements that we need and could not find in the literature. Many of our results appear in one form or another in the standard category theory sources (especially Kelly [17] and Borceux [1]) and in the model theoretic work of Dugger, Hovey, Schwede, and Shipley [5, 6, 12, 30, 31, 32]. Although the latter papers largely focus on simplicial contexts, they contain the original versions and forerunners of many of our results.

The first two sections set out the basic framework. We discuss enriched model categories in general in §1 and discuss enriched diagram categories in §2. Aside perhaps from §2.2 and §2.3, which describe ways of constructing maps from small

$\mathscr{V}$-categories into full $\mathscr{V}$-subcategories of $\mathscr{V}$ or more generally $\mathscr{M}$, these sections contain little that is new.

Our main focus, to which we turn in §3, is the comparison between given enriched categories and related diagram categories. We will discuss answers to the following questions in general terms. They are natural variants on the theme of understanding the relationship between model categories in general and model categories of enriched presheaves. When $\mathscr{V}$ is the category $sSet$ of simplicial sets, a version of the first question was addressed by Dwyer and Kan [7]. Again when $\mathscr{V} = sSet$, a question related to the second was addressed by Dugger [3, 4]. When $\mathscr{V}$ is the category $\Sigma\mathscr{S}$ of symmetric spectra, the third question was addressed by Schwede and Shipley [31]. In all four questions, $\mathscr{D}$ denotes a small $\mathscr{V}$-category. The only model structure on presheaf categories that concerns us in these questions is the projective or level model structure induced from a given model structure on $\mathscr{V}$.

Question 0.1. *Suppose that $\mathscr{M}$ is a $\mathscr{V}$-category and $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$ is a $\mathscr{V}$-functor. When can one use δ to define a $\mathscr{V}$-model structure on $\mathscr{M}$ such that $\mathscr{M}$ is Quillen equivalent to the $\mathscr{V}$-model category $\mathscr{V}^{\mathscr{D}}$ of enriched presheaves $\mathscr{D}^{op} \longrightarrow \mathscr{V}$?*

Question 0.2. *Suppose that $\mathscr{M}$ is a $\mathscr{V}$-model category. When is $\mathscr{M}$ Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$, where $\mathscr{D}$ is the full $\mathscr{V}$-subcategory of $\mathscr{M}$ given by some well chosen set of objects of $\mathscr{M}$?*

Question 0.3. *Suppose that $\mathscr{M}$ is a $\mathscr{V}$-model category, where $\mathscr{V}$ is a* stable *model category. When is $\mathscr{M}$ Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$, where $\mathscr{D}$ is the full $\mathscr{V}$-subcategory of $\mathscr{M}$ given by some well chosen set of objects of $\mathscr{M}$?*

Question 0.4. *More generally, we can ask Questions 0.2 and 0.3, but seeking a Quillen equivalence between $\mathscr{M}$ and $\mathscr{V}^{\mathscr{D}}$ for some $\mathscr{V}$-functor $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$, not necessarily the inclusion of a full $\mathscr{V}$-subcategory.*

In §4 and §5, we give a variety of results that show how to change $\mathscr{D}$, $\mathscr{M}$, and $\mathscr{V}$ without changing the Quillen equivalence class of the model categories we are interested in. Many of these results are technical variants or generalizations (or sometimes just helpful specializations) of results of Dugger, Hovey, Schwede, and Shipley [5, 6, 12, 30, 31, 32]. Some of these results are needed for the sequel [10] and others are not, but we feel that a reasonably thorough compendium in one place may be a service to others. The results in this direction are scattered in the literature, and they are important in applications of model category theory in a variety of contexts. The new notion of a tensored adjoint pair is implicit but not explicit in the literature and captures a commonly occurring phenomenon of enriched adjunction. The new notions of weakly unital $\mathscr{V}$-categories and presheaves describe a phenomenon that appears categorically when the unit $\mathbf{I}$ of the symmetric monoidal model category $\mathscr{V}$ is not cofibrant and appears topologically in connection with Atiyah duality, as we will explain in [10].

In §6, we give a general discussion of equivariant model categories, beginning with a discussion of the double enrichment inherent in the equivariant context. Here the literature needed rethinking. For a general $\mathscr{V}$-category $\mathscr{M}$, the fixed point and orbit objects of a G-object in $\mathscr{M}$ should lie in $\mathscr{M}$, and then the relevant presheaves must take values in $\mathscr{M}$ rather than in $\mathscr{V}$. Thus the first question to ask is a version of Question 0.1 for the $\mathscr{V}$-category $G\mathscr{M}$, but with presheaves taking values in $\mathscr{M}$. This is discussed in §6.2. Here there is already a surprise. There are two choices of a relevant presheaf category, and it is perhaps the less obvious one that leads to the correct generalization of the standard Quillen equivalence relating G-spaces to space-valued functors defined on the orbit category of G.

One can also ask appropriate versions of Questions 0.2 and 0.3, starting from $G\mathscr{M}$ and considering presheaves in $\mathscr{V}$. This is discussed in §6.3. It turns out that these questions have immediate equivariant answers when there is a nonequivariant answer for $\mathscr{M}$, but again there is a surprise. The answers that one get appear naturally as answers to Question 0.4, not as answers to the questions as originally asked. That is, the domain category for the presheaves needed to model $G\mathscr{M}$ is not a full subcategory of $G\mathscr{M}$.

We are in large part motivated by questions about G-spectra, and most of the generalities in this paper will come into play in the sequel [10]. For example, for naive G-spectra, the results of §6.3 apply as they stand. However, we are interested in genuine G-spectra, and for that this paper only gives the starting point.

Both equivariantly and in general, the idea is that $\mathscr{V}$ is in practice a well understood model category, as are presheaf categories with values in $\mathscr{V}$. Modelling a general model category $\mathscr{M}$ in terms of such a presheaf category, with its elementary levelwise model structure, can be very useful in practice, as many papers in the literature make clear.

1. Enriched model categories

1.1. **Standing assumptions on $\mathscr{V}$.** Throughout this paper, $\mathscr{V}$ will be a closed symmetric monoidal category that is also a cofibrantly generated proper monoidal model category, as specified in [12, 4.2.6] or [11, 11.1.2]; see Definition 1.17 and Theorem 1.20 below. While it is sensible to require $\mathscr{V}$ to be proper, as we have done, we shall not make essential use of that assumption.

Remark 1.1. Although we will use the standard phrase "cofibrantly generated", we really have in mind "compactly generated" model categories, or a hybrid for which the cofibrations are compactly generated but the acyclic cofibrations may be cofibrantly generated. Compact generation, when applicable, allows one to use ordinary sequential cell complexes, without recourse to transfinite considerations. The cell objects are then much closer to the applications and intuitions than are the transfinite cell objects that are standard in the model category literature. Full details of this variant are in [24]; see also [25].

We write $V \otimes W$ or $V \otimes_{\mathscr{V}} W$ for the product and $\underline{\mathscr{V}}(V, W)$ for the internal hom in $\mathscr{V}$, and we write $\mathscr{V}(V, W)$ for the set of morphisms $V \longrightarrow W$ in $\mathscr{V}$. Thus

$$\mathscr{V}(V \otimes W, Z) \cong \mathscr{V}(V, \underline{\mathscr{V}}(W, Z)) \tag{1.2}$$

as sets and

$$\underline{\mathscr{V}}(V \otimes W, Z) \cong \underline{\mathscr{V}}(V, \underline{\mathscr{V}}(W, Z)) \tag{1.3}$$

in $\mathscr{V}$. We let $\mathbf{I}$ denote the unit object of $\mathscr{V}$.

Remark 1.4. It is often assumed that $\mathbf{I}$ is cofibrant. This is true in the most commonly used enriching categories, and we assumed it in earlier drafts. However, in the sequel [10], the assumption is not always satisfied, and it becomes important to know where it is needed and where not. We will be careful to show the places where it comes into play. Where it is not mentioned it is not needed.

It is usual to take $\mathscr{V}$ to be the cartesian monoidal category $sSet$ of simplicial sets, but it could equally well be the cartesian monoidal category $\mathscr{U}$ of compactly generated topological spaces, the cartesian monoidal Cat of small categories, or the category $\mathscr{M}_k$ of chain complexes over a commutative ring k under the tensor product. In stable homotopy theory, it could be the category $\Sigma\mathscr{S}$ or $\mathscr{I}\mathscr{S}$ of symmetric or orthogonal spectra or the category $\mathscr{Z}$ of EKMM S-modules under the smash product. The unit object is cofibrant in $\Sigma\mathscr{S}$ and $\mathscr{I}\mathscr{S}$, but not in $\mathscr{Z}$, but $\mathscr{Z}$ has the compensating advantage that all objects are fibrant. We will need to use both $\mathscr{I}\mathscr{S}$ and $\mathscr{Z}$ in the sequel [10], and this paper has a number of results that are needed for the comparisons there.

1.2. **Enriched categories and standing assumptions on $\mathscr{M}$.** We assume familiarity with the definitions of enriched categories, enriched functors, and enriched natural transformations [1, 17]. A brief elementary account is given in [24, Ch. 16]. We refer to these as $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations.

Throughout this paper, we let $\mathscr{M}$ be a bicomplete $\mathscr{V}$-category. We will explain the bicompleteness assumption in §1.3. We let $\underline{\mathscr{M}}(M,N)$ denote the enriched hom object in $\mathscr{V}$ between objects M and N of $\mathscr{M}$. We clarify the notation to $\underline{\mathscr{M}}_{\mathscr{V}}(M,N)$ when we consider changes of enriching category. In contrast, we let $\mathscr{M}(M,N)$ denote the hom set in the underlying category of $\mathscr{M}$, which has the same objects but only sets of morphisms. By definition,

$$\mathscr{M}(M,N) = \mathscr{V}(\mathbf{I}, \underline{\mathscr{M}}(M,N)). \tag{1.5}$$

We regard the underlying category as part of the structure of $\mathscr{M}$. Philosophically, if we think of the underlying category as the primary structure, we think of "enriched" as an adjective modifying the term category. If we think of the entire structure as fundamental, we think of "enriched category" as a noun (see [24]).

Remark 1.6. In the categorical literature, it is standard to let $\mathscr{M}_0$ denote the underlying category. Then $\mathscr{M}_0(M,N)$ denotes a morphism set of $\mathscr{M}_0$ and $\mathscr{M}(M,N)$ denotes a hom object in $\mathscr{V}$. This notation is logical, and our first draft used it. However, its conflict with standard practice in the rest of mathematics was obtrusive, so we have changed to a notation that is closer to that of the topological and model categorical literature. However, when we deal with presheaf categories, we will not be interested in the underlying category of the domain category $\mathscr{D}$. We shall therefore use the notation $\mathscr{D}$ rather than $\underline{\mathscr{D}}$ in that special case, using the notation $\mathscr{D}_0$ for the underlying category whenever it appears. We also use the notation $\mathscr{D}_0$ for unenriched domain categories for presheaves.

We recall two standard notions of enriched category theory. A $\mathscr{V}$-adjunction

$$\mathscr{N} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{M}$$

between $\mathscr{V}$-functors $\mathbb{T}$ and $\mathbb{U}$ is given by a binatural isomorphism

$$\underline{\mathscr{M}}(\mathbb{T}N, M) \cong \underline{\mathscr{N}}(N, \mathbb{U}M) \tag{1.7}$$

in $\mathscr{V}$. Applying $\mathscr{V}(\mathbf{I}, -)$, it induces an adjunction $\mathscr{M}(\mathbb{T}N, M) \cong \mathscr{N}(N, \mathbb{U}M)$ on underlying categories. One characterization is that a $\mathscr{V}$-functor $\mathbb{T}$ has a right $\mathscr{V}$-adjoint if and only if $\mathbb{T}$ preserves tensors (see below) and its underlying functor has a right adjoint in the usual set-based sense [1, II.6.7.6]. The dual characterization holds for the existence of a left adjoint to $\mathbb{U}$. The isomorphisms (1.3) are examples of $\mathscr{V}$-adjunctions. We shall give a generalization of the notion of an enriched adjunction that allows for a change of $\mathscr{V}$ in §5.5.

The $\mathscr{V}$-product $\otimes$ between $\mathscr{V}$-categories $\mathscr{M}$ and $\mathscr{N}$ has objects the pairs of objects (M, N) and has hom objects in $\mathscr{V}$

$$\underline{\mathscr{M}\otimes\mathscr{N}}((M,N),(M',N')) = \underline{\mathscr{M}}(M,M')\otimes\underline{\mathscr{N}}(N,N'),$$

with units and composition induced in the evident way from those of $\mathscr{M}$ and $\mathscr{N}$.

1.3. **Tensors, cotensors, and bicompleteness.** The bicompleteness assumption on $\mathscr{M}$ means that $\mathscr{M}$ has all weighted limits and colimits [17]. Equivalently, $\mathscr{M}$ is bicomplete in the usual set-based sense, and $\mathscr{M}$ has tensors $M \odot V$ and cotensors $\Phi(V, M)$ for $V \in \mathscr{V}$ and $M \in \mathscr{M}$. These notations are not standard. The standard notation for $\odot$ is $\otimes$, with obvious ambiguity. The usual, if not standard, notations for $\Phi(V, M)$ are $[V, M]$, M^V, or $F(V, M)$, none of which seems entirely satisfactory. By definition, tensors and cotensors are given by $\mathscr{V}$-bifunctors

$$\odot\colon \mathscr{M}\otimes\mathscr{V}\longrightarrow\mathscr{M} \quad \text{and} \quad \Phi\colon \mathscr{V}^{\mathrm{op}}\otimes\mathscr{M}\longrightarrow\mathscr{M}$$

that take part in $\mathscr{V}$-adjunctions

$$\underline{\mathscr{M}}(M\odot V, N)\cong\underline{\mathscr{V}}(V,\underline{\mathscr{M}}(M,N))\cong\underline{\mathscr{M}}(M,\Phi(V,N)). \tag{1.8}$$

We shall often write tensors as $V \odot M$ instead of $M \odot V$. In principle, since tensors are defined by a universal property and are therefore only defined up to isomorphism, there is no logical preference. However, in practice, we usually have explicit canonical constructions which differ by an interchange isomorphism. When $\mathscr{M} = \mathscr{V}$, we have the tensors and cotensors

$$V\odot W = V\otimes W \quad \text{and} \quad \Phi(V,W)=\underline{\mathscr{V}}(V,W).$$

While (1.8) is the correct categorical definition [1, 17], one sometimes sees the definition given in the unenriched sense of ordinary adjunctions

$$\mathscr{M}(M\odot V, N)\cong\mathscr{V}(V,\underline{\mathscr{M}}(M,N))\cong\mathscr{M}(M,\Phi(V,N)). \tag{1.9}$$

These follow by applying the functor $\mathscr{V}(I, -)$ to the adjunctions in (1.8). There is a partial converse to this implication. It is surely known, but we have not seen it in the literature.

Lemma 1.10. *Assume that we have the first of the ordinary adjunctions (1.9). Then we have the first of the enriched adjunctions (1.8) if and only if we have a natural isomorphism*

$$(M\odot V)\odot W\cong M\odot(V\otimes W). \tag{1.11}$$

Dually, assume that we have the second of the ordinary adjunctions (1.9). Then we have the second of the enriched adjunctions (1.8) if and only if we have a natural isomorphism

$$\Phi(V,\Phi(W,M))\cong\Phi(V\otimes W,M). \tag{1.12}$$

Proof. For objects N of $\mathscr{M}$, we have natural isomorphisms

$$\mathscr{M}((M\odot V)\odot W,N)\cong\mathscr{V}(W,\underline{\mathscr{M}}(M\odot V,N))$$

and

$$\mathscr{M}(M\odot(V\otimes W),N)\cong\mathscr{V}(V\otimes W,\underline{\mathscr{M}}(M,N))\cong\mathscr{V}(W,\underline{\mathscr{V}}(V,\underline{\mathscr{M}}(M,N))).$$

The first statement follows from the Yoneda lemma. The proof of the second statement is dual. □

Since we take (1.8) as a standing assumption, we have the isomorphisms (1.9), (1.11), (1.12). We will use some other standard maps and isomorphisms without comment. For example, there is a natural map, sometimes an isomorphism,

$$\omega\colon \underline{\mathscr{M}}(M,N)\otimes V \longrightarrow \underline{\mathscr{M}}(M,N\odot V). \tag{1.13}$$

This map in $\mathscr{V}$ is adjoint to the map in $\mathscr{M}$ given by the evident evaluation map

$$M\odot(\underline{\mathscr{M}}(M,N)\otimes V)\cong (M\odot \underline{\mathscr{M}}(M,N))\odot V\longrightarrow N\odot V.$$

1.4. **Model theoretic terminology.** Now suppose that $\mathscr{M}$ is a model category. Note that the weak equivalences, fibrations, and cofibrations live in the underlying category. We say that $\mathscr{M}$ is a $\mathscr{V}$-model category if, for a cofibration $i\colon A\longrightarrow X$ and fibration $p\colon E\longrightarrow B$ in $\mathscr{M}$, the induced map

$$\underline{\mathscr{M}}(i^*,p_*)\colon \underline{\mathscr{M}}(X,E)\longrightarrow \underline{\mathscr{M}}(A,E)\times_{\underline{\mathscr{M}}(A,B)}\underline{\mathscr{M}}(X,B) \tag{1.14}$$

is a fibration in $\mathscr{V}$ which is a weak equivalence if either i or p is a weak equivalence. The relationship of (1.14) with lifting properties should be clear.

By adjunction, as in [12, 4.2.2], the following two conditions are each equivalent to the properties required of (1.14). First, for a cofibration $i\colon A\longrightarrow X$ in $\mathscr{V}$ and a cofibration $j\colon B\longrightarrow Y$ in $\mathscr{M}$, the pushout product

$$i\Box j\colon A\odot Y\cup_{A\odot B} X\odot B\longrightarrow X\odot Y \tag{1.15}$$

is a cofibration in $\mathscr{M}$ which is a weak equivalence if either i or j is a weak equivalence. Second, for a cofibration $i\colon A\longrightarrow X$ in $\mathscr{V}$ and a fibration $p\colon E\longrightarrow B$ in $\mathscr{M}$, the induced map

$$\Phi(i^*,p_*)\colon \Phi(X,E)\longrightarrow \Phi(A,E)\times_{\Phi(A,B)}\Phi(X,B) \tag{1.16}$$

is a fibration in $\mathscr{M}$ which is a weak equivalence if either i or p is a weak equivalence.

Definition 1.17. The model structure on $\mathscr{V}$ is said to be monoidal if the equivalent conditions (1.14), (1.15), and (1.16) hold when $\mathscr{M}=\mathscr{V}$ and, in addition, the map $V\otimes Q\mathbf{I}\longrightarrow V\otimes \mathbf{I}\cong V$ is a weak equivalence for all cofibrant $V\in\mathscr{V}$, where $Q\mathbf{I}\longrightarrow \mathbf{I}$ is a cofibrant approximation of $\mathbf{I}$. When $\mathscr{V}$ is monoidal, we say that a $\mathscr{V}$-category $\mathscr{M}$ is a $\mathscr{V}$-model category if (1.14), (1.15), and (1.16) hold and, in addition, the map $M\odot Q\mathbf{I}\longrightarrow M\odot\mathbf{I}\cong M$ induced by $Q\mathbf{I}\longrightarrow\mathbf{I}$ is a weak equivalence for all cofibrant $M\in\mathscr{M}$.

When $\mathscr{M}$ is a $\mathscr{V}$-model category, the homotopy category $\mathrm{Ho}\mathscr{M}$ is enriched over $\mathrm{Ho}\mathscr{V}$, with $\underline{Ho\mathscr{M}}(M,N)$ represented by the object $\underline{\mathscr{M}}(M,N)$ of $\mathscr{V}$ when M and N are bifibrant (cofibrant and fibrant). We shall write $[M,N]_{\mathscr{M}}$ for the hom sets in $\mathrm{Ho}\mathscr{M}$, and similarly for $\mathscr{V}$. We then have

$$[M,N]_{\mathscr{M}}=[\mathbf{I},\underline{Ho\mathscr{M}}(M,N)]_{\mathscr{V}}. \tag{1.18}$$

The additional unit assumptions of Definition 1.17 are necessary for the proof.

We characterized enriched adjunctions categorically in §1.2. Model categorically, we are interested in Quillen $\mathscr{V}$-adjunctions.

Definition 1.19. A Quillen $\mathscr{V}$-adjunction is a $\mathscr{V}$-adjunction such that the induced adjunction on underlying model categories is a Quillen adjunction in the usual sense. (In the language of [12, 4.2.18], $\mathbb{T}$ is then called a $\mathscr{V}$-Quillen functor.) A Quillen $\mathscr{V}$-equivalence is a Quillen $\mathscr{V}$-adjunction such that the induced adjunction on underlying model categories is a Quillen equivalence in the usual sense.

We shall usually be dealing with cofibrantly generated model categories. We sometimes write $\mathcal{I}_{\mathscr{M}}$ and $\mathcal{J}_{\mathscr{M}}$ for given sets of generators for the cofibrations and acyclic cofibrations in $\mathscr{M}$, usually deleting the subscript when $\mathscr{M} = \mathscr{V}$. We recall one of the many variants of the standard characterization of such model categories ([11, 11.3.1], [24, 15.2.3], [25, 4.5.6]). The latter two sources include the compactly generated variant. We assume familiarity with the small object argument, which applies to the construction of both compactly and cofibrantly generated model categories ([11, 24, 25]). Recall that, for a set of maps $\mathcal{I}$, a relative $\mathcal{I}$-cell complex is a map $A \longrightarrow X$ such that X is a possibly transfinite colimit of objects X_i such that $X_0 = A$. For a limit ordinal β, $X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha$. For a successor ordinal $\alpha + 1$, $X_{\alpha+1}$ is the pushout of a coproduct of maps in $\mathcal{I}$ along a map from the domain of the coproduct into X_α. In the compact variant, we place no restrictions on the cardinality of the coproducts and only use countable sequences $\{X_i\}$. (In the usual theory, some sources reindex so that only one cell is attached at each stage, but there is no mathematical point in doing so, as is clarified in [24]).

A subcategory $\mathscr{W}$ of a category $\mathscr{C}$ is a category of weak equivalences if it contains all isomorphisms, is closed under retracts, and satisfies the two out of three property.

Theorem 1.20. *Let $\mathscr{W}$ be a subcategory of weak equivalences in a bicomplete category $\mathscr{M}$ and let $\mathcal{I}$ and $\mathcal{J}$ be sets of maps which permit the small object argument. Then $\mathscr{M}$ is a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$ if the following two conditions hold.*

(i) Every relative $\mathcal{J}$-cell complex $A \longrightarrow X$ is in $\mathscr{W}$; we then say that $\mathcal{J}$ is a good[1] *set of $\mathscr{W}$-acyclic maps.*
(ii) A map has the RLP with respect to $\mathcal{I}$ if and only if it is a weak equivalence and has the RLP with respect to $\mathcal{J}$.

We have in mind the underlying category of an enriched category, but of course enrichment is irrelevant to this result. In practice, since coproducts and sequential colimits generally preserve weak equivalences, the proof that a given set $\mathcal{J}$ is good boils down to showing that a pushout of a map in $\mathcal{J}$ is a weak equivalence. The verification may be technically different in different contexts. In topological situations, a general discussion and precise axiomatizations of how this property can be verified are given in [25, 4.5.8, 5.46], which apply to all topological situations the authors have encountered.

Henceforward, let $\mathcal{I}$ and $\mathcal{J}$ denote given sets of generating cofibrations and generating acyclic cofibrations for $\mathscr{V}$. Of course, we take them to satisfy (i) and (ii). We shall be constructing new model categories by use of left adjoints defined on $\mathscr{V}$. In such constructions, condition (ii) is inherited by adjunction from $\mathscr{V}$, and the adjunction reduces the small object argument hypothesis to a smallness condition in $\mathscr{V}$ that is usually easy to verify. Then the only substantive condition is the goodness condition in (i).

[1]Here and in the cognate notions of "goodness" below, these notions focus attention on what we view to be the main issue in the rigorous verification of the model axioms.

2. Enriched diagram categories

2.1. **Categories of enriched presheaves.** For a small $\mathscr{V}$-category $\mathscr{D}$, let $\mathscr{M}^{\mathscr{D}}$ denote the $\mathscr{V}$-category of enriched presheaves in the bicomplete $\mathscr{V}$-category $\mathscr{M}$. Good references for the general structure of such categories are [1, 6, 32, 33]. As noted before, we have no interest in the underlying category of $\mathscr{D}$ and write $\mathscr{D}(d,e)$ rather than $\underline{\mathscr{D}}(d,e)$ for its hom objects in $\mathscr{V}$ and write $\mathscr{D}_0(d,e)$ for hom sets when necessary. We write $\mathscr{M}_0^{\mathscr{D}}$ for the underlying category of a category of enriched presheaves, and we sometimes also write $\mathscr{M}_0^{\mathscr{D}}$ for presheaf categories when there is no enrichment in sight.

When considering presheaf categories, we are interested primarily in the case $\mathscr{M} = \mathscr{V}$. However, the general case is no more difficult and will be needed when considering equivariant examples. It is standard, especially in additive situations, to think of a small $\mathscr{V}$-category $\mathscr{D}$ as a kind of categorical "ring with many objects", and to think of (contravariant) $\mathscr{V}$-functors defined on $\mathscr{D}$ as (right) $\mathscr{D}$-modules. Many ideas and proofs become more transparent when first translated to the language of rings and modules.

The objects of $\mathscr{M}^{\mathscr{D}}$ are the $\mathscr{V}$-functors $X\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{M}$. On objects, we write $d \mapsto X_d$, and X is then given by maps

$$X = X(d,e)\colon \mathscr{D}(d,e) \longrightarrow \underline{\mathscr{M}}(X_e, X_d)$$

in $\mathscr{V}$. The morphisms $\alpha\colon X \longrightarrow Y$ in the underlying category of $\mathscr{M}^{\mathscr{D}}$ are the $\mathscr{V}$-natural transformations. Explicitly, they are given by maps $\alpha_d\colon X_d \longrightarrow Y_d$ in $\mathscr{M}$ such that the following diagrams commute in $\mathscr{V}$.

$$\begin{array}{ccc} \mathscr{D}(d,e) & \xrightarrow{\;X\;} & \underline{\mathscr{M}}(X_e, X_d) \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle (\alpha_d)_*} \\ \underline{\mathscr{M}}(Y_e, Y_d) & \xrightarrow[(\alpha_e)^*]{} & \underline{\mathscr{M}}(X_e, Y_d). \end{array}$$

The enriched hom $\underline{\mathscr{M}}^{\mathscr{D}}(X,Y)$ is the equalizer in $\mathscr{V}$ displayed in the diagram

$$\underline{\mathscr{M}}^{\mathscr{D}}(X,Y) \longrightarrow \textstyle\prod_d \underline{\mathscr{M}}(X_d, Y_d) \rightrightarrows \prod_{d,e} \underline{\mathscr{M}}(\mathscr{D}(e,d) \odot X_d, Y_e). \tag{2.1}$$

The parallel arrows are defined using the evaluation maps

$$\mathscr{D}(e,d) \odot X_d \longrightarrow X_e \quad\text{and}\quad \mathscr{D}(e,d) \odot Y_d \longrightarrow Y_e$$

of the $\mathscr{V}$-functors X and Y, in the latter case after composition with

$$\mathscr{D}(e,d) \odot (-)\colon \underline{\mathscr{M}}(X_d, Y_d) \longrightarrow \underline{\mathscr{M}}(\mathscr{D}(e,d) \odot X_d, \mathscr{D}(e,d) \odot Y_d).$$

The $\mathscr{V}$-category $\mathscr{M}^{\mathscr{D}}$ is bicomplete, with colimits, limits, tensors and cotensors defined in the evident objectwise fashion; in particular,

$$(X \odot V)_d = X_d \odot V \quad\text{and}\quad \Phi(V,X)_d = \Phi(V, X_d).$$

For clarity below, The reader should verify that we may identify $(\mathscr{M}^{\mathrm{op}})^{\mathscr{D}}$ with the opposite of the $\mathscr{V}$-category $\mathscr{M}^{\mathscr{D}^{\mathrm{op}}}$ and so think of it as a category of presheaves in $\mathscr{M}$. Of course, this category must not be confused with $(\mathscr{M}^{\mathscr{D}})^{\mathrm{op}}$. Applied levelwise, the functors $(-) \odot M\colon \mathscr{V} \longrightarrow \mathscr{M}$ and $\Phi(-,M)\colon \mathscr{V}^{\mathrm{op}} \longrightarrow \mathscr{M}$ for varying M induce $\mathscr{V}$-functors

$$\odot\colon \mathscr{V}^{\mathscr{D}} \otimes \mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{D}} \quad\text{and}\quad \Phi\colon \mathscr{V}^{\mathscr{D}^{\mathrm{op}}} \otimes \mathscr{M} \longrightarrow (\mathscr{M}^{\mathscr{D}})^{\mathrm{op}}.$$

Similarly, the functors $\underline{\mathscr{M}}(M,-)$ and $\underline{\mathscr{M}}(-,M)$ induce $\mathscr{V}$-functors

$$\mathscr{M}^{\mathrm{op}} \otimes \mathscr{M}^{\mathscr{D}} \longrightarrow \mathscr{V}^{\mathscr{D}} \quad \text{and} \quad \mathscr{M}^{\mathscr{D}^{\mathrm{op}}} \otimes \mathscr{M} \longrightarrow (\mathscr{V}^{\mathscr{D}})^{\mathrm{op}}.$$

We denote both of these by $\underline{\mathscr{M}}(-,-)$, relying on context to distinguish them.

Let $X \in \mathscr{V}^{\mathscr{D}}$, $Y \in \mathscr{M}^{\mathscr{D}}$, and $Z \in \mathscr{M}^{\mathscr{D}^{\mathrm{op}}}$. The categorical tensor product (specializing left Kan extension) of the contravariant functor X and the covariant functor Z on $\mathscr{D}$ gives the object $X \odot_{\mathscr{D}} Z \in \mathscr{M}$ displayed in the coequalizer diagram

$$\textstyle\coprod_{d,e} X_e \otimes \mathscr{D}(d,e) \odot Z_d \rightrightarrows \coprod_d X_d \odot Z_d \longrightarrow X \odot_{\mathscr{D}} Z. \tag{2.2}$$

The parallel arrows are defined using the evaluation maps of X and Z and the isomorphism (1.11). Similarly, the categorical hom of the contravariant functors X and Y gives the object $\Phi_{\mathscr{D}}(X,Y) \in \mathscr{M}$ displayed in the equalizer diagram

$$\Phi_{\mathscr{D}}(X,Y) \longrightarrow \textstyle\prod_d \Phi(X_d, Y_d) \rightrightarrows \prod_{d,e} \Phi(\mathscr{D}(e,d) \otimes X_d, Y_e) \tag{2.3}$$

analogous to (2.1). With these constructions, we have adjunctions analogous to those of (1.8).

$$\underline{\mathscr{M}}(X \odot_{\mathscr{D}} Z, M) \cong \underline{\mathscr{V}}^{\mathscr{D}}(X, \underline{\mathscr{M}}(Z,M)) \cong \underline{\mathscr{M}}^{\mathscr{D}^{\mathrm{op}}}(Z, \Phi(X,M)) \tag{2.4}$$

and

$$\underline{\mathscr{M}}^{\mathscr{D}}(X \odot M, Y) \cong \underline{\mathscr{V}}^{\mathscr{D}}(X, \underline{\mathscr{M}}(M,Y)) \cong \underline{\mathscr{M}}(M, \Phi_{\mathscr{D}}(X,Y)). \tag{2.5}$$

Applying $\mathscr{V}(\mathbf{I},-)$, there result ordinary adjunctions, with hom sets replacing hom objects in $\mathscr{V}$, that are analogous to those displayed in (1.9).

Notation 2.6. For $d \in \mathscr{D}$, let $\mathbb{D}(d)$ denote the presheaf in $\mathscr{V}$ represented by d, so that $\mathbb{D}(d)_e = \mathscr{D}(e,d)$. Then $\mathbb{D}$ is the object function of a $\mathscr{V}$-functor $\mathbb{D}\colon \mathscr{D} \longrightarrow \mathscr{V}^{\mathscr{D}}$. Let $\mathrm{ev}_d\colon \mathscr{M}^{\mathscr{D}} \longrightarrow \mathscr{M}$ denote the d^{th} object $\mathscr{V}$-functor, which sends X to X_d.

Proposition 2.7. *The $\mathscr{V}$-functor ev_d has the left $\mathscr{V}$-adjoint $F_d\colon \mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{D}}$ defined on objects by $F_d M = \mathbb{D}(d) \odot M$ for $M \in \mathscr{M}$. Dually, $\mathrm{ev}_d\colon \mathscr{M}^{\mathscr{D}^{op}} \longrightarrow \mathscr{M}$ has the right $\mathscr{V}$-adjoint G_d defined on objects by $G_d M = \Phi(\mathbb{D}(d), M)$.*

Proof. By (2.5) and the enriched Yoneda lemma [1, 6.3.5],

$$\begin{aligned} \underline{\mathscr{M}}^{\mathscr{D}}(F_d M, Y) &\cong \underline{\mathscr{V}}^{\mathscr{D}}(\mathbb{D}(d), \underline{\mathscr{M}}(M,Y)) \\ &\cong \underline{\mathscr{M}}(M, Y_d) = \underline{\mathscr{M}}(M, \mathrm{ev}_d(Y)). \end{aligned}$$

Dually, by (2.4) and the enriched Yoneda lemma,

$$\begin{aligned} \underline{\mathscr{M}}^{\mathscr{D}^{\mathrm{op}}}(Z, G_d M) &\cong \underline{\mathscr{V}}^{\mathscr{D}}(\mathbb{D}(d), \underline{\mathscr{M}}(Z,M)) \\ &\cong \underline{\mathscr{M}}(Z_d, M) = \underline{\mathscr{M}}(\mathrm{ev}_d(Z), M). \quad \square \end{aligned}$$

Since limits, colimits, tensors, and cotensors in $\mathscr{M}^{\mathscr{D}}$ are defined levelwise, the functors ev_d preserve all of these, and so do their adjoints F_d and G_d.

2.2. **Constructing $\mathscr{V}$-categories over a full $\mathscr{V}$-subcategory of $\mathscr{V}$.** In the sequel [10] we are especially interested in finding calculationally accessible domain $\mathscr{V}$-categories $\mathscr{C}$ for categories of presheaves $\mathscr{V}^{\mathscr{C}}$ that are equivalent to categories of presheaves $\mathscr{V}^{\mathscr{D}}$, where $\mathscr{D}$ is a well chosen full $\mathscr{V}$-subcategory of an ambient $\mathscr{V}$-category $\mathscr{M}$. Of course, for that purpose we are not at all concerned with the underlying categories of $\mathscr{C}$ and $\mathscr{D}$. In §2.3, we shall give a theoretical description of all such $\mathscr{V}$-maps $\mathscr{C} \longrightarrow \mathscr{D}$, where $\mathscr{D}$ is preassigned.

Here we restrict attention to $\mathscr{M} = \mathscr{V}$ and give a simple general way of constructing a $\mathscr{V}$-map $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D}$ where $\mathscr{C}$ is a small $\mathscr{V}$-category and $\mathscr{D}$ is a full $\mathscr{V}$-category of $\mathscr{V}$ whose objects are specified in terms of $\mathscr{C}$. Despite its simplicity, this example will play a key role in the sequel [10].

Construction 2.8. Fix an object $e \in \mathscr{C}$. In the applications, e is a distinguished object with favorable properties. Let $\mathscr{D}$ be the full $\mathscr{V}$-subcategory of $\mathscr{V}$ whose objects are the $\mathscr{C}(e,c)$ for $c \in \mathscr{C}$. We define a $\mathscr{V}$-functor $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D}$ such that $\gamma(c) = \mathscr{C}(e,c)$ on objects. The map

$$\gamma\colon \mathscr{C}(b,c) \longrightarrow \mathscr{D}(b,c) = \underline{\mathscr{V}}(\mathscr{C}(e,b), \mathscr{C}(e,c))$$

in $\mathscr{V}$ is the adjoint of the composition

$$\circ\colon \mathscr{C}(b,c) \otimes \mathscr{C}(e,b) \longrightarrow \mathscr{C}(e,c).$$

The diagrams

$$\begin{array}{ccc} & \mathbf{I} & \\ {}^{\eta}\swarrow & & \searrow^{\eta} \\ \mathscr{C}(c,c) & \xrightarrow[\gamma]{} & \underline{\mathscr{V}}(\mathscr{C}(e,c), \mathscr{C}(e,c)) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{C}(b,c) \otimes \mathscr{C}(a,b) & \xrightarrow{\gamma\otimes\gamma} & \underline{\mathscr{V}}(\mathscr{C}(e,b), \mathscr{C}(e,c)) \otimes \underline{\mathscr{V}}(\mathscr{C}(e,a), \mathscr{C}(e,b)) \\ \circ\downarrow & & \downarrow\circ \\ \mathscr{C}(a,c) & \xrightarrow[\gamma]{} & \underline{\mathscr{V}}(\mathscr{C}(e,a), \mathscr{C}(e,c)) \end{array}$$

commute since their adjoints

$$\begin{array}{ccc} & \mathbf{I} \otimes \mathscr{C}(e,c) & \\ {}^{\eta\otimes\mathrm{id}}\swarrow & & \searrow^{\cong} \\ \mathscr{C}(c,c) \otimes \mathscr{C}(e,c) & \xrightarrow[\circ]{} & \mathscr{C}(e,c)) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{C}(b,c) \otimes \mathscr{C}(a,b) \otimes \mathscr{C}(e,a) & \xrightarrow{\mathrm{id}\,\otimes\circ} & \mathscr{C}(b,c) \otimes \mathscr{C}(e,b) \\ \circ\otimes\mathrm{id}\downarrow & & \downarrow\circ \\ \mathscr{C}(a,b) \otimes \mathscr{C}(e,a) & \xrightarrow[\circ]{} & \mathscr{C}(a,c) \end{array}$$

commute. To see that the last diagram is adjoint to the second, observe that $\gamma \otimes \gamma = (\gamma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma)$.

Remark 2.9. If $\mathscr{C}$ is a symmetric monoidal $\mathscr{V}$-category with unit object e and product $\Box$, then the composite $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D} \subset \mathscr{V}$ is a lax symmetric monoidal $\mathscr{V}$-functor. The data showing this are the unit map $\mathbf{I} \longrightarrow \mathscr{C}(e,e) = \gamma(e)$ and the product map

$$\Box\colon \gamma(b) \otimes \gamma(c) = \mathscr{C}(e,b) \otimes \mathscr{C}(e,c) \longrightarrow \mathscr{C}(e, b\Box c) = \gamma(b\Box c),$$

where we have used the canonical isomorphism $e\Box e \cong e$.

2.3. **Characterizing $\mathscr{V}$-categories over a full $\mathscr{V}$-subcategory of $\mathscr{M}$.** Here we use the first adjunction of (1.8) to characterize the $\mathscr{V}$-categories $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D}$ over any full $\mathscr{V}$-subcategory $\mathscr{D}$ of a $\mathscr{V}$-category $\mathscr{M}$. Technically, we do not assume that $\mathscr{M}$ is bicomplete, but we do assume the adjunction, so that we have tensors; we write them as $V \odot M$. Let $\mathscr{V}$-$\mathscr{C}at/\mathscr{D}$ be the category whose objects are the $\mathscr{V}$-functors $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D}$ that are the identity on objects and whose morphisms are the $\mathscr{V}$-natural transformations $\alpha\colon \mathscr{C} \longrightarrow \mathscr{C}'$ such that $\gamma' \circ \alpha = \gamma$.

Consider the following data.

(i) For each pair (d, e) of objects of $\mathscr{D}$, an object $\mathscr{C}(d, e)$ of $\mathscr{V}$ and an "evaluation map" $\varepsilon\colon \mathscr{C}(d, e) \odot d \longrightarrow e$ in $\mathscr{M}$ with adjoint map (in $\mathscr{V}$)

$$\gamma\colon \mathscr{C}(d, e) \longrightarrow \mathscr{D}(d, e) = \underline{\mathscr{M}}(d, e).$$

We require the following associativity diagram to commute for $(b, c, d, e) \in \mathscr{D}$.

$$\begin{array}{ccc}
(\mathscr{C}(d,e) \otimes \mathscr{C}(c,d) \otimes \mathscr{C}(b,c)) \odot b & \xrightarrow{(\mathrm{id}\otimes\mu)\odot\mathrm{id}} & (\mathscr{C}(d,e) \otimes (\mathscr{C}(b,d)) \odot b \\
\downarrow{\scriptstyle (\mu\otimes\mathrm{id})\odot\mathrm{id}} & & \downarrow{\scriptstyle \cong} \\
(\mathscr{C}(c,e) \otimes \mathscr{C}(b,c)) \odot b & & \mathscr{C}(d,e) \odot (\mathscr{C}(b,d) \odot b) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}\odot\varepsilon} \\
\mathscr{C}(c,e) \odot (\mathscr{C}(b,c) \odot b) & & \mathscr{C}(d,e) \odot d \\
\downarrow{\scriptstyle \mathrm{id}\odot\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
\mathscr{C}(c,e) \odot c & \xrightarrow[\varepsilon]{} & e
\end{array}$$

Diagram chasing then shows that, under the canonical isomorphism of their sources, the composites in the diagram agree with the following composite of evaluation maps.

$$\mathscr{C}(d,e) \odot (\mathscr{C}(c,d) \odot (\mathscr{C}(b,c) \odot b) \longrightarrow \mathscr{C}(d,e) \odot (\mathscr{C}(c,d) \odot c) \longrightarrow \mathscr{C}(d,e) \odot d \longrightarrow e.$$

(ii) For each object d of $\mathscr{D}$ a "unit map" $\eta\colon \mathbf{I} \longrightarrow \mathscr{C}(d, d)$ in $\mathscr{V}$ such that the following diagram commutes.

$$\begin{array}{ccc}
 & \mathbf{I} \odot d & \\
{\scriptstyle \eta\odot\mathrm{id}}\swarrow & & \searrow{\scriptstyle \cong} \\
\mathscr{C}(d,d) \odot d & \xrightarrow[\varepsilon]{} & d
\end{array}$$

Using the isomorphism (1.11), which depends on having the enriched adjunction (1.8) in $\mathscr{M}$, we define composition maps

$$\mu\colon \mathscr{C}(d,e) \otimes \mathscr{C}(c,d) \longrightarrow \mathscr{C}(c,e)$$

in $\mathscr{V}$ as the adjoints of the following composites of evaluation maps in $\mathscr{M}$.

$$(\mathscr{C}(d,e) \otimes \mathscr{C}(c,d)) \odot c \cong \mathscr{C}(d,e) \odot (\mathscr{C}(c,d) \odot c) \xrightarrow{\mathrm{id}\odot\varepsilon} \mathscr{C}(d,e) \odot d \xrightarrow{\varepsilon} e.$$

Proposition 2.10. *There is an isomorphism between $\mathscr{V}$-$\mathscr{C}at/\mathscr{D}$ and the category whose objects consist of the data specified in (i) and (ii) above and whose morphisms*

$\alpha\colon \mathscr{C} \longrightarrow \mathscr{C}'$ are given by maps $\alpha\colon \mathscr{C}(d,e) \longrightarrow \mathscr{C}'(d,e)$ such that the following diagrams commute (in $\mathscr{M}$ and $\mathscr{V}$ respectively).

$$\begin{array}{ccc} \mathscr{C}(d,e)\odot d & \xrightarrow{\alpha\odot id} & \mathscr{C}'(c,d)\odot e \\ & {\scriptstyle\varepsilon}\searrow \quad \swarrow{\scriptstyle\varepsilon'} & \\ & e & \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} & \mathbf{I} & \\ & {\scriptstyle\eta}\swarrow \quad \searrow{\scriptstyle\eta'} & \\ \mathscr{C}(d,d) & \xrightarrow[\alpha]{} & \mathscr{C}'(d,d) \end{array}$$

Proof. For an object $\mathscr{C}$ of the category defined in the statement we easily verify from the given data that $\mathscr{C}$ is a $\mathscr{V}$-category with the specified unit and composition maps and that the maps γ together with the identity function on objects specify a $\mathscr{V}$-functor $\mathscr{C} \longrightarrow \mathscr{D}$. Conversely, for a $\mathscr{V}$-functor $\gamma\colon \mathscr{C} \longrightarrow \mathscr{D}$ that is the identity on objects, we obtain data as in (i) and (ii) by use of the adjunction (1.8). This correspondence between objects carries over to a correspondence between morphisms. □

2.4. **The level, or projective, model structure on good presheaf categories.** With these preliminaries out of the way, we return to model category theory.

Definition 2.11. Let $\mathscr{M}$ be a cofibrantly generated $\mathscr{V}$-model category. A map $\alpha\colon X \longrightarrow Y$ in $\mathscr{M}^{\mathscr{D}}$ is a level weak equivalence or level fibration if each map $\alpha_d\colon X_d \longrightarrow Y_d$ is a weak equivalence or fibration in $\mathscr{M}$. Define $F\mathcal{I}_{\mathscr{M}}$ and $F\mathcal{J}_{\mathscr{M}}$ to be the sets of all maps $F_d i$ and $F_d j$ in $\mathscr{M}^{\mathscr{D}}$, where $d \in \mathscr{D}$, $i \in \mathcal{I}_{\mathscr{M}}$, and $j \in \mathcal{J}_{\mathscr{M}}$. Say that $\mathscr{D}$ is $\mathscr{M}$-good if $F\mathcal{I}_{\mathscr{M}}$ and $F\mathcal{J}_{\mathscr{M}}$ admit the small object argument and $F\mathcal{J}_{\mathscr{M}}$ is a good set of level acyclic maps in the sense of Theorem 1.20(i)). Say that $\mathscr{D}$ is $\mathscr{M}$-Quillen if the functors $\mathscr{D}(e,d)\odot(-)$ on $\mathscr{M}$ are Quillen left adjoints (equivalently, the functors $\Phi(\mathscr{D}(d,e),-)$ are Quillen right adjoints); this implies that $\mathscr{D}$ is $\mathscr{M}$-good.

Theorem 2.12. *If $\mathscr{D}$ is $\mathscr{M}$-good, then $\mathscr{M}^{\mathscr{D}}$ is a cofibrantly generated $\mathscr{V}$-model category under the level weak equivalences and level fibrations; the sets $F\mathcal{I}_{\mathscr{M}}$ and $F\mathcal{J}_{\mathscr{M}}$ are the generating cofibrations and generating acyclic cofibrations. If $\mathscr{M}$ is proper, so is $\mathscr{M}^{\mathscr{D}}$. The adjunctions (F_d, ev_d) are Quillen adjunctions. If the functors $\mathscr{D}(e,d)\odot -$ preserve cofibrations, as holds if $\mathscr{D}$ is $\mathscr{M}$-Quillen, then cofibrations in $\mathscr{M}^{\mathscr{D}}$ are level cofibrations, hence cofibrant objects are level cofibrant.*

Proof. We have assumed the smallness condition and condition (i) of Theorem 1.20, and condition (ii) is inherited by adjunction from $\mathscr{M}$. Since pushouts, pullbacks, and weak equivalences are defined levelwise, $\mathscr{M}^{\mathscr{D}}$ is proper when $\mathscr{M}$ is. To see that $\mathscr{M}^{\mathscr{D}}$ is a $\mathscr{V}$-model category, it suffices to verify the pushout product characterization (1.15) of $\mathscr{V}$-model categories, and this follows by adjunction from the fact that $\mathscr{M}$ is a $\mathscr{V}$-model category. By definition, the functors ev_d preserve fibrations and weak equivalences. For the last statement, we may as well replace $\mathscr{D}$ by $\mathscr{D}^{\mathrm{op}}$ and consider the model structure on $\mathscr{M}^{\mathscr{D}^{\mathrm{op}}}$. By Proposition 2.7, its evaluation functor ev_d has right adjoint $\Phi(\mathbb{D}(d),-)$. Since the functors $\mathscr{D}(e,d)\odot -$ preserve cofibrations, the adjunction (1.8) shows that the functors $\Phi(\mathscr{D}(e,d),-)$ and therefore $\Phi(\mathbb{D}(d),-)$ preserve acyclic fibrations. In turn, this implies that ev_d preserves cofibrations. □

The model structure of the theorem is usually called the projective model structure, since there is an evident dual notion of an injective model structure, but we shall just call it the level model structure.

Remark 2.13. By adjunction, since acyclic fibrations are level acyclic fibrations, if $i\colon M \longrightarrow N$ is a cofibration in $\mathscr{M}$ then $F_d i\colon F_d M \longrightarrow F_d N$ is a cofibration in $\mathscr{M}^{\mathscr{D}}$ for any $d \in \mathscr{D}$. Therefore, if M is cofibrant, then each $F_d M$ is cofibrant. In particular, if $\mathscr{M} = \mathscr{V}$ and $\mathbf{I}$ is cofibrant, then each represented presheaf $\mathbb{D}(d)$ is cofibrant in $\mathscr{V}^{\mathscr{D}}$. This need not hold in general, and that gives one persuasive reason for preferring to enrich in monoidal categories $\mathscr{V}$ with cofibrant unit object.

The $\mathscr{M}$-goodness of $\mathscr{D}$ is usually not hard to verify. By adjunction, the smallness condition means that the domains of maps $i \in \mathcal{I}_{\mathscr{M}}$ and $j \in \mathcal{J}_{\mathscr{M}}$ are small with respect to the level maps $A_d \longrightarrow X_d$ of a relative $F\mathcal{I}_{\mathscr{M}}$ or $F\mathcal{J}_{\mathscr{M}}$ cell complex $A \longrightarrow X$ in $\mathscr{M}^{\mathscr{D}}$. In practice, or so it seems to us, this condition is not the main issue, and we generally assume that it holds without further comment. It certainly holds if $\mathscr{D}$ is $\mathscr{M}$-Quillen, since then the $A_d \longrightarrow X_d$ are relative cell complexes. Of course, it also holds when the domains of the maps i and j are small with respect to all of $\mathscr{M}$. In topological situations, the smallness condition often follows from the compactness of such domains.

The acyclicity condition has more substance but also usually holds in practice. It holds if the functors $\mathbb{D}(d) \odot (-)$ take generating level acyclic cofibrations to weak equivalences and the colimits appearing in the definition of relative cell objects preserve weak equivalences. In particular, it holds if the functors $\mathbb{D}(d)\odot(-)$ preserve acyclic cofibrations, hence it holds if $\mathscr{D}$ is $\mathscr{M}$-Quillen.

Remark 2.14. Since $\mathscr{M}$ is a $\mathscr{V}$-model category, $\mathscr{D}$ is $\mathscr{M}$-Quillen if all $\mathscr{D}(d,e)$ are cofibrant in $\mathscr{V}$. This is one advantage to focusing on simplicial enrichment, since all simplicial sets are cofibrant. The monoid axiom on $\mathscr{V}$ of [30, 3.3] says that tensoring an acyclic cofibration with any object gives an acyclic cofibration, and if that holds then any $\mathscr{D}$ is $\mathscr{V}$-good. That gives a persuasive reason for preferring to enrich in monoidal model categories $\mathscr{V}$ that satisfy the monoid axiom. When the monoid axiom holds, Theorem 2.12 is [32, 6.1]; see also [32, 7.2] for stable situations.

The monoid axiom is immediate when all objects of $\mathscr{V}$ are cofibrant [30, 3.4], but it often holds when that fails. In fact, in topological situations, $\mathscr{D}$ is often $\mathscr{M}$-good and Theorem 2.12 applies even when the $\mathscr{D}(d,e)$ are not cofibrant, the monoid axiom fails for $\mathscr{V}$, and the functors $\mathscr{D}(d,e) \odot (-)$ do not preserve level acyclic cofibrations. As already noted, axiomatizations of exactly what is needed to ensure this are given in [25, 4.5.8, 5.4.6], which apply to all situations we have encountered. The essential point is that in topology (and in homological algebra), one has both classical cofibrations (HEP) and the cofibrations of the Quillen model structure, and one can use the more general classical cofibrations to check goodness.

Remark 2.15. One often starts with a plain unenriched (or discrete) category $\mathscr{D} = \mathscr{D}_0$. In that case, let $\mathbf{I}[-]$ be the functor from sets to $\mathscr{V}$ that sends a set K to the coproduct of copies of $\mathbf{I}$ indexed by the elements of K and sends a function $f\colon K \longrightarrow L$ to the map that sends the kth copy of $\mathbf{I}$ by the identity map to the $f(k)^{\text{th}}$ copy of $\mathbf{I}$. The functor $\mathbf{I}[-]$ is strong monoidal via the evident isomorphisms $\mathbf{I}[*] \cong \mathbf{I}$ and $\mathbf{I}[K] \otimes \mathbf{I}[L] \cong \mathbf{I}[K \times L]$. Define $\mathbf{I}[\mathscr{D}]$ to be the $\mathscr{V}$-category with the same object set as $\mathscr{D}$ and with morphism objects $\mathbf{I}[\mathscr{D}(d,e)]$. The composition is induced from that of $\mathscr{D}$. If $\mathbf{I}$ is cofibrant in $\mathscr{V}$, then $\mathbf{I}[K]$ is cofibrant for all K and $\mathbf{I}[\mathscr{D}]$ is $\mathscr{M}$-Quillen for any $\mathscr{M}$. Using that $\mathbf{I}[K] \odot V$ is the coproduct of copies of V indexed by the elements of K, we see that the ordinary category $\mathscr{M}_0^{\mathscr{D}}$ of unenriched presheaves in $\mathscr{M}$ can be identified with the underlying category of the enriched category $\mathscr{M}^{\mathbf{I}[\mathscr{D}]}$:

$$\mathscr{M}_0^{\mathscr{D}} \cong \mathscr{M}_0^{\mathbf{I}[\mathscr{D}]}.$$

In model category theory, diagram categories with discrete domain categories $\mathscr{D}$ are often used to study homotopy limits and colimits [2, 11, 12]. Shulman [33] has given a study of enriched homotopy limits and colimits in $\mathscr{V}$-model categories $\mathscr{M}$, starting in the same general framework in which we are working.

3. Comparisons between model categories $\mathscr{M}$ and $\mathscr{V}^{\mathscr{D}}$

3.1. **The categorical context for the comparison.** Of course, Theorem 2.12 is the starting point for a great deal of work in many directions. For example, model diagram categories are the starting point for several constructions of stable homotopy categories [14, 20, 21] and for Voevodsky's homotopical algebraic geometry [15, 26, 34]. In these applications, the level model structure is just a step on the way to the definition of a more sophisticated model structure, but we will be interested primarily in applications in which the level model structure is itself the one of interest.

We have so far assumed no relationship between $\mathscr{D}$ and $\mathscr{M}$, and in practice one encounters different interesting contexts. For example, one might have functors $\phi\colon \mathscr{D} \longrightarrow \mathscr{V}$ and $\psi\colon \mathscr{N} \longrightarrow \mathscr{M}$ such that the cotensors $\psi\Phi(\phi d, N)$ for fixed $N \in \mathscr{N}$ and varying $d \in \mathscr{D}$ give rise to a functor $\mathbb{U}\colon \mathscr{N} \longrightarrow \mathscr{M}^{\mathscr{D}}$ and so implicitly establish such a relationship. The idea is simple, but does not seem to be discussed in the literature. We will not discuss this context in general, but we will see that examples of it play a key role in equivariant enriched model category theory.

We shall focus on cases where $\mathscr{D}$ and $\mathscr{M}$ are more directly related. We are especially interested in the restricted kind of $\mathscr{V}$-categories $\mathscr{D}$ that are given by full embeddings $\mathscr{D} \subset \mathscr{M}$, but we shall see that it is worth working more generally with a fixed $\mathscr{V}$-functor $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$ as starting point. We set up the relevant formal context before returning to model theoretic considerations.

Notations 3.1. We fix a small $\mathscr{V}$-category $\mathscr{D}$ and a $\mathscr{V}$-functor $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$, writing $(\mathscr{D}, \delta)$ for the pair. As a case of particular interest, for a fixed set $\mathscr{D}$ (or $\mathscr{D}_{\mathscr{M}}$) of objects of $\mathscr{M}$, we let $\mathscr{D}$ also denote the full $\mathscr{V}$-subcategory of $\mathscr{M}$ with object set $\mathscr{D}$, and we then implicitly take δ to be the inclusion.

We wish to compare $\mathscr{M}$ with the category $\mathscr{V}^{\mathscr{D}}$ of enriched presheaves in $\mathscr{V}$. There are two relevant frameworks. In one, $\mathscr{D}$ is given a priori and $\mathscr{M}$ is defined in terms related to $\mathscr{D}$ and $\mathscr{V}$. In the other, $\mathscr{M}$ is given a priori and $\mathscr{D}$ is defined in terms of $\mathscr{M}$. In any case, there is an enriched adjunction relating $\mathscr{M}$ and $\mathscr{V}^{\mathscr{D}}$.

Definition 3.2. Define a $\mathscr{V}$-functor $\mathbb{U}\colon \mathscr{M} \longrightarrow \mathscr{V}^{\mathscr{D}}$ by letting $\mathbb{U}(M)\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{V}$ be the $\mathscr{V}$-functor represented by M, so that $\mathbb{U}(M)_e = \underline{\mathscr{M}}(\delta e, M)$. The evaluation maps of this presheaf are

$$\underline{\mathscr{M}}(\delta e, M) \otimes \mathscr{D}(d, e) \xrightarrow{\mathrm{id}\otimes\delta} \underline{\mathscr{M}}(\delta e, M) \otimes \underline{\mathscr{M}}(\delta d, \delta e) \xrightarrow{\circ} \underline{\mathscr{M}}(\delta d, M).$$

When δ is a full embedding, $\mathbb{U}(d)$ is the represented presheaf $\mathbb{D}(d)$ used in Proposition 2.7, and $\mathbb{U}$ extends the represented presheaf functor $d \mapsto \mathbb{D}(d)$ from $\mathscr{D}$ to $\mathscr{M}$.

Proposition 3.3. *The $\mathscr{V}$-functor $\mathbb{U}$ has a left $\mathscr{V}$-adjoint $\mathbb{T}$.*

Proof. For $X \in \mathscr{V}^{\mathscr{D}}$, define $\mathbb{T}X = X \odot_{\mathscr{D}} \delta$. This is an example of (2.2), and the $\mathscr{V}$-adjunction

$$\underline{\mathscr{M}}(\mathbb{T}X, M) \cong \underline{\mathscr{V}^{\mathscr{D}}}(X, \mathbb{U}M)$$

is a special case of (2.4). □

We need some observations about the unit $\eta\colon \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{T}$ and counit $\varepsilon\colon \mathbb{T}\mathbb{U} \longrightarrow \mathrm{Id}$ of the adjunction $(\mathbb{T}, \mathbb{U})$.

Proposition 3.4. *Let $d \in \mathscr{D}$ and $V \in \mathscr{V}$. Then $\mathbb{T}(\mathbb{D}(d) \odot V)$ is naturally isomorphic to $\delta d \odot V$ and*

$$\eta\colon F_d V = \mathbb{D}(d) \odot V \longrightarrow \mathbb{U}\mathbb{T}(\mathbb{D}(d) \odot V) \cong \mathbb{U}(\delta d \odot V), \tag{3.5}$$

evaluated at $e \in \mathscr{D}$, is the map

$$\mathscr{D}(e, d) \otimes V \longrightarrow \underline{\mathscr{M}}(\delta e, \delta d) \otimes V \longrightarrow \underline{\mathscr{M}}(\delta e, \delta d \odot V)$$

given by δ and ω (1.13). If $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$ is the inclusion of a full subcategory and we take $V = \mathbf{I}$, then $\eta\colon \mathscr{D}(e, d) \longrightarrow \underline{\mathscr{M}}(e, d)$ is the identity and $\varepsilon\colon \mathbb{T}\mathbb{U}(d) = \mathbb{T}\mathbb{D}(d) \to d$ is an isomorphism.

Proof. For the first statement, for any $M \in \mathscr{M}$ we have

$$\begin{aligned} \underline{\mathscr{M}}(\mathbb{T}(\mathbb{D}(d) \odot V), M) &\cong \underline{\mathscr{V}^{\mathscr{D}}}(\mathbb{D}(d) \odot V, \mathbb{U}(M)) \\ &\cong \underline{\mathscr{V}}(V, \underline{\mathscr{V}^{\mathscr{D}}}(\mathbb{D}(d), \mathbb{U}(M))) \\ &\cong \underline{\mathscr{V}}(V, \underline{\mathscr{M}}(\delta d, M)) \\ &\cong \underline{\mathscr{M}}(\delta d \odot V, M), \end{aligned}$$

by Proposition 3.3, two uses of (1.8), and the Yoneda lemma. This implies the claimed isomorphism. The descriptions of ε and η follow easily. □

Remark 3.6. There is another way of viewing the pair $(\mathscr{D}, \delta)$. We take $\mathscr{D}_{\mathscr{M}}$ to be the full $\mathscr{V}$-subcategory of $\mathscr{M}$ with objects the δd. Then δ factors as the composite of a $\mathscr{V}$-functor $\nu\colon \mathscr{D} \longrightarrow \mathscr{D}_{\mathscr{M}}$ and the inclusion $\iota\colon \mathscr{D}_{\mathscr{M}} \subset \mathscr{M}$. The $\mathscr{V}$-adjunction $(\mathbb{T}, \mathbb{U})$ factors as the composite of $\mathscr{V}$-adjunctions

$$\nu_!\colon \mathscr{V}^{\mathscr{D}} \rightleftarrows \mathscr{V}^{\mathscr{D}_{\mathscr{M}}} : \nu^* \quad \text{and} \quad \mathbb{T}\colon \mathscr{V}^{\mathscr{D}_{\mathscr{M}}} \rightleftarrows \mathscr{M} : \mathbb{U}$$

(see Proposition 4.4 below). As suggested by the notation, the same $\mathscr{D}$ can relate to different categories $\mathscr{M}$. However, the composite Quillen adjunction can be a Quillen equivalence even though neither of the displayed Quillen adjunctions is so. See Theorem 6.20 below for an important class of examples.

Our results in this paper, like nearly all of the results in the literature on replacing given model categories by equivalent presheaf categories, will ignore any given multiplicative structure on $\mathscr{M}$. The following observations give a starting point for a study of products, but we shall not pursue this further here. There are several problems. For starters, the hypotheses in the following remark are natural categorically, but they are seldom satisfied in the applications. Moreover, the assumption here that δ is op-lax clashes with the conclusion that γ is lax in Remark 2.9. In practice, it cannot be expected that either is strong symmetric monoidal.

Remark 3.7. Suppose that $\mathscr{D}$ is symmetric $\mathscr{V}$-monoidal with product $\oplus$ and unit object e. Then $\mathscr{V}^{\mathscr{D}}$ is symmetric $\mathscr{V}$-monoidal with product $\otimes$ and unit object $\mathbb{D}(e)$. For $X, Y \in \mathscr{V}^{\mathscr{D}}$, we have the evident external product $\bar{\otimes}\colon \mathscr{D} \otimes \mathscr{D} \longrightarrow \mathscr{V}$, which is given on objects by $(X\bar{\otimes}Y)(b,c) = X(b)\otimes Y(c)$, and left Kan extension along $\oplus$ gives the product $X\otimes Y \in \mathscr{V}^{\mathscr{D}}$. It is characterized by the adjunction

$$\mathscr{V}^{\mathscr{D}}(X\otimes Y, Z) \cong \mathscr{V}^{\mathscr{D}\otimes\mathscr{D}}(X\bar{\otimes}Y, Z\circ\oplus).$$

Now suppose further that $\mathscr{M}$ is symmetric $\mathscr{V}$-monoidal with product $\Box$ and unit object $\mathbf{J}$ and that the $\mathscr{V}$-functor $\delta\colon \mathscr{D}\longrightarrow\mathscr{M}$ is op-lax symmetric $\mathscr{V}$-monoidal, so that we are given a map $\zeta\colon \delta e\longrightarrow \mathbf{J}$ in $\mathscr{M}$ and a natural $\mathscr{V}$-map

$$\psi\colon \delta(c\oplus d)\longrightarrow \delta c\Box\delta d.$$

Then the functor $\mathbb{U}\colon \mathscr{M}\longrightarrow\mathscr{V}^{\mathscr{D}}$ is lax symmetric monoidal and therefore its left adjoint $\mathbb{T}$ is op-lax symmetric monoidal. The data showing this are a unit map $\eta\colon \mathbb{D}(e)\longrightarrow \mathbb{U}\mathbf{J}$ in $\mathscr{V}^{\mathscr{D}}$ and a natural $\mathscr{V}$-map $\phi\colon \mathbb{U}M\otimes\mathbb{U}N\longrightarrow\mathbb{U}(M\Box N)$. Recall that $\mathbb{U}M(d) = \underline{\mathscr{M}}(\delta d, M)$. The map η is given by the composite maps

$$\mathbb{D}(e)(d) = \mathscr{D}(d,e)\longrightarrow \underline{\mathscr{M}}(\delta d,\delta e)\longrightarrow\underline{\mathscr{M}}(\delta d,\mathbf{J})$$

induced by δ and ζ. The natural $\mathscr{V}$-map ϕ is adjoint to the natural $\mathscr{V}$-map

$$\mathbb{U}M\bar{\otimes}\mathbb{U}N\longrightarrow\mathbb{U}(M\Box N)\circ\oplus$$

given by the composite maps

$$\underline{\mathscr{M}}(\delta c, M)\otimes\underline{\mathscr{M}}(\delta d, N)\longrightarrow\underline{\mathscr{M}}(\delta c\Box\delta d, M\Box N)\longrightarrow\underline{\mathscr{M}}(\delta(c\oplus d), M\Box N)$$

induced by $\Box$ and ϕ.

3.2. **When does $\mathscr{V}^{\mathscr{D}}$ induce an equivalent model structure on $\mathscr{M}$?** With the details of context in hand, we return to the questions in the introduction. Letting $\mathscr{M}$ be a bicomplete $\mathscr{V}$-category, we repeat the first question. Here we start with a diagram model category $\mathscr{V}^{\mathscr{D}}$ and try to create a Quillen equivalent model structure on $\mathscr{M}$. Here and in the later questions, we are interested in Quillen $\mathscr{V}$-adjunctions and Quillen $\mathscr{V}$-equivalences, as defined in Definition 1.19.

Question 3.8. *For which $\delta\colon\mathscr{D}\longrightarrow\mathscr{M}$ can one define a $\mathscr{V}$-model structure on $\mathscr{M}$ such that $\mathscr{M}$ is Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$?*

Perhaps more sensibly, we can first ask this question for full embeddings corresponding to chosen sets of objects of $\mathscr{M}$ and then look for more calculable smaller categories $\mathscr{D}$, using Remark 3.6 to break the question into two steps.

When $\mathscr{V} = sSet$, Question 3.8 in roughly this form was first considered by Dwyer and Kan [7]. In some of their examples, they chose to turn given topological categories into simplicial ones. As we have already said, from our point of view it does not seem natural to focus solely on simplicially enriched categories and on presheaves with values in $sSet$. It is worth repeating a nice joke from John Baez, on being given a simplicial answer to a topological question:

> "The folklore is fine as long as we really *can* interchange topological spaces and simplicial sets. Otherwise it's a bit like this:
>
> 'It doesn't matter if you take a cheese sandwich or a ham sandwich; they're equally good.'
> 'Okay, I'll take a ham sandwich.'
> 'No! Take a cheese sandwich - they're equally good.'
>
> One becomes suspicious..."

A relevant mathematical point is that model categories in which every object is cofibrant, like $sSet$, behave quite differently from model categories in which every object is fibrant, like $\mathscr{U}$, even when they are Quillen equivalent. Topological model categories are studied in general in [25]. An early topological example where

Question 3.8 has an interesting positive answer is that of G-spaces (Piacenza [29], [22, Ch. VI]), which we recall in Example 6.1.

The general answer to Question 3.8 starts from a model structure on $\mathscr{M}$ that is defined in terms of $\mathscr{D}$.

Definition 3.9. A map $f\colon M \longrightarrow N$ in $\mathscr{M}$ is a $\mathscr{D}$-equivalence or $\mathscr{D}$-fibration if $f_*\colon \underline{\mathscr{M}}(\delta d, M) \longrightarrow \underline{\mathscr{M}}(\delta d, N)$ is a weak equivalence or fibration in $\mathscr{V}$ for all $d \in \mathscr{D}$; f is a $\mathscr{D}$-cofibration if it satisfies the LLP with respect to the $\mathscr{D}$-acyclic $\mathscr{D}$-fibrations. Define $\mathbb{T}F\mathcal{I}$ and $\mathbb{T}F\mathcal{J}$ to be the sets of maps in $\mathscr{M}$ obtained by applying $\mathbb{T}$ to the sets $F\mathcal{I}$ and $F\mathcal{J}$ in $\mathscr{V}^{\mathscr{D}}$. Say that $\mathscr{D}$ is $(\mathscr{M}, \mathscr{V})$-good if

(i) $\mathscr{D}$ is $\mathscr{V}$-good in the sense of Definition 2.11 (a condition independent of $\mathscr{M}$),
(ii) $\mathbb{T}F\mathcal{I}$ and $\mathbb{T}F\mathcal{J}$ satisfy the small object argument, and
(iii) $\mathbb{T}F\mathcal{J}$ is a good set of $\mathscr{D}$-acyclic maps in $\mathscr{M}$.

By adjunction, the smallness condition holds if the domains of maps in $\mathcal{I}$ or $\mathcal{J}$ are small with respect to the maps $\underline{\mathscr{M}}(\delta d, A) \longrightarrow \underline{\mathscr{M}}(\delta d, X)$, where $A \longrightarrow X$ is a $\mathbb{T}F\mathcal{I}$ or $\mathbb{T}F\mathcal{J}$ cell object in $\mathscr{M}$. This condition is usually not hard to verify in practice. The condition (iii) that a relative $\mathbb{T}F\mathcal{J}$-cell complex $A \longrightarrow X$ is a $\mathscr{D}$-equivalence is more substantial. It holds if and only if $\mathbb{U}$ carries relative $\mathbb{T}F\mathcal{J}$-cell complexes to level equivalences.

Theorem 3.10. *If $\mathscr{D}$ is $(\mathscr{M}, \mathscr{V})$-good, then $\mathscr{M}$ is a cofibrantly generated $\mathscr{V}$-model category under the $\mathscr{D}$-classes of maps, and $(\mathbb{T}, \mathbb{U})$ is a Quillen $\mathscr{V}$-adjunction. It is a Quillen $\mathscr{V}$-equivalence if and only if the unit map $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is a weak equivalence in $\mathscr{V}^{\mathscr{D}}$ for all cofibrant objects X.*

Proof. Since $\mathscr{D}$ is $\mathscr{V}$-good, $\mathscr{V}^{\mathscr{D}}$ is a $\mathscr{V}$-model category by Theorem 2.12. As in [11, 11.3.2], $\mathscr{M}$ inherits its $\mathscr{V}$-model structure from $\mathscr{V}^{\mathscr{D}}$, via Theorem 1.20. Since $\mathbb{U}$ creates the $\mathscr{D}$-equivalences and $\mathscr{D}$-fibrations in $\mathscr{M}$, $(\mathbb{T}, \mathbb{U})$ is a Quillen $\mathscr{V}$-adjunction. The last statement holds by [12, 1.3.16] or [21, A.2]. □

Remark 3.11. Since $\mathscr{V}$ is right proper and the right adjoints $\mathscr{M}(\delta d, -)$ preserve pullbacks, it is clear that $\mathscr{M}$ is right proper. It is not clear that $\mathscr{M}$ is left proper. Since we have assumed that $\mathscr{V}$ is left proper, $\mathscr{M}$ is left proper provided that, for a cofibration $M \longrightarrow N$ and a weak equivalence $M \longrightarrow Q$, the maps

$$\underline{\mathscr{M}}(\delta d, M) \longrightarrow \underline{\mathscr{M}}(\delta d, N)$$

are cofibrations in $\mathscr{V}$ and the canonical maps

$$\underline{\mathscr{M}}(\delta d, N) \cup_{\underline{\mathscr{M}}(\delta d, M)} \underline{\mathscr{M}}(\delta d, Q) \longrightarrow \underline{\mathscr{M}}(\delta, N \cup_M Q)$$

are weak equivalences in $\mathscr{V}$. Again, in topological situations, left properness can often be shown in situations where it is not obviously to be expected; see [21, 6.5] or [25, 5.5.1], for example.

Remark 3.12. To prove that $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is a weak equivalence when X is cofibrant, one may assume that X is an $F\mathcal{I}$-cell complex. When $X = F_d V$, the maps

$$\underline{\mathscr{M}}(e, d) \otimes V \longrightarrow \underline{\mathscr{M}}(e, d \odot V)$$

are usually quite explicit, and sometimes even isomorphisms, and one first checks that they are weak equivalences when V is the source or target of a map in $\mathcal{I}$. One then uses that cell complexes are built up as (transfinite) sequential colimits of pushouts of coproducts of maps in $F\mathcal{I}$. There are two considerations in play.

First, one needs $\mathscr{V}$ to be sufficiently well behaved that the relevant colimits preserve weak equivalences. Second, one needs $\mathscr{M}$ and $\mathscr{D}$ to be sufficiently well behaved that the right adjoint $\mathbb{U}$ preserves the relevant categorical colimits, at least up to weak equivalence. Formally, if X is a relevant categorical colimit, $\operatorname{colim} X_s$ say, then $\eta\colon X_d \longrightarrow \underline{\mathscr{M}}(\delta d, \mathbb{T}X)$ factors as the composite

$$\operatorname{colim}(X_s)_d \longrightarrow \operatorname{colim} \underline{\mathscr{M}}(\delta d, \mathbb{T}X_s) \longrightarrow \underline{\mathscr{M}}(\delta d, \operatorname{colim} \mathbb{T}X_s),$$

and a sensible strategy is to prove that these two maps are each weak equivalences, the first as a colimit of weak equivalences in $\mathscr{V}$ and the second by a preservation of colimits result for $\mathbb{U}$. Suitable compactness (or smallness) of the objects d can reduce the problem to the pushout case, which can be dealt with using an appropriate version of the gluing lemma asserting that a pushout of weak equivalences is a weak equivalence. Again, we prefer not to give a formal axiomatization since the relevant verifications can be technically quite different in different contexts.

3.3. **When is a given model category $\mathscr{M}$ equivalent to some $\mathscr{V}^{\mathscr{D}}$?** We are more interested in the second question in the introduction, which we repeat. Changing focus, we now start with a given model structure on $\mathscr{M}$.

Question 3.13. *Suppose that $\mathscr{M}$ is a $\mathscr{V}$-model category. When is $\mathscr{M}$ Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$, where $\mathscr{D} = \mathscr{D}_{\mathscr{M}}$ is the full sub $\mathscr{V}$-category of $\mathscr{M}$ given by some well chosen set of objects?*

Assumptions 3.14. Since we want $\underline{\mathscr{M}}(d, e)$ to be homotopically meaningful, we require henceforward that the objects in our chosen set of objects of $\mathscr{M}$ be bifibrant. Since we want $\mathscr{V}^{\mathscr{D}}$ to have a level $\mathscr{V}$-model structure, we also require henceforward that the $\mathscr{V}$-category $\mathscr{D}$ be $\mathscr{V}$-good in the sense of Definition 2.11. Remember that this often holds for any $\mathscr{D}$. We occasionally want to give $\mathscr{M}$ its $\mathscr{D}$-model structure, in order to compare it with the model structure we started with, and we then require $\mathscr{D}$ to be $(\mathscr{M}, \mathscr{V})$-good in the sense of Definition 3.9.

This question does not seem to have been asked before in quite this form and level of generality. Working simplicially, Dugger [4] studied a related question, asking when a given model category is Quillen equivalent to some localization of a presheaf category. He called such an equivalence a "presentation" of a model category, viewing the localization as specifying the relations. That is an interesting point of view for theoretical purposes, since the result can be used to deduce formal properties of $\mathscr{M}$ from formal properties of presheaf categories and localization. However, the relevant domain categories $\mathscr{D}$ are not intended to be small and calculationally accessible.

Working simplicially with stable model categories enriched over symmetric spectra, Schwede and Shipley made an extensive study of essentially this question in a series of papers, starting with [31]. The question is simpler to answer stably than in general, and we shall return to this in §3.4.

Of course, sometimes the given model structure on $\mathscr{M}$ will be a $\mathscr{D}$-model structure from Theorem 3.10, as in Example 6.1 below, and then nothing more needs to be said. However, when that is not the case, the answer can be much less obvious. We offer a general approach to the question. The following starting point is immediate from the definitions and Assumptions 3.14.

Proposition 3.15. $(\mathbb{T}, \mathbb{U})$ *is a Quillen adjunction between the $\mathscr{V}$-model categories $\mathscr{M}$ and $\mathscr{V}^{\mathscr{D}}$.*

Proof. Applied to the cofibrations $\emptyset \longrightarrow d$ given by our assumption that the objects $d \in \mathscr{M}$ are cofibrant, the definition of a $\mathscr{V}$-model structure implies that $p_*\colon \underline{\mathscr{M}}(d,E) \longrightarrow \underline{\mathscr{M}}(d,B)$ is a fibration or acyclic fibration in $\mathscr{V}$ if $p\colon E \longrightarrow B$ is a fibration or acyclic fibration in $\mathscr{M}$. □

Clearly, we cannot expect $\mathscr{M}$ to be Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$ or to itself with the $\mathscr{D}$-model structure (if present) unless the $\mathscr{D}$-equivalences are closely related to the class $\mathscr{W}$ of weak equivalences in the given model structure on $\mathscr{M}$. For example, $\mathscr{D}$ might be $\mathscr{W}$-good, meaning that the set $\mathbb{T}F\mathcal{J}$ is a good set of $\mathscr{W}$-acyclic maps, as specified in Theorem 1.20(i).

Definition 3.16. Let $\mathscr{D}$ be a set of objects of $\mathscr{M}$ satisfying Assumptions 3.14.

(i) Say that $\mathscr{D}$ is a reflecting set if $\mathbb{U}$ reflects weak equivalences between fibrant objects of $\mathscr{M}$; this means that if M and N are fibrant and $f\colon M \longrightarrow N$ is a map in $\mathscr{M}$ such that $\mathbb{U}f$ is a weak equivalence, then f is a weak equivalence.
(ii) Say that $\mathscr{D}$ is a creating set if $\mathbb{U}$ creates the weak equivalences in $\mathscr{M}$; this means that a map $f\colon M \longrightarrow N$ in $\mathscr{M}$ is a weak equivalence if and only if $\mathbb{U}f$ is a weak equivalence, so that $\mathscr{W}$ coincides with the $\mathscr{D}$-equivalences.

Remark 3.17. Since the functor $\mathbb{U}$ preserves acyclic fibrations between fibrant objects, it preserves weak equivalences between fibrant objects [12, 1.1.12]. Therefore, if $\mathscr{D}$ is a reflecting set, then $\mathbb{U}$ creates the weak equivalences between the fibrant objects of $\mathscr{M}$.

Observe that Theorem 3.10 requires $\mathscr{D}$ to be a creating set. However, when one starts with a given model structure on $\mathscr{M}$, there are many examples where no reasonably small set $\mathscr{D}$ creates all of the weak equivalences in $\mathscr{M}$, rather than just those between fibrant objects. On the other hand, in many topological situations all objects are fibrant, and then there is no distinction. By [12, 1.3.16] (and [21, A.2]), we have the following criteria for $(\mathbb{T},\mathbb{U})$ to be a Quillen equivalence.

Theorem 3.18. *Let $\mathscr{M}$ be a $\mathscr{V}$-model category and $\mathscr{D} \subset \mathscr{M}$ be a small full subcategory such that Assumptions 3.14 are satisfied.*

(i) $(\mathbb{T},\mathbb{U})$ *is a Quillen equivalence if and only if $\mathscr{D}$ is a reflecting set and the composite*

$$X \xrightarrow{\eta} \mathbb{U}\mathbb{T}X \xrightarrow{\mathbb{U}\lambda} \mathbb{U}R\mathbb{T}X$$

is a weak equivalence in $\mathscr{V}^{\mathscr{D}}$ for every cofibrant object X. Here η is the unit of the adjunction and $\lambda\colon \mathrm{Id} \longrightarrow R$ is a fibrant replacement functor in $\mathscr{M}$.
(ii) When $\mathscr{D}$ is a creating set, $(\mathbb{T},\mathbb{U})$ *is a Quillen equivalence if and only if the map $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is a weak equivalence for every cofibrant X.*
(iii) $(\mathbb{T},\mathbb{U})$ *is a Quillen equivalence if and only if it induces an adjoint equivalence of homotopy categories.*

Of course, (iii) is a general criterion, valid for any Quillen adjunction $(\mathbb{T},\mathbb{U})$. We conclude that, in favorable situations, $\mathscr{M}$ is Quillen equivalent to the presheaf category $\mathscr{V}^{\mathscr{D}}$, but this can only happen when $\mathscr{D}$ is a reflecting set. In outline, the verification of (i) or (ii) of Theorem 3.18 proceeds along much the same lines as in Remark 3.12, and again we see little point in an axiomatization. Whether or not the conclusion holds, we have the following observation.

Proposition 3.19. *Let $\mathscr{D}$ be an $(\mathscr{M},\mathscr{V})$-good and creating set of objects of $\mathscr{M}$. Then the identity functor on $\mathscr{M}$ is a left Quillen equivalence from the $\mathscr{D}$-model structure on $\mathscr{M}$ to the given model structure, and $(\mathbb{T},\mathbb{U})$ is a Quillen equivalence with respect to one of these model structures if and only if it is a Quillen equivalence with respect to the other.*

Proof. The weak equivalences of the two model structures on $\mathscr{M}$ are the same, and since $\mathbb{T}$ is a Quillen left adjoint for both model structures, the relative $\mathbb{T}F\mathcal{J}-$cell complexes are acyclic cofibrations in both. Their retracts give all of the $\mathscr{D}$-cofibrations, but perhaps only some of the cofibrations in the given model structure, which therefore might have more fibrations and so also more acyclic fibrations. □

A general difficulty in using a composite such as that in Theorem 3.18(i) to prove a Quillen equivalence is that the fibrant approximation R is almost never a $\mathscr{V}$-functor and need not behave well with respect to colimits. The following observation is relevant (and so is Baez's joke).

Remark 3.20. In topological situations, one often encounters Quillen equivalent model categories $\mathscr{M}$ and $\mathscr{N}$ with different advantageous features. Thus suppose that $(\mathbb{F},\mathbb{G})$ is a Quillen equivalence $\mathscr{M} \longrightarrow \mathscr{N}$ such that $\mathscr{M}$ but not necessarily $\mathscr{N}$ is a $\mathscr{V}$-model category and every object of $\mathscr{N}$ is fibrant. Let X be a cofibrant object of $\mathscr{V}^{\mathscr{D}}$, as in Theorem 3.18(i), and consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & \mathbb{U}\mathbb{T}X & \xrightarrow{\mathbb{U}\lambda} & \mathbb{U}R\mathbb{T}X \\ & \searrow & \downarrow{\scriptstyle \mathbb{U}\zeta} & & \simeq \downarrow{\scriptstyle \mathbb{U}\zeta} \\ & & \mathbb{U}\mathbb{G}\mathbb{F}\mathbb{T}X & \underset{\mathbb{U}\mathbb{G}\mathbb{F}\lambda}{\xrightarrow{\simeq}} & \mathbb{U}\mathbb{G}\mathbb{F}R\mathbb{T}X, \end{array}$$

where ζ is the unit of $(\mathbb{F},\mathbb{G})$. The arrows labeled $\simeq$ are weak equivalences because $R\mathbb{T}X$ is fibrant and cofibrant in $\mathscr{M}$ and $\mathbb{G}\mathbb{F}R\mathbb{T}X$ is fibrant in $\mathscr{M}$. Therefore the top composite is a weak equivalence, as desired, if and only if the diagonal arrow $\mathbb{U}\zeta\circ\eta$ is a weak equivalence. In effect, $\mathbb{G}\mathbb{F}\mathbb{T}X$ is a fibrant approximation of $\mathbb{T}X$, eliminating the need to consider R. It can happen that $\mathbb{G}$ has better behavior on colimits than R does, and this can simplify the required verifications.

3.4. **Stable model categories are categories of module spectra.** In [31], which has the same title as this section, Schwede and Shipley define a "spectral category" to be a small category enriched in the category $\Sigma\mathscr{S}$ of symmetric spectra, and they understand a "category of module spectra" to be a presheaf category of the form $\Sigma\mathscr{S}^{\mathscr{D}}$ for some spectral category $\mathscr{D}$. Up to notation, their context is the same as the context of our §1 and §2, but restricted to $\mathscr{V} = \Sigma\mathscr{S}$. In particular, they give an answer to that case of Question 0.3, which we repeat.

Question 3.21. *Suppose that $\mathscr{M}$ is a $\mathscr{V}$-model category, where $\mathscr{V}$ is a stable model category. When is $\mathscr{M}$ Quillen equivalent to $\mathscr{V}^{\mathscr{D}}$, where $\mathscr{D}$ is the full $\mathscr{V}$-subcategory of $\mathscr{M}$ given by some well chosen set of objects?*

To say that $\mathscr{V}$ is stable just means that $\mathscr{V}$ is pointed and that the suspension functor Σ on $\mathrm{Ho}\mathscr{V}$ is an equivalence. It follows that $\mathrm{Ho}\mathscr{V}$ is triangulated [12, §7.2]. It also follows that any $\mathscr{V}$-model category $\mathscr{M}$ is again stable and therefore $\mathrm{Ho}\mathscr{M}$ is triangulated. This holds since the suspension functor Σ on $\mathrm{Ho}\mathscr{M}$ is equivalent to the derived tensor with the invertible object $\Sigma\mathbf{I}$ of $\mathrm{Ho}\mathscr{V}$.

We here reconsider the work of Schwede and Shipley [31] and the later related work of Dugger [5] from our perspective. They start with a stable model category $\mathscr{M}$. They do not assume that it is a $\Sigma\mathscr{S}$-model category (which they call a "spectral model category"). Under appropriate hypotheses on $\mathscr{M}$, Hovey [12] defined the category $\Sigma\mathscr{M}$ of symmetric spectra in $\mathscr{M}$ and proved both that it is a $\Sigma\mathscr{S}$-model category and that it is Quillen equivalent to $\mathscr{M}$ [12, 8.11, 9.1]. Under significantly weaker hypotheses on $\mathscr{M}$, Dugger [5, 5.5] observed that an application of his earlier work on presentations of model categories [4] implies that $\mathscr{M}$ is Quillen equivalent to a model category $\mathscr{N}$ that satisfies the hypotheses needed for Hovey's results.

By the main result of Schwede and Shipley, [31, 3.9.3], when $\mathscr{M}$ and hence $\mathscr{N}$ has a compact set of generators (see Definition 3.22 below), $\Sigma\mathscr{N}$ is Quillen equivalent to a presheaf category $\Sigma\mathscr{S}^{\mathscr{E}}$ for a full $\Sigma\mathscr{S}$-subcategory $\mathscr{E}$ of $\Sigma\mathscr{N}$. Dugger proves that one can pull back the $\Sigma\mathscr{S}$-enrichment of $\Sigma\mathscr{N}$ along the two Quillen equivalences to obtain a $\Sigma\mathscr{S}$-model category structure on $\mathscr{M}$ itself. Pulling back $\mathscr{E}$ gives a full $\Sigma\mathscr{S}$-subcategory $\mathscr{D}$ of $\mathscr{M}$ such that $\mathscr{M}$ is Quillen equivalent to $\Sigma\mathscr{S}^{\mathscr{D}}$. In a sequel to [31], Schwede and Shipley [32] show that the conclusion can be transported along changes of $\mathscr{V}$ to any of the other standard modern model categories of spectra.

We are especially interested in explicit identification of the relevant domain categories $\mathscr{D}$, and for that we want to start with a given enrichment on $\mathscr{M}$ itself, not on some enriched category that is Quillen equivalent to $\mathscr{M}$. Philosophically, it seems to us that when one starts with a nice $\mathscr{V}$-enriched model category $\mathscr{M}$, there is little if any gain in switching from $\mathscr{V}$ to $\Sigma\mathscr{S}$ or to any other preconceived choice. In fact, with the switch, it is not obvious how to compare an intrinsic $\mathscr{V}$-category $\mathscr{D}$ living in $\mathscr{M}$ to the associated spectral category living in $\Sigma\mathscr{M}$. When $\mathscr{V}$ is $\Sigma\mathscr{S}$ itself, this point is addressed in [31, A.2.4], and it is addressed more generally in [5, 6].

We shall turn to the study of comparisons of this sort in §5. However, it is sensible to avoid unnecessary comparisons by working with given enrichments whenever possible. In particular, stable model categories $\mathscr{M}$ very often appear in nature as $\mathscr{V}$-enriched in an appropriate stable category $\mathscr{V}$ other than $\Sigma\mathscr{S}$, and we shall work from that starting point. The preference becomes important mathematically when one tries to find simplified models for the relevant full subcategories $\mathscr{D}$ of $\mathscr{M}$.

This perspective allows us to avoid the particular technology of symmetric spectra, which is at the technical heart of [31] and [5]. The price is a loss of generality, since we ignore the problem of how to enrich a given stable model category if it does not happen to come in nature with a suitable enrichment: as our sketch above indicates, that problem is a major focus of [5, 31].

In any context, working stably makes it very much simpler to prove Quillen equivalences. We give a $\mathscr{V}$-analogue of [31, Thm 3.3.3(iii)], after some recollections about triangulated categories that explain how such arguments work in general.

Definition 3.22. Let $\mathscr{A}$ be a triangulated category with coproducts. An object X of $\mathscr{A}$ is compact (or small) if the natural map $\oplus\mathscr{A}(X,Y_i) \longrightarrow \mathscr{A}(X,\amalg Y_i)$ is an isomorphism for every set of objects Y_i. A set $\mathscr{D}$ of objects generates $\mathscr{A}$ if a map $f\colon X \longrightarrow Y$ is an isomorphism if and only if $f_*\colon \mathscr{A}(d,X)_* \longrightarrow \mathscr{A}(d,Y)_*$ is an isomorphism for all $d \in \mathscr{D}$. Here we write $\mathscr{A}(-,-)$ and $\mathscr{A}(-,-)_*$ for the maps and graded maps in $\mathscr{A}$, so that generating sets need not be closed under Σ. We say that $\mathscr{D}$ is compact if each $d \in \mathscr{D}$ is compact.

We emphasize the distinction between generating sets in triangulated categories and the sets of domains (or cofibers) of generating sets of cofibrations in model categories. The former generating sets can be much smaller. For example, in a good model category of spectra, one must use all spheres S^n to obtain a generating set of cofibrations, but a generating set for the homotopy category need only contain $S = S^0$. The difference is much more striking for parametrized spectra [25, 13.1.16].

The following result is due to Neeman [28, 3.2]. Recall that a localizing subcategory of a triangulated category is a triangulated subcategory that is closed under coproducts; it is necessarily also closed under isomorphisms.

Lemma 3.23. *The smallest localizing subcategory of $\mathscr{A}$ that contains a compact generating set $\mathscr{D}$ is $\mathscr{A}$ itself.*

This result is used in tandem with the following one to prove equivalences.

Lemma 3.24. *Let $E, F\colon \mathscr{A} \longrightarrow \mathscr{B}$ be exact and coproduct-preserving functors between triangulated categories and let $\phi\colon E \longrightarrow F$ be a natural transformation that commutes with Σ. Then the full subcategory of $\mathscr{A}$ consisting of those objects X for which ϕ is an isomorphism is localizing.*

When proving adjoint equivalences, the exact and coproduct-preserving hypotheses in the previous result are dealt with using the following observations (see [27, 3.9 and 5.1] and [8, 7.4]). Of course, a left adjoint obviously preserves coproducts.

Lemma 3.25. *Let (L, R) be an adjunction between triangulated categories $\mathscr{A}$ and $\mathscr{B}$. Then L is exact if and only if R is exact. Assume that L is additive and $\mathscr{A}$ has a compact set of generators $\mathscr{D}$. If R preserves coproducts, then L preserves compact objects. Conversely, if $L(d)$ is compact for $d \in \mathscr{D}$, then R preserves coproducts.*

Returning to our model theoretic context, let $\mathscr{D}$ be any small $\mathscr{V}$-category, not necessarily related to any given $\mathscr{M}$. To apply the results above, we need a compact generating set in $\mathrm{Ho}\mathscr{V}^{\mathscr{D}}$, and for that we need a compact generating set in $\mathrm{Ho}\mathscr{V}$. It is usually the case in applications that the unit object $\mathbf{I}$ is itself a compact generating set, but it is harmless to work more generally. We have in mind equivariant applications where this would fail.

Lemma 3.26. *Let $Ho\mathscr{V}$ have a compact generating set $\mathscr{C}$ and define $F\mathscr{C}$ to be the set of objects $F_d c \in Ho\mathscr{V}^{\mathscr{D}}$, where $c \in \mathscr{C}$ and $d \in \mathscr{D}$. Assume either that cofibrant presheaves are levelwise cofibrant or that any coproduct of weak equivalences in $\mathscr{V}$ is a weak equivalence. Then $F\mathscr{C}$ is a compact generating set.*

Proof. Since this is a statement about homotopy categories, we may assume without loss of generality that each $c \in \mathscr{C}$ is cofibrant in $\mathscr{V}$. Since the weak equivalences and fibrations in $\mathscr{V}^{\mathscr{D}}$ are defined levelwise, they are preserved by ev_d and (F_d, ev_d) is a Quillen adjunction. Therefore the adjunction passes to homotopy categories. Since coproducts in $\mathscr{V}^{\mathscr{D}}$ are defined levelwise, they commute with ev_d. Therefore the map

$$\oplus_i \mathrm{Ho}\mathscr{V}^{\mathscr{D}}(F_d c, Y_i) \longrightarrow \mathrm{Ho}\mathscr{V}^{\mathscr{D}}(F_d c, \amalg_i Y_i)$$

can be identified by adjunction with the isomorphism

$$\oplus_i \mathrm{Ho}\mathscr{V}(c, \mathrm{ev}_d Y_i) \longrightarrow \mathrm{Ho}\mathscr{V}(c, \amalg_i \mathrm{ev}_d Y_i),$$

where the Y_i are bifibrant presheaves. The identification of sources is immediate. For the identification of targets, our assumption ensures that the coproduct $\amalg \mathrm{ev}_d Y_i$ in $\mathscr{V}$ represents the derived coproduct $\amalg \mathrm{ev}_d Y_i$ in $\mathrm{Ho}\mathscr{V}$. Since the functors ev_d create the weak equivalences in $\mathscr{V}^{\mathscr{D}}$, it is also clear by adjunction that $F\mathscr{C}$ generates $\mathrm{Ho}\mathscr{V}^{\mathscr{D}}$ since $\mathscr{C}$ generates $\mathrm{Ho}\mathscr{V}$. $\square$

By Proposition 2.7, if $\mathscr{C} = \{\mathbf{I}\}$, then $F\mathscr{C}$ can be identified with $\{\mathbb{D}(d)\}$. Switching context from the previous section by replacing reflecting sets by generating sets, we have the following result. When $\mathscr{V}$ is the category of symmetric spectra, it is Schwede and Shipley's result [31, 3.9.3(iii)]. We emphasize for use in the sequel [10] that our general version can apply even when $\mathbf{I}$ is not cofibrant and $\mathscr{V}$ does not satisfy the monoid axiom. We fix a cofibrant approximation $\mathbb{Q}\mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 3.27. *Let $\mathscr{M}$ be a $\mathscr{V}$-model category, where $\mathscr{V}$ is stable and $Ho\mathscr{V}$ has $\{\mathbf{I}\}$ as a compact generating set. Let $\mathscr{D}$ be a $\mathscr{V}$-good full $\mathscr{V}$-subcategory of $\mathscr{M}$ given by a set of bifibrant objects d that form a compact generating set in $Ho\mathscr{M}$. Assume the following two conditions.*

- *(i)* Either $\mathbf{I}$ *is cofibrant in* $\mathscr{V}$ or *every object of* $\mathscr{M}$ *is fibrant and the induced map* $F_d\mathbb{Q}\mathbf{I} \longrightarrow F_d\mathbf{I}$ *is a weak equivalence for each* $d \in \mathscr{D}$.
- *(ii)* Either *cofibrant presheaves are level cofibrant* or *coproducts of weak equivalences in* $\mathscr{V}$ *are weak equivalences.*

Then $(\mathbb{T}, \mathbb{U})$ *is a Quillen equivalence between* $\mathscr{V}^{\mathscr{D}}$ *and* $\mathscr{M}$.

Proof. In view of what we have already proven, it only remains to show that the derived adjunction $(\mathbb{T}, \mathbb{U})$ on homotopy categories is an adjoint equivalence. The distinguished triangles in Ho$\mathscr{M}$ and Ho$\mathscr{V}^{\mathscr{D}}$ are generated by the cofibrations in the underlying model categories. Since $\mathbb{T}$ preserves cofibrations, its derived functor is exact, and so is the derived functor of $\mathbb{U}$. We claim that Lemma 3.25 applies to show that $\mathbb{U}$ preserves coproducts. By Lemma 3.26 and hypothesis, $\{F_d\mathbf{I}\}$ is a compact set of generators for Ho$\mathscr{V}^{\mathscr{D}}$. To prove the claim, we must show that $\{\mathbb{T}F_d\mathbf{I}\}$ is a compact set of generators for Ho$\mathscr{M}$. It suffices to show that $\mathbb{T}F_d\mathbf{I} \cong d$ in Ho$\mathscr{M}$, and Proposition 3.4 gives that $\mathbb{T}F_d\mathbf{I} \cong d$ in $\mathscr{M}$. If $\mathbf{I}$ is cofibrant, this is an isomorphism between cofibrant objects of $\mathscr{M}$. If not, the unit axiom for the $\mathscr{V}$-model category $\mathscr{M}$ gives that the induced map $d \odot \mathbb{Q}\mathbf{I} \longrightarrow d \odot \mathbf{I} \cong d$ is a weak equivalence for $d \in \mathscr{D}$. Since $\mathbb{T}F_dV \cong d \odot V$ for $V \in \mathscr{V}$, this is a weak equivalence $\mathbb{T}F_dQ\mathbf{I} \longrightarrow \mathbb{T}F_d\mathbf{I}$. Either way, we have the required isomorphism in Ho$\mathscr{M}$.

Now, in view of Lemmas 3.23, 3.24, and 3.26, we need only show that the isomorphisms $\eta\colon F_d\mathbf{I} \longrightarrow \mathbb{U}\mathbb{T}F_d\mathbf{I}$ in $\mathscr{V}^{\mathscr{D}}$ and $\varepsilon\colon \mathbb{T}\mathbb{U}d \longrightarrow d$ in $\mathscr{M}$ given in Proposition 3.4 imply that their derived maps are isomorphisms in the respective homotopy categories Ho$\mathscr{V}^{\mathscr{D}}$ and Ho$\mathscr{M}$. Assume first that $\mathbf{I}$ is cofibrant. Then the former implication is immediate and, since $\mathbb{U}(d) = F_d(\mathbf{I})$ is cofibrant, so is the latter.

Thus assume that $\mathbf{I}$ is not cofibrant. Then to obtain η on the homotopy category Ho$\mathscr{V}^{\mathscr{D}}$, we must replace $\mathbf{I}$ by $Q\mathbf{I}$ before applying the map η in $\mathscr{V}$. By (3.5), when we apply $\eta\colon \mathrm{Id} \longrightarrow \mathbb{U}\mathbb{T}$ to F_dV for $V \in \mathscr{V}$ and evaluate at e, we get a natural map

$$\eta\colon \mathscr{D}(e,d) \otimes V = \underline{\mathscr{M}}(e,d) \otimes V \longrightarrow \underline{\mathscr{M}}(e, d \odot V)$$

that is an isomorphism when $V = \mathbf{I}$. We must show that it is a weak equivalence when $V = \mathbb{Q}\mathbf{I}$. To see this, replace V by the weak equivalence $\mathbb{Q}\mathbf{I} \longrightarrow \mathbf{I}$. On the left, the resulting map is a weak equivalence $\delta(e,d) \otimes \mathbb{Q}\mathbf{I} \longrightarrow \delta(e,d) \otimes \mathbf{I}$ by assumption. On the right, the resulting map $\underline{\mathscr{M}}(e, d \odot \mathbb{Q}\mathbf{I}) \longrightarrow \underline{\mathscr{M}}(e, d \odot \mathbf{I})$ is a weak equivalence by Lemma 4.13 below and our assumption that all objects of $\mathscr{M}$ are fibrant. Therefore η is a weak equivalence when $V = \mathbb{Q}\mathbf{I}$. Similarly, to pass to the homotopy category Ho$\mathscr{M}$, we must replace $\mathbb{U}(d) = F_d(\mathbf{I})$ by a cofibrant approximation before applying ε in $\mathscr{M}$. By assumption, $F_d\mathbb{Q}\mathbf{I} \longrightarrow F_d\mathbf{I}$ is such a cofibrant approximation. Up to isomorphism, $\mathbb{T}$ takes this map to the weak equivalence $d \odot \mathbb{Q}\mathbf{I} \longrightarrow d \odot \mathbf{I} \cong d$, and the conclusion follows. □

Remark 3.28. Since the functor F_d is strong symmetric monoidal, the assumption that $F_d\mathbb{Q}\mathbf{I} \longrightarrow F_d\mathbf{I}$ is a weak equivalence says that (F_d, ev_d) is a monoidal Quillen adjunction in the sense of Definition 5.6 below. The assumption holds by the unit axiom for the $\mathscr{V}$-model category $\mathscr{M}$ if the objects $\mathscr{D}(d,e)$ are cofibrant in $\mathscr{V}$.

Remark 3.29. More generally, if $\mathrm{Ho}\mathscr{V}$ has a compact generating set $\mathscr{C}$, then Theorem 3.27 will hold as stated provided that $\eta\colon F_dc \longrightarrow \mathbb{U}\mathbb{T}F_dc$ is an isomorphism in $\mathrm{Ho}\mathscr{V}^{\mathscr{D}}$ for all $c \in \mathscr{C}$.

Remark 3.30. If we assume further that $\mathscr{D}$ is $(\mathscr{M},\mathscr{V})$-good (Definition 3.9) and its objects form a creating set in $\mathscr{M}$, then the identity functor of $\mathscr{M}$ is a Quillen equivalence from the $\mathscr{D}$-model structure to the given model structure on $\mathscr{M}$, by Proposition 3.19. In practice, the creating set hypothesis never applies when working in a simplicial context, but it can apply when working in topological or homological contexts (at the price of increasing the size of $\mathscr{D}$).

Thus, stably, the crux of the answer to Question 3.21 is to identify appropriate compact generating sets in $\mathscr{M}$. The utility of the answer depends on understanding the associated hom objects, with their composition, in $\mathscr{V}$.

4. Changing the categories $\mathscr{D}$ and $\mathscr{M}$, keeping $\mathscr{V}$ fixed

With our standing assumptions on $\mathscr{V}$ and $\mathscr{M}$ from §1.1 and §1.2, we assume once and for all that all categories in this section and the next satisfy the hypotheses of Theorem 2.12. This ensures that $\mathscr{M}$, $\mathscr{V}^{\mathscr{D}}$, and $\mathscr{M}^{\mathscr{D}}$ are cofibrantly generated $\mathscr{V}$-model categories.

4.1. Changing $\mathscr{D}$.

In applications, we are especially interested in changing a given diagram category $\mathscr{D}$ to a more calculable equivalent. We might also be interested in changing the $\mathscr{V}$-category $\mathscr{M}$ to a Quillen equivalent $\mathscr{V}$-category $\mathscr{N}$, with $\mathscr{D}$ fixed, but the way that change works is evident from our levelwise definitions.

Proposition 4.1. *For a $\mathscr{V}$-functor $\xi\colon \mathscr{M} \longrightarrow \mathscr{N}$ and any small $\mathscr{V}$-category $\mathscr{D}$, there is an induced $\mathscr{V}$-functor $\xi_*\colon \mathscr{M}^{\mathscr{D}} \longrightarrow \mathscr{N}^{\mathscr{D}}$, and it induces an equivalence of homotopy categories if ξ does so. A Quillen adjunction or Quillen equivalence between $\mathscr{M}$ and $\mathscr{N}$ induces a Quillen adjunction or Quillen equivalence between $\mathscr{M}^{\mathscr{D}}$ and $\mathscr{N}^{\mathscr{D}}$.*

We have several easy observations about changing $\mathscr{D}$, with $\mathscr{M}$ fixed. Before returning to model categories, we record a categorical observation. In the rest of this subsection, $\mathscr{M}$ is any $\mathscr{V}$-category, but our main interest is in the case $\mathscr{M} = \mathscr{V}$.

Lemma 4.2. *Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be a $\mathscr{V}$-functor and $\mathscr{M}$ be a $\mathscr{V}$-category. Then there is a $\mathscr{V}$-adjunction $(\nu_!, \nu^*)$ between $\mathscr{M}^{\mathscr{D}}$ and $\mathscr{M}^{\mathscr{E}}$.*

Proof. The $\mathscr{V}$-functor ν^* restricts a presheaf Y on $\mathscr{E}$ to the presheaf $Y \circ \nu$ on $\mathscr{D}$. Its left adjoint $\nu_!$ sends a presheaf X on $\mathscr{D}$ to its left Kan extension, or prolongation, along ν (e.g. [21, 23.1]). Explicitly, $(\nu_!X)_e = X \otimes_{\mathscr{D}} \nu_e$, where $\nu_e\colon \mathscr{D} \longrightarrow \mathscr{V}$ is given on objects by $\nu_e(d) = \mathscr{E}(e, \nu d)$ and on hom objects by the adjoints of the composites

$$\mathscr{D}(d,d') \otimes \mathscr{E}(e,\nu d) \xrightarrow{\nu\otimes\mathrm{id}} \mathscr{E}(\nu d, \nu d') \otimes \mathscr{E}(e,\nu d) \xrightarrow{\circ} \mathscr{E}(e, \nu d').$$

□

Definition 4.3. Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be a $\mathscr{V}$-functor and let $\mathscr{M}$ be a $\mathscr{V}$-model category.

(i) ν is weakly full and faithful if each $\nu\colon \mathscr{D}(d,d') \longrightarrow \mathscr{E}(\nu d, \nu d')$ is a weak equivalence in $\mathscr{V}$.
(ii) ν is essentially surjective if each object $e \in \mathscr{E}$ is isomorphic (in the underlying category of $\mathscr{E}$) to an object νd for some $d \in \mathscr{D}$
(iii) ν is a weak equivalence if it is weakly full and faithful and essentially surjective.
(iv) ν is an $\mathscr{M}$-weak equivalence if $\mathscr{D}$ and $\mathscr{E}$ are $\mathscr{M}$-good (Definition 2.11),

$$\nu \odot \mathrm{id}\colon \mathscr{D}(d,d') \odot M \longrightarrow \mathscr{E}(\nu d, \nu d') \odot M$$

is a weak equivalence in $\mathscr{M}$ for all cofibrant $M \in \mathscr{M}$, and ν is essentially surjective.

Proposition 4.4. *Let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be a $\mathscr{V}$-functor and let $\mathscr{M}$ be a $\mathscr{V}$-model category. If ν is essentially surjective, then $(\nu_!, \nu^*)$ is a Quillen adjunction, and it is a Quillen equivalence if ν is an $\mathscr{M}$-weak equivalence.*

Proof. If ν is essentially surjective, then easy diagram chases show that ν^* creates the level fibrations and weak equivalences of $\mathscr{M}^{\mathscr{E}}$, so that we have a Quillen $\mathscr{V}$-adjunction. Clearly $\nu_! F_d$ is the left adjoint $F_{\nu d}$ of $ev_d \circ \nu^*$, and $\eta\colon X \longrightarrow \nu^*\nu_! X$ is given on objects $X = F_d M$ by maps of the form that we require to be weak equivalences when ν is an $\mathscr{M}$-weak equivalence. The functor ν^* preserves colimits, since these are defined levelwise, and the relevant colimits (those used to construct cell objects) preserve weak equivalences. Thus η is a weak equivalence when X is cofibrant and ν is an $\mathscr{M}$-weak equivalence. This implies that $(\nu_!, \nu^*)$ is a Quillen equivalence (see [12, 1.3.16] or [21, A.2]). □

Remark 4.5. Let $\mathscr{D} \subset \mathscr{E}$ be sets of bifibrant objects in $\mathscr{M}$ and let $\nu\colon \mathscr{D} \longrightarrow \mathscr{E}$ be the corresponding inclusion of full $\mathscr{V}$-subcategories of $\mathscr{M}$. If $\mathscr{D}$ is a reflecting or creating set of objects in the sense of Definition 3.16 or if $\mathscr{D}$ is a generating set in the sense of Definition 3.22, then so is $\mathscr{E}$. Therefore, if Theorem 3.18 or Theorem 3.27 applies to prove that $\mathbb{U}\colon \mathscr{M} \longrightarrow \mathscr{V}^{\mathscr{D}}$ is a right Quillen equivalence, then it also applies to prove that $\mathbb{U}\colon \mathscr{M} \longrightarrow \mathscr{V}^{\mathscr{E}}$ is a right Quillen equivalence. Since $\nu^*\mathbb{U} = \mathbb{U}$, this implies that $\nu^*\colon \mathscr{V}^{\mathscr{E}} \longrightarrow \mathscr{V}^{\mathscr{D}}$ is a Quillen equivalence. In this context, the "essentially surjective" hypothesis in Proposition 4.4 generally fails.

4.2. **Quasi-equivalences and changes of $\mathscr{D}$.** Here we describe a Morita type criterion for when two $\mathscr{V}$-categories $\mathscr{D}$ and $\mathscr{E}$ are connected by a zigzag of weak equivalences. This generalizes work along the same lines of Keller [16], Schwede and Shipley [31], and Dugger [5], which deal with particular enriching categories, and we make no claim to originality. It can be used in tandem with Proposition 4.4 to obtain zigzags of weak equivalences between categories of presheaves.

Recall that we have the $\mathscr{V}$-product $\mathscr{D}^{\mathrm{op}} \otimes \mathscr{E}$ between the $\mathscr{V}$-categories $\mathscr{D}^{\mathrm{op}}$ and $\mathscr{E}$. The objects of $\mathscr{V}^{\mathscr{D}^{\mathrm{op}} \otimes \mathscr{E}}$ are often called "distributors" in the categorical literature, but we follow [31] and call them $(\mathscr{D}, \mathscr{E})$-bimodules. Thus a $(\mathscr{D}, \mathscr{E})$-bimodule $\mathscr{F}$ is a contravariant $\mathscr{V}$-functor $\mathscr{D}^{\mathrm{op}} \otimes \mathscr{E} \longrightarrow \mathscr{V}$. It is convenient to write the action of $\mathscr{D}$ on the left (since it is covariant) and the action of $\mathscr{E}$ on the right. We write $\mathscr{F}(d, e)$ for the object in $\mathscr{V}$ that $\mathscr{F}$ assigns to the object (d, e). The definition encodes three associativity diagrams

$$\begin{array}{ccc}
\mathscr{D}(e,f)\otimes\mathscr{D}(d,e)\otimes\mathscr{F}(c,d) & \longrightarrow & \mathscr{D}(d,f)\otimes\mathscr{F}(c,d)\\
\downarrow & & \downarrow\\
\mathscr{D}(e,f)\otimes\mathscr{F}(c,e) & \longrightarrow & \mathscr{F}(c,f)
\end{array}$$

$$\begin{array}{ccc}
\mathscr{D}(e,f)\otimes\mathscr{F}(d,e)\otimes\mathscr{E}(c,d) & \longrightarrow & \mathscr{F}(d,f)\otimes\mathscr{E}(c,d)\\
\downarrow & & \downarrow\\
\mathscr{D}(e,f)\otimes\mathscr{F}(c,e) & \longrightarrow & \mathscr{F}(c,f)
\end{array}$$

$$\begin{array}{ccc}
\mathscr{F}(e,f)\otimes\mathscr{E}(d,e)\otimes\mathscr{E}(c,d) & \longrightarrow & \mathscr{F}(d,f)\otimes\mathscr{E}(c,d)\\
\downarrow & & \downarrow\\
\mathscr{F}(e,f)\otimes\mathscr{E}(c,e) & \longrightarrow & \mathscr{F}(c,f)
\end{array}$$

and two unit diagrams

$$\begin{array}{ccc}
\mathbf{I}\otimes\mathscr{F}(c,d) & \longrightarrow & \mathscr{D}(d,d)\otimes\mathscr{F}(c,d)\\
& \searrow^{\cong} & \downarrow\\
& & \mathscr{F}(c,d)
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{F}(d,e)\otimes\mathbf{I} & \longrightarrow & \mathscr{F}(d,e)\otimes\mathscr{E}(d,d)\\
& \searrow^{\cong} & \downarrow\\
& & \mathscr{F}(d,e).
\end{array}$$

The following definition and proposition are adapted from work of Schwede and Shipley [31]; see also [5]. They encode and exploit two further unit conditions.

Definition 4.6. Let $\mathscr{D}$ and $\mathscr{E}$ have the same sets of objects, denoted $\mathbb{O}$. Define a quasi-equivalence between $\mathscr{D}$ and $\mathscr{E}$ to be a $(\mathscr{D},\mathscr{E})$-bimodule $\mathscr{F}$ together with a map $\zeta_d\colon \mathbf{I}\longrightarrow\mathscr{F}(d,d)$ for each $d\in\mathbb{O}$ such that for all pairs $(d,e)\in\mathbb{O}$, the maps

$$(4.7)\qquad (\zeta_d)^*\colon \mathscr{D}(d,e)\longrightarrow\mathscr{F}(d,e) \quad\text{and}\quad \mathscr{F}(d,e)\longleftarrow\mathscr{E}(d,e)\colon(\zeta_e)_*$$

in $\mathscr{V}$ given by composition with ζ_d and ζ_e are weak equivalences. Given $\mathscr{F}$ and the maps ζ_d, define a new $\mathscr{V}$-category $\mathscr{G}(\mathscr{F},\zeta)$ with object set $\mathbb{O}$ by letting $\mathscr{G}(\mathscr{F},\zeta)(d,e)$ be the pullback in $\mathscr{V}$ displayed in the diagram

$$(4.8)\qquad \begin{array}{ccc}
\mathscr{G}(\mathscr{F},\zeta)(d,e) & \longrightarrow & \mathscr{E}(d,e)\\
\downarrow & & \downarrow{\scriptstyle (\zeta_e)_*}\\
\mathscr{D}(d,e) & \underset{(\zeta_d)^*}{\longrightarrow} & \mathscr{F}(d,e)
\end{array}$$

Its units and composition are induced from those of $\mathscr{D}$ and $\mathscr{E}$ and the bimodule structure on $\mathscr{F}$ by use of the universal property of pullbacks. The unlabelled arrows specify $\mathscr{V}$-functors

$$(4.9)\qquad \mathscr{D}\longleftarrow\mathscr{G}(\mathscr{F},\zeta) \quad\text{and}\quad \mathscr{G}(\mathscr{F},\zeta)\longrightarrow\mathscr{E}.$$

Proposition 4.10. *Assume that the unit $\mathbf{I}$ is cofibrant in $\mathscr{V}$. If $\mathscr{D}$ and $\mathscr{E}$ are quasi-equivalent, then there is a chain of weak equivalences connecting $\mathscr{D}$ and $\mathscr{E}$.*

Proof. Choose a quasi-equivalence $(\mathscr{F},\zeta)$. If either all $(\zeta_d)^*$ or all $(\zeta_e)_*$ are acyclic fibrations, then all four arrows in (4.8) are weak equivalences and (4.9) displays a zigzag of weak equivalences between $\mathscr{D}$ and $\mathscr{E}$. We shall reduce the general case to two applications of this special case. Observe that by taking a fibrant replacement in the category $\mathscr{V}^{\mathscr{D}^{\mathrm{op}}\otimes\mathscr{E}}$, we may assume without loss of generality that our given $(\mathscr{D},\mathscr{E})$-bimodule $\mathscr{F}$ is fibrant, so that each $\mathscr{F}(d,e)$ is fibrant in $\mathscr{V}$.

For fixed e, the adjoint of the right action of $\mathscr{E}$ on $\mathscr{F}$ gives maps

$$\mathscr{E}(d,d') \longrightarrow \underline{\mathscr{V}}(\mathscr{F}(d',e),\mathscr{F}(d,e))$$

that allow us to view the functor $\mathbb{F}(e)_d = \mathscr{F}(d,e)$ as an object of $\mathscr{V}^{\mathscr{E}}$; it is fibrant since each $\mathscr{F}(d,e)$ is fibrant in $\mathscr{V}$. Fixing e and letting d vary, the maps $(\zeta_e)_*$ of (4.7) specify a map $\mathbb{E}(e) \longrightarrow \mathbb{F}(e)$ in $\mathscr{V}^{\mathscr{E}}$. By hypothesis, this map is a level weak equivalence, and it is thus a weak equivalence in $\mathscr{V}^{\mathscr{E}}$. Factor it as the composite of an acyclic cofibration $\iota(e)\colon \mathbb{E}(e) \longrightarrow X(e)$ and a fibration $\rho(e)\colon X(e) \longrightarrow \mathbb{F}(e)$. Then $\rho(e)$ is acyclic by the two out of three property. By Remark 2.13, our assumption that $\mathbf{I}$ is cofibrant implies that $\mathbb{E}(e)$ and therefore $X(e)$ is cofibrant in $\mathscr{V}^{\mathscr{E}}$, and $X(e)$ is fibrant since $\mathbb{F}(e)$ is fibrant. Let $\mathscr{E}nd(X)$ denote the full subcategory of $\mathscr{V}^{\mathscr{E}}$ whose objects are the bifibrant presheaves $X(e)$.

Now use (2.1) to define

$$\underline{\mathscr{Y}}(d,e) = \underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{E}(d),X(e)) \cong X(e)_d \quad \text{and} \quad \underline{\mathscr{Z}}(d,e) = \underline{\mathscr{V}}^{\mathscr{E}}(X(d),\mathbb{F}(e)),$$

where the isomorphism is given by the enriched Yoneda lemma. Composition in $\underline{\mathscr{V}}^{\mathscr{E}}$ gives a left action of $\mathscr{E}nd(X)$ on $\mathscr{Y}$ and a right action of $\mathscr{E}nd(X)$ on $\mathscr{Z}$. Evaluation $\underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{E}(d),X(e))\otimes\mathbb{E}(d) \longrightarrow X(e)$ gives a right action of $\mathscr{E}$ on $\mathscr{Y}$. The action of $\mathscr{D}$ on $\mathscr{F}$ gives maps $\mathscr{D}(e,f) \longrightarrow \underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{F}(e),\mathbb{F}(f))$ and these together with composition in $\underline{\mathscr{V}}^{\mathscr{E}}$ give a left action of $\mathscr{D}$ on $\mathscr{Z}$. These actions make $\mathscr{Y}$ an $(\mathscr{E}nd(X),\mathscr{E})$-bimodule and $\mathscr{Z}$ a $(\mathscr{D},\mathscr{E}nd(X))$-bimodule. We may view the weak equivalences $\iota(e)$ as maps $\iota_e\colon \mathbf{I} \longrightarrow \underline{\mathscr{Y}}(e,e)$ and the weak equivalences $\rho(e)$ as maps $\rho_e\colon \mathbf{I} \longrightarrow \underline{\mathscr{Z}}(e,e)$. We claim that $(\mathscr{Y},\iota)$ and $(\mathscr{Z},\rho)$ are quasi-equivalences to which the acyclic fibration special case applies to give a zigzag of weak equivalences

$$\mathscr{E} \longleftarrow \mathscr{G}(\mathscr{Y},\iota) \longrightarrow \mathscr{E}nd(X) \longleftarrow \mathscr{G}(\mathscr{Z},\rho) \longrightarrow \mathscr{D}. \tag{4.11}$$

The maps

$$(\iota)_*\colon \mathbb{E}(e)_d = \mathscr{E}(d,e) \longrightarrow \mathscr{Y}(d,e) = \underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{E}(d),X(e)) \cong X(e)_d$$

are the weak equivalences $\iota\colon \mathbb{E}(e)_d \longrightarrow X(e)_d$. The maps

$$(\iota_d)^*\colon \underline{\mathscr{V}}^{\mathscr{E}}(X(d),X(e)) \longrightarrow \underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{E}(d),X(e))$$

are acyclic fibrations since ι_d is an acyclic cofibration and $X(e)$ is fibrant. This gives the first two weak equivalences in the zigzag (4.11). The maps

$$(\rho_d)^*\colon \mathbb{D}(e)_d = \mathscr{D}(d,e) \longrightarrow \mathscr{Z}(d,e) = \underline{\mathscr{V}}^{\mathscr{E}}(X(d),\mathbb{F}(e))$$

are weak equivalences since their composites with the maps

$$(\iota_d)^*\colon \underline{\mathscr{V}}^{\mathscr{E}}(X(d),\mathbb{F}(e)) \longrightarrow \underline{\mathscr{V}}^{\mathscr{E}}(\mathbb{E}(d),\mathbb{F}(e)) \cong \mathbb{F}(e)_d$$

are the original weak equivalences $(\zeta_d)^*$. The maps

$$(\rho_e)_*\colon \underline{\mathscr{V}}^{\mathscr{E}}(X(d),X(e)) \longrightarrow \underline{\mathscr{V}}^{\mathscr{E}}(X(d),\mathbb{F}(e))$$

are acyclic fibrations since ρ_e is an acyclic fibration and $X(d)$ is cofibrant. This gives the second two weak equivalences in the zigzag (4.11). ∎

Remark 4.12. The assumption that $\mathbf{I}$ is cofibrant is only used to ensure that the represented presheaves $\mathbb{E}(e)$ are cofibrant. If we know that in some other way, then we need not assume that $\mathbf{I}$ is cofibrant.

4.3. **Changing full subcategories $\mathscr{D}$ of Quillen equivalent categories $\mathscr{M}$.** In the following three results, we do *not* assume that the unit $\mathbf{I}$ of $\mathscr{V}$ is cofibrant. We show how to obtain quasi-equivalences between full subcategories of Quillen equivalent $\mathscr{V}$-model categories $\mathscr{M}$ and $\mathscr{N}$. When $\mathbf{I}$ is cofibrant, Proposition 4.10 applies to obtain weak equivalences to which Proposition 4.4 can be applied. As explained in Remark 4.16 below, in favorable circumstances these results can be used in tandem with Propositions 4.10 and 4.4 even when $\mathbf{I}$ is not cofibrant. This will be helpful in the sequel [10].

We will use the following standard invariance lemma to deduce an invariance statement that applies to full subcategories of $\mathscr{V}$-model categories $\mathscr{M}$.

Lemma 4.13. *Let $\mathscr{M}$ be a $\mathscr{V}$-model category, let M and M' be cofibrant objects of $\mathscr{M}$, and let N and N' be fibrant objects of $\mathscr{M}$. If $\eta\colon M \longrightarrow M'$ and $\zeta\colon N \longrightarrow N'$ are weak equivalences in $\mathscr{M}$, then the induced maps*

$$\eta^*\colon \underline{\mathscr{M}}(M', N) \longrightarrow \underline{\mathscr{M}}(M, N) \quad \text{and} \quad \underline{\mathscr{M}}(M, N') \longleftarrow \underline{\mathscr{M}}(M, N)\colon \zeta_*$$

are weak equivalences in $\mathscr{V}$.

Proof. We prove the result for ζ_*. The proof for η^* is dual. Consider the functor $\underline{\mathscr{M}}(M, -)$ from $\mathscr{M}$ to $\mathscr{V}$. By Ken Brown's lemma [12, 1.1.12] and our assumption that N and N' are fibrant, it suffices to prove that ζ_* is a weak equivalence when ζ is an acyclic fibration. If $V \longrightarrow W$ is a cofibration in $\mathscr{V}$, then $M \odot V \longrightarrow M \odot W$ is a cofibration in $\mathscr{M}$ since M is cofibrant and $\mathscr{M}$ is a $\mathscr{V}$-model category. Therefore the adjunction (1.8) that defines $\odot$ implies that if ζ is an acyclic fibration in $\mathscr{M}$, then ζ_* is an acyclic fibration in $\mathscr{V}$ and thus a weak equivalence in $\mathscr{V}$. □

Corollary 4.14. *Let $(\mathbb{T}, \mathbb{U})$ be a Quillen $\mathscr{V}$-equivalence between $\mathscr{V}$-model categories $\mathscr{M}$ and $\mathscr{N}$. Let $\{M_d\}$ be a set of bifibrant objects of $\mathscr{M}$ and $\{N_d\}$ be a set of bifibrant objects of $\mathscr{N}$ with the same indexing set $\mathbb{O}$. Suppose given weak equivalences $\zeta_d\colon \mathbb{T}M_d \longrightarrow N_d$ for all d. Let $\mathscr{D}$ and $\mathscr{E}$ be the full subcategories of $\mathscr{M}$ and $\mathscr{N}$ with objects $\{M_d\}$ and $\{N_d\}$. Then the $\mathscr{V}$-categories $\mathscr{D}$ and $\mathscr{E}$ are quasi-equivalent.*

Proof. Define

$$\mathscr{F}(d, e) = \underline{\mathscr{N}}(\mathbb{T}M_d, N_e) \cong \underline{\mathscr{M}}(M_d, \mathbb{U}N_e).$$

Composition in $\underline{\mathscr{N}}$ and $\underline{\mathscr{M}}$ gives $\mathscr{F}$ an $(\mathscr{E}, \mathscr{D})$-bimodule structure. The given weak equivalences ζ_d are maps $\zeta_d\colon \mathbf{I} \longrightarrow \mathscr{F}(d, d)$, and we also write ζ_d for the adjoint weak equivalences $M_d \longrightarrow \mathbb{U}N_d$. By Lemma 4.13, the maps

$$(\zeta_d)^*\colon \underline{\mathscr{N}}(N_d, N_e) \longrightarrow \underline{\mathscr{N}}(\mathbb{T}M_d, N_e) \quad \text{and} \quad \underline{\mathscr{M}}(M_d, \mathbb{U}N_e) \longleftarrow \underline{\mathscr{M}}(M_d, M_e)\colon (\zeta_e)_*$$

are weak equivalences since the sources are cofibrant and the targets are fibrant. □

The case $\mathscr{M} = \mathscr{N}$ is of particular interest.

Corollary 4.15. *If $\{M_d\}$ and $\{N_d\}$ are two sets of bifibrant objects of $\mathscr{M}$ such that M_d is weakly equivalent to N_d for each d, then the full $\mathscr{V}$-subcategories of $\mathscr{M}$ with object sets $\{M_d\}$ and $\{N_d\}$ are quasi-equivalent.*

Remark 4.16. While Proposition 4.10 requires $\mathbf{I}$ to be cofibrant, that result is independent of anything about enriched model categories $\mathscr{M}$. In stable homotopy theory, we encounter model categories $\mathscr{V}$ and $\mathscr{V}_+$ with the same underlying symmetric monoidal category and the same weak equivalences such that the identity functor $\mathscr{V}_+ \longrightarrow \mathscr{V}$ is a left Quillen equivalence. The unit object $\mathbf{I}$ is cofibrant in $\mathscr{V}$ but not in $\mathscr{V}_+$. We sometimes encounter interesting $\mathscr{V}$-enriched categories $\mathscr{M}$ that are $\mathscr{V}_+$-model categories but that are *not* $\mathscr{V}$-model categories. Since the weak equivalences in $\mathscr{V}$ and $\mathscr{V}_+$ are the same, we can apply the previous three results with $\mathscr{V}$ replaced by $\mathscr{V}_+$ to obtain quasi-equivalences to which Proposition 4.10 applies. Then Proposition 4.4 applies to give Quillen equivalences between corresponding categories of presheaves.

5. Changing the categories $\mathscr{V}$, $\mathscr{D}$, and $\mathscr{M}$

5.1. **Changing the enriching category $\mathscr{V}$.** Let us return to Baez's joke and compare simplicial and topological enrichments, among other things. We work categorically until otherwise specified, ignoring model categorical structure. We also ignore presheaf categories in this subsection. The category theorists were here first, and the following result is due to Eilenberg and Kelly [9, 6.3]. Recall that a functor $\mathbb{T}\colon \mathscr{V} \longrightarrow \mathscr{W}$ between symmetric monoidal categories is lax symmetric monoidal if we have a map $\nu\colon \mathbf{I}_{\mathscr{W}} \longrightarrow \mathbb{T}\mathbf{I}_{\mathscr{V}}$ and a natural map

$$\omega\colon \mathbb{T}V \otimes \mathbb{T}V' \longrightarrow \mathbb{T}(V \otimes V')$$

that are compatible with the coherence data (unit, associativity, and symmetry isomorphisms); $\mathbb{T}$ is op-lax monoidal if the arrows point the other way, and $\mathbb{T}$ is strong symmetric monoidal if ν is an isomorphism and ω is a natural isomorphism. Suppose that $\mathbb{T}$ has a right adjoint $\mathbb{U}$. If $\mathbb{U}$ is lax symmetric monoidal, then $\mathbb{T}$ is op-lax symmetric monoidal via the adjoint of $\mathbf{I}_{\mathscr{V}} \longrightarrow \mathbb{U}\mathbf{I}_{\mathscr{W}}$ and the adjoint of the natural composite

$$V \otimes V' \longrightarrow \mathbb{U}\mathbb{T}V \otimes \mathbb{U}\mathbb{T}V' \longrightarrow \mathbb{U}(\mathbb{T}V \otimes \mathbb{T}V').$$

The dual also holds. It follows that if $\mathbb{T}$ is strong symmetric monoidal, then $\mathbb{U}$ is lax symmetric monoidal.

Proposition 5.1. *Let*

$$\mathscr{V} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}$$

be an adjoint pair between symmetric monoidal categories and let $\mathscr{M}$ be a bicomplete $\mathscr{W}$-category. Assume that $\mathbb{U}$ is lax symmetric monoidal and that the adjoint $\mathbb{T}\mathbf{I}_{\mathscr{V}} \longrightarrow \mathbf{I}_{\mathscr{W}}$ of the unit comparison map $\mathbf{I}_{\mathscr{V}} \longrightarrow \mathbb{U}\mathbf{I}_{\mathscr{W}}$ is an isomorphism. Then $\mathscr{M}$ is a $\mathscr{V}$-category with

$$\underline{\mathscr{M}}_{\mathscr{V}}(M,N) = \mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,N).$$

If, further, $\mathbb{T}$ is strong and not just op-lax symmetric monoidal, then $\mathscr{M}$ is a bicomplete $\mathscr{V}$-category[2] *with*

$$M \odot V = M \odot \mathbb{T}(V) \quad \text{and} \quad \Phi(V,M) = \Phi(\mathbb{T}V, M).$$

[2]If the functor $\mathscr{V}(\mathbf{I}_{\mathscr{V}}, -)\colon \mathscr{V} \longrightarrow \mathrm{Set}$ is conservative (reflects isomorphisms), as holds for example when $\mathscr{V} = \mathrm{Mod}_k$, then $\mathscr{M}$ becomes a bicomplete $\mathscr{V}$-category without the assumption that $\mathbb{T}$ is strong symmetric monoidal.

Proof. Using the product comparison map

$$\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,N)\otimes\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(L,M)\longrightarrow\mathbb{U}(\underline{\mathscr{M}}_{\mathscr{W}}(M,N)\otimes\underline{\mathscr{M}}_{\mathscr{W}}(L,M)),$$

we see that the composition functors for $\underline{\mathscr{M}}_{\mathscr{W}}$ induce composition functors for $\underline{\mathscr{M}}_{\mathscr{V}}$; similarly, the composites of the unit comparison map and the unit maps $\mathbf{I}_{\mathscr{W}}\longrightarrow\underline{\mathscr{M}}_{\mathscr{W}}(M,M)$ in $\mathscr{W}$ induce unit maps $\mathbf{I}_{\mathscr{V}}\longrightarrow\mathscr{M}_{\mathscr{V}}(M,M)$ in $\mathscr{V}$. This much makes sense even without the adjoint $\mathbb{T}$ and would apply equally well with the roles of $\mathbb{U}$ and $\mathbb{T}$ reversed. However, when we say that $\mathscr{M}$ is a $\mathscr{V}$-category, we mean that the new underlying category $\mathscr{M}_{\mathscr{V}}$, for which

$$\mathscr{M}_{\mathscr{V}}(M,N)=\mathscr{V}(\mathbf{I}_{\mathscr{V}},\underline{\mathscr{M}}_{\mathscr{V}}(M,N)),$$

must be naturally isomorphic to the originally given underlying category $\mathscr{M}=\mathscr{M}_{\mathscr{W}}$, for which

$$\mathscr{M}_{\mathscr{W}}(M,N)=\mathscr{W}(\mathbf{I}_{\mathscr{W}},\underline{\mathscr{M}}_{\mathscr{W}}(M,N)).$$

The given adjunction specializes to give

$$\mathscr{W}(\mathbb{T}\mathbf{I}_{\mathscr{V}},\underline{\mathscr{M}}_{\mathscr{W}}(M,N))\cong\mathscr{V}(\mathbf{I}_{\mathscr{V}},\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,N))=\mathscr{V}(\mathbf{I}_{\mathscr{V}},\underline{\mathscr{M}}_{\mathscr{V}}(M,N)),$$

and our assumed isomorphism $\mathbb{T}\mathbf{I}_{\mathscr{V}}\longrightarrow\mathbf{I}_{\mathscr{W}}$ induces the required isomorphism.

Suppose now that $\mathbb{T}$ is strong symmetric monoidal. For each $V\in\mathscr{V}$ and $W\in\mathscr{W}$, a Yoneda argument provides an isomorphism

$$\underline{\mathscr{V}}(V,\mathbb{U}W)\cong\mathbb{U}\underline{\mathscr{W}}(\mathbb{T}V,W)$$

that makes the pair of functors $(\mathbb{T},\mathbb{U})$ into a $\mathscr{V}$-adjoint pair (1.7). In particular, this gives an isomorphism

$$\underline{\mathscr{V}}(V,\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,N))\cong\mathbb{U}\underline{\mathscr{W}}(\mathbb{T}V,\underline{\mathscr{M}}_{\mathscr{W}}(M,N)).$$

By the adjunctions that define $\mathscr{W}$-tensors and $\mathscr{W}$-cotensors in $\mathscr{M}$, this gives natural isomorphisms

$$\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M\odot\mathbb{T}V,N)\cong\underline{\mathscr{V}}(V,\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,N))\cong\mathbb{U}\underline{\mathscr{M}}_{\mathscr{W}}(M,\Phi(\mathbb{T}V,Y))$$

which imply the claimed identification of $\mathscr{V}$-tensors and $\mathscr{V}$-cotensors in $\mathscr{M}$. □

Example 5.2. Recall from Remark 2.14 that we have a strong monoidal functor $\mathbf{I}_{\mathscr{V}}[-]\colon \mathrm{Set}\to\mathscr{V}$. It is left adjoint to $\mathscr{V}(\mathbf{I}_{\mathscr{V}},-)\colon\mathscr{V}\to\mathrm{Set}$. The change of enrichment given by Proposition 5.1 produces the underlying category of a $\mathscr{V}$-category.

The following example says that simplicial enrichment is more general than topological enrichment. Therefore, if one insists on uniformity, simplicial enrichment is to be preferred.

Example 5.3. Consider the adjunction

$$sSet \underset{S}{\overset{T}{\rightleftarrows}} \mathscr{U},$$

where T and S are the geometric realization and total singular complex functors. Then T and S are both strong symmetric monoidal, and we conclude from Proposition 5.1 that any category enriched and bitensored over $\mathscr{U}$ is canonically enriched and bitensored over $sSet$.

Remark 5.4. In (1.7), we considered enriched adjunctions between categories both enriched over a fixed $\mathscr{V}$. One can ask what it should mean for an adjunction

$$\mathscr{V} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}$$

to be enriched. A reasonable answer is that there should be unit and counit maps

$$\underline{\mathscr{V}}(V,V') \longrightarrow \underline{\mathscr{V}}(\mathbb{U}\mathbb{T}V, \mathbb{U}\mathbb{T}V') \text{ and } \underline{\mathscr{W}}(\mathbb{T}\mathbb{U}W, \mathbb{T}\mathbb{U}W') \longrightarrow \underline{\mathscr{W}}(W,W')$$

in $\mathscr{V}$ and $\mathscr{W}$, respectively. This fails for (T,S) since the function

$$\mathscr{U}(TSX, TSY) \longrightarrow \mathscr{U}(X,Y)$$

induced by the counit is not continuous.

Proposition 5.1 is relevant to many contexts in which we use two related enrichments simultaneously. Such double enrichment is intrinsic to equivariant theory, as we explain in §5.1 below, and to the relationship between spectra and spaces.

Example 5.5. Let $\mathscr{T}$ be the closed symmetric monoidal category of nondegenerately based spaces in $\mathscr{U}$ and let $\mathscr{S}$ be some good closed symmetric monoidal category of spectra, such as the categories of symmetric or orthogonal spectra. While interpretations vary with the choice of $\mathscr{S}$, we always have a zeroth space (or zeroth simplicial set) functor, which we denote by ev_0. It has a left adjoint, which we denote by F_0. We might also write $F_0 = \Sigma^\infty$ and $\mathrm{ev}_0 = \Omega^\infty$, but homotopical understanding requires fibrant and/or cofibrant approximation, depending on the choice of $\mathscr{S}$.[3] We assume that F_0 is strong symmetric monoidal, as holds for symmetric and orthogonal spectra [21, 1.8]. By Proposition 5.1, $\mathscr{S}$ is then enriched over $\mathscr{T}$ as well as over itself. The based space $\mathscr{S}(X,Y)$ of maps $X \longrightarrow Y$ is

$$\mathscr{S}(X,Y) = \mathrm{ev}_0(\underline{\mathscr{S}}(X,Y)).$$

Returning to model category theory, suppose that we are in the situation of Proposition 5.1 and that $\mathscr{V}$ and $\mathscr{W}$ are monoidal model categories and $\mathscr{M}$ is a $\mathscr{W}$-model category. It is natural to ask under what conditions on the adjunction $(\mathbb{T},\mathbb{U})$ the resulting $\mathscr{V}$-category $\mathscr{M}_{\mathscr{V}}$ becomes a $\mathscr{V}$-model category. Recall the following definition from [12, 4.2.16].

Definition 5.6. A monoidal Quillen adjunction $(\mathbb{T},\mathbb{U})$ between symmetric monoidal model categies is a Quillen adjunction in which the left adjoint $\mathbb{T}$ is strong symmetric monoidal and the map $\mathbb{T}(Q\mathbf{I}_{\mathscr{V}}) \to \mathbb{T}(\mathbf{I}_{\mathscr{V}})$ is a weak equivalence.

The following result is essentially the same as [5, A.5] (except that the compatibility of $\mathbb{T}$ with a cofibrant replacement of $\mathbf{I}_V$ is not mentioned there).

Proposition 5.7. *Let* $\mathscr{V} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}$ *be a monoidal Quillen adjunction between symmetric monoidal model categories. Suppose that* $\mathscr{M}$ *is a* $\mathscr{W}$*-model category. Then the enrichment of* $\mathscr{M}$ *in* $\mathscr{V}$ *of Proposition 5.1 makes* $\mathscr{M}$ *into a* $\mathscr{V}$*-model category.*

Corollary 5.8. *Any topological model category has a canonical structure of a simplicial model category.*

[3] Care is needed to avoid Lewis's pitfall [18], which was discussed recently in [23, §11]. An exhaustive study of how to achieve homotopically meaningful adjunctions between $\mathscr{T}$ and various choices of $\mathscr{S}$ is given in Lind [19].

5.2. **The model category $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$.** As a preliminary to change results for $\mathscr{V}$ and $\mathscr{D}$, we need a model category of domain $\mathscr{V}$-categories for categories of presheaves in $\mathscr{V}$. In this subsection and the next, all domain $\mathscr{V}$-categories $\mathscr{D}$ have the same set of objects $\mathbb{O} = \{d\}$. Let $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ be the category of $\mathscr{V}$-categories with object set $\mathbb{O}$ and $\mathscr{V}$-functors that are the identity on objects. The following result is [31, 6.3], and we just sketch the proof. Recall our standing hypothesis that $\mathscr{V}$ is a cofibrantly generated monoidal model category (§1.1 and Definition 1.17). To avoid yet another "good" definition, we assume in addition that $\mathscr{V}$ satisfies the monoid axiom; as in Remark 2.14, less stringent hypotheses suffice.

Theorem 5.9. *The category $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ is a cofibrantly generated model category in which a map $\alpha\colon \mathscr{D} \longrightarrow \mathscr{E}$ is a weak equivalence or fibration if each $\alpha\colon \mathscr{D}(d,e) \longrightarrow \mathscr{E}(d,e)$ is a weak equivalence or fibration in $\mathscr{V}$; α is a cofibration if it satisfies the LLP with respect to the acyclic fibrations. If α is a cofibration and either $\mathbf{I}$ or each $\mathscr{D}(d,e)$ is cofibrant in $\mathscr{V}$, then each $\alpha\colon \mathscr{D}(d,e) \longrightarrow \mathscr{E}(d,e)$ is a cofibration.*

Sketch proof. Define the category $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ to be the product of copies of $\mathscr{V}$ indexed on the set $\mathbb{O} \times \mathbb{O}$. Thus an object is a set $\{\mathscr{C}(d,e)\}$ of objects of $\mathscr{V}$. As a product of model categories, $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ is a model category. A map is a weak equivalence, fibration or cofibration if each of its components is so. Say that $\mathscr{C}$ is concentrated at (d,e) if $\mathscr{C}(d',e') = \phi$, the initial object, for $(d',e') \neq (d,e)$. For $V \in \mathscr{V}$, write $V(d,e)$ for the graph concentrated at (d,e) with value V there. The model category $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ is cofibrantly generated. Its generating cofibrations and acyclic cofibrations are the maps $\alpha(d,e)\colon V(d,e) \longrightarrow W(d,e)$ specified by generating cofibrations or generating acyclic cofibrations $V \longrightarrow W$ in $\mathscr{V}$.

The category $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ is monoidal with product denoted $\Box$. The $(d,e)^{th}$ object of $\mathscr{D}\Box\mathscr{E}$ is the coproduct over $c \in \mathbb{O}$ of $\mathscr{E}(c,e) \otimes \mathscr{D}(d,c)$. The unit object is the $\mathscr{V}\mathbb{O}$-graph $\mathbf{I}$ with $\mathbf{I}(d,d) = \mathbf{I}$ and $\mathbf{I}(d,e) = \phi$ if $d \neq e$. The category $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ is the category of monoids in $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$, hence there is a forgetful functor

$$\mathbb{U}\colon \mathscr{V}\mathbb{O}\text{-}\mathscr{C}at \longrightarrow \mathscr{V}\mathbb{O}\text{-}\mathscr{G}raph$$

This functor has a left adjoint $\mathbb{F}$ that constructs the free $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ generated by a $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ $\mathscr{C}$. The construction is analogous to the construction of a tensor algebra. The $\mathscr{V}$-category $\mathbb{F}\mathscr{C}$ is the coproduct of its homogeneous parts $\mathbb{F}_p\mathscr{C}$ of "degree p monomials". Explicitly, $\mathbb{F}_0\mathscr{C} = \mathbf{I}[\mathbb{O}] = \amalg\ \mathbf{I}(d,d)$, $(\mathbb{F}_1\mathscr{C})(d,e) = \mathscr{C}(d,e)$, and, for $p > 1$,

$$(\mathbb{F}_p\mathscr{C})(d,e) = \amalg \mathscr{C}(d_{p-1},e) \otimes \mathscr{C}(d_{p-2},d_{p-1}) \otimes \cdots \otimes \mathscr{C}(d_1,d_2) \otimes \mathscr{C}(d,d_1).$$

The unit map $\mathbf{I} \longrightarrow \mathbb{F}(d,d)$ is given by the identity map $\mathbf{I} \longrightarrow \mathbf{I}(d,d) \subset (\mathbb{F}\mathscr{C})(d,d)$. The composition is given by the evident $\otimes$-juxtaposition maps.

The generating cofibrations and acyclic cofibrations are obtained by applying $\mathbb{F}$ to the generating cofibrations and acyclic cofibrations of $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$. A standard implication of Theorem 1.20 applies to the adjunction $(\mathbb{F},\mathbb{U})$. The assumed applicability of the small object argument to the generating cofibrations and acyclic cofibrations in $\mathscr{V}$ implies its applicability to the generating cofibrations and acyclic cofibrations in $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$, and condition (ii) of Theorem 1.20 is a formal consequence of its analogue for $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$. Thus to prove the model axioms it remains only to verify (i) of Theorem 1.20. The relevent cell complexes are defined using coproducts, pushouts, and sequential colimits in $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$, and the proof reduces to consideration of pushouts.

The rest is essentially the same as in the one object case, which is treated in [30, 6.2], with objects $\mathscr{D}(d,e)$ replacing copies of a monoid in $\mathscr{V}$ in the argument. The proof relies on combinatorial analysis of the relevant pushouts. As noted in the proof of [32, 6.3], there is a slight caveat to account for the fact that [30, 6.2] worked with a symmetric monoidal category, whereas the product $\Box$ on $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ is not symmetric. However, the levelwise definition of the model structure on $\mathscr{V}\mathbb{O}$-$\mathscr{G}raph$ allows use of the symmetry in $\mathscr{V}$ at the relevant place in the proof. $\Box$

5.3. **Categorical changes of $\mathscr{V}$ and $\mathscr{D}$.** We return to the context of §5.1 and suppose given an adjunction

$$\mathscr{V} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}. \tag{5.10}$$

We assume that $\mathbb{T}$ is strong symmetric monoidal and therefore $\mathbb{U}$ is lax symmetric monoidal. We consider changes of presheaf categories in this context, working categorically in this subsection and model categorically in the next.

We need some elementary formal structure that relates categories of presheaves whose domain $\mathscr{V}$-categories or $\mathscr{W}$-categories have a common fixed object set $\mathbb{O}$. To see that the formal structure really is elementary, it is helpful to think of $\mathscr{V}$ and $\mathscr{W}$ as the categories of modules over commutative rings R and S, and consider base change functors associated to a ring homomorphism $\phi\colon R \longrightarrow S$. To ease the translation, think of presheaves $\mathscr{D}^{op} \longrightarrow \mathscr{V}$ as right $\mathscr{D}$-modules and covariant functors $\mathscr{D} \longrightarrow \mathscr{V}$ as left $\mathscr{D}$-modules.

We have two adjunctions induced by (5.10). The first is obvious, namely

$$\mathscr{V}\mathbb{O}\text{-}\mathscr{C}at \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}\mathbb{O}\text{-}\mathscr{C}at. \tag{5.11}$$

This adjunction is implicit in Proposition 5.1. The functors $\mathbb{T}$ and $\mathbb{U}$ on presheaf categories are obtained by applying the functors $\mathbb{T}$ and $\mathbb{U}$ of (5.10) objectwise.

The second is a little less obvious. Consider $\mathscr{D} \in \mathscr{V}\mathbb{O}$-$\mathscr{C}at$ and $\mathscr{E} \in \mathscr{W}\mathbb{O}$-$\mathscr{C}at$ and let $\phi\colon \mathscr{D} \longrightarrow \mathbb{U}\mathscr{E}$ be a map of $\mathscr{V}$-categories; equivalently, we could start with the adjoint $\tilde{\phi}\colon \mathbb{T}\mathscr{D} \longrightarrow \mathscr{E}$. We then have an induced adjunction

$$\mathscr{V}^{\mathscr{D}} \underset{\mathbb{U}_\phi}{\overset{\mathbb{T}_\phi}{\rightleftarrows}} \mathscr{W}^{\mathscr{E}}. \tag{5.12}$$

To see this, consider $X \in \mathscr{V}^{\mathscr{D}}$ and $Y \in \mathscr{W}^{\mathscr{E}}$. The presheaf $\mathbb{U}_\phi Y\colon \mathscr{D}^{op} \longrightarrow \mathscr{V}$ is defined via the adjoints of the following maps in $\mathscr{V}$.

$$\mathscr{D}(d,e) \otimes_{\mathscr{V}} \mathbb{U}Y_e \xrightarrow{\phi\otimes\mathrm{id}} \mathbb{U}\mathscr{E}(d,e) \otimes_{\mathscr{V}} \mathbb{U}Y_e \longrightarrow \mathbb{U}(\mathscr{E}(d,e) \otimes_{\mathscr{W}} Y_e) \longrightarrow \mathbb{U}Y_d.$$

The presheaf $\mathbb{T}_\phi X\colon \mathscr{E}^{op} \longrightarrow \mathscr{W}$ is obtained by an extension of scalars that can be written conceptually as $\mathbb{T}X \otimes_{\mathbb{T}\mathscr{D}} \mathbb{E}$. To make sense of this, recall that we have the represented presheaves $\mathbb{E}(e)$ such that $\mathbb{E}(e)_d = \mathscr{E}(d,e)$. As e-varies, these define a covariant $\mathscr{W}$-functor $\mathbb{E}\colon \mathscr{E} \longrightarrow \mathscr{W}^{\mathscr{E}}$. Pull this back via ϕ to obtain a covariant $\mathscr{W}$-functor $\mathbb{T}\mathscr{D} \longrightarrow \mathscr{W}^{\mathscr{E}}$. The tensor product is the coequalizer

$$\textstyle\coprod_{d,e} \mathbb{T}X_e \otimes_{\mathscr{W}} \mathbb{T}\mathscr{D}(d,e) \otimes_{\mathscr{W}} \mathbb{E}(d) \rightrightarrows \coprod_d \mathbb{T}X_d \otimes_{\mathscr{W}} \mathbb{E}(d) \longrightarrow \mathbb{T}X \otimes_{\mathbb{T}\mathscr{D}} \mathbb{E} \equiv \mathbb{T}_\phi X,$$

where the parellel arrows are given by the functors $\mathbb{T}X$ and $\mathbb{E}$. Composition on the right makes this a contravariant functor $\mathscr{E} \longrightarrow \mathscr{W}$.

There are two evident special cases, which are treated in [6, App A]. They are obtained by starting with either $\mathscr{E}$ or $\mathscr{D}$ and taking ϕ to be either

$$\mathrm{id}\colon \mathbb{U}\mathscr{E} \longrightarrow \mathbb{U}\mathscr{E} \quad \text{or} \quad \eta\colon \mathscr{D} \longrightarrow \mathbb{U}\mathbb{T}\mathscr{D}.$$

These specializations give adjunctions

$$\mathscr{V}^{\mathbb{U}\mathscr{E}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}^{\mathscr{E}}. \tag{5.13}$$

and

$$\mathscr{V}^{\mathscr{D}} \underset{\mathbb{U}_\eta}{\overset{\mathbb{T}_\eta}{\rightleftarrows}} \mathscr{W}^{\mathbb{T}\mathscr{D}}. \tag{5.14}$$

As in algebraic instances of the one object special case, the adjunction (5.12) factors as the composite of the adjunction (5.13) and an adjunction $(\phi_!, \phi^*)$:

$$\mathscr{V}^{\mathscr{D}} \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathscr{V}^{\mathbb{U}\mathscr{E}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}^{\mathscr{E}}. \tag{5.15}$$

This follows from the observation that the right adjoints in (5.12) and (5.15) are the same.

5.4. **Model categorical changes of $\mathscr{V}$ and $\mathscr{D}$.** We want a result to the effect that if $(\mathbb{T}, \mathbb{U})$ in (5.10) is a Quillen equivalence, then $(\mathbb{T}_\phi, \mathbb{U}_\phi)$ in (5.12) is also a Quillen equivalence. As in Remark 4.16, we set up a general context that will be encountered in the sequel [10]; it is a variant of the context of [32, §6]. We assume that the identity functor is a left Quillen equivalence $\mathscr{V}_+ \longrightarrow \mathscr{V}$ for two model structures on $\mathscr{V}$ with the same weak equivalences, where the unit $\mathbf{I}$ is cofibrant in $\mathscr{V}$ but not necessarily in $\mathscr{V}_+$. Similarly, we assume that $\mathscr{V}$ but not necessarily $\mathscr{V}_+$ satisfies the monoid axiom, which ensures that all $\mathscr{V}$-categories $\mathscr{D}$ are $\mathscr{V}$-good. We do not assume that $\mathscr{W}$ satisfies the monoid axiom, but we do assume that all $\mathscr{W}$-categories $\mathscr{E}$ in sight are $\mathscr{W}$-good and all weak equivalences $\mathscr{E} \longrightarrow \mathscr{E}'$ in sight are $\mathscr{W}$-weak equivalences in the sense of Definition 4.3; compare Remark 2.14.

Categorically, the adjunction (5.10) is independent of model structures. However, we assume that

$$\mathscr{V}_+ \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}. \tag{5.16}$$

is a Quillen equivalence in which $\mathbb{U}$ creates the weak equivalences in $\mathscr{V}$ and that the unit $\eta\colon V \longrightarrow \mathbb{U}\mathbb{T}V$ of the adjunction is a weak equivalence for all cofibrant V in $\mathscr{V}$ (not just in $\mathscr{V}_+$). With the level model structures that we are considering, the right adjoint $\mathbb{U}_\phi$ in the adjunction

$$\mathscr{V}_+^{\mathscr{D}} \underset{\mathbb{U}_\phi}{\overset{\mathbb{T}_\phi}{\rightleftarrows}} \mathscr{W}^{\mathscr{E}} \tag{5.17}$$

then creates the weak equivalences and fibrations in $\mathscr{W}^{\mathscr{E}}$, so that (5.17) is again a Quillen adjunction. With these assumptions, we have the following variant of theorems in [6, 32].

Theorem 5.18. *If $(\mathbb{T}, \mathbb{U})$ in (5.16) is a Quillen equivalence and $\phi\colon \mathscr{D} \longrightarrow \mathbb{U}\mathscr{E}$ is a weak equivalence, then $(\mathbb{T}_\phi, \mathbb{U}_\phi)$ in (5.17) is a Quillen equivalence.*

Proof. We have a factorization of (5.17) as in (5.15), and $(\phi_!, \phi^*)$ is a Quillen equivalence by Proposition 4.4. Therefore it suffices to consider the special case $\phi = \mathrm{id}\colon \mathbb{U}\mathscr{E} \longrightarrow \mathbb{U}\mathscr{E}$.

Let $\gamma\colon \mathbb{Q}\mathbb{U}\mathscr{E} \longrightarrow \mathbb{U}\mathscr{E}$ be a cofibrant approximation in the model structure on $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ of Theorem 5.9. Since $\mathbf{I}$ is cofibrant in $\mathscr{V}$, each $\mathbb{Q}\mathbb{U}\mathscr{E}(d,e)$ is cofibrant and thus, by assumption, each map $\eta\colon \mathbb{Q}\mathbb{U}\mathscr{E}(d,e) \longrightarrow \mathbb{U}\mathbb{T}\mathbb{Q}\mathbb{U}\mathscr{E}(d,e)$ is a weak equivalence. Let $\tilde{\gamma}\colon \mathbb{T}\mathbb{Q}\mathbb{U}\mathscr{E} \longrightarrow \mathscr{E}$ be the adjoint of γ obtained from the adjunction (5.11). Since the weak equivalence γ is the composite

$$\mathbb{Q}\mathbb{U}\mathscr{E} \xrightarrow{\ \eta\ } \mathbb{U}\mathbb{T}\mathbb{Q}\mathbb{U}\mathscr{E} \xrightarrow{\ \mathbb{U}\tilde{\gamma}\ } \mathbb{U}\mathscr{E}$$

and η is a weak equivalence, $\mathbb{U}\tilde{\gamma}$ is a weak equivalence by the two out of three property. Since $\mathbb{U}$ creates the weak equivalences, $\tilde{\gamma}$ is a weak equivalence.

The identity $\mathbb{U}\tilde{\gamma}\circ\eta = \gamma$ leads to a commutative square of right Quillen adjoints

$$\begin{array}{ccc} \mathscr{W}^{\mathscr{E}} & \xrightarrow{\ \tilde{\gamma}^*\ } & \mathscr{W}^{\mathbb{T}\mathbb{Q}\mathbb{U}\mathscr{E}} \\ {\scriptstyle \mathbb{U}}\downarrow & & \downarrow{\scriptstyle \mathbb{U}_\eta} \\ \mathscr{V}_+^{\mathbb{U}\mathscr{E}} & \xrightarrow[\ \gamma^*\ ]{} & \mathscr{V}_+^{\mathbb{Q}\mathbb{U}\mathscr{E}}. \end{array}$$

By Proposition 4.4 (or Quillen invariance) the horizontal arrows are the right adjoints of Quillen equivalences. Therefore it suffices to prove that the right vertical arrow is the right adjoint of a Quillen equivalence.

To see this, start more generally with a cofibrant object $\mathscr{D}$ in $\mathscr{V}\mathbb{O}$-$\mathscr{C}at$ and consider the Quillen adjunction

$$\mathscr{V}_+^{\mathscr{D}} \underset{\mathbb{U}_\eta}{\overset{\mathbb{T}_\eta}{\rightleftarrows}} \mathscr{W}^{\mathbb{T}\mathscr{D}} \tag{5.19}$$

It suffices to prove that the unit $X \longrightarrow \mathbb{U}_\eta\mathbb{T}_\eta X$ is a weak equivalence for any cofibrant X in $\mathscr{V}_+^{\mathscr{D}}$. Since X is also cofibrant in $\mathscr{V}^{\mathscr{D}}$ and each $\mathscr{D}(d,e)$ is cofibrant in $\mathscr{V}$, each X_d is cofibrant in $\mathscr{V}$ by Theorem 2.12. Our assumption that $\eta\colon V \longrightarrow \mathbb{U}\mathbb{T}V$ is a weak equivalence for all cofibrant V gives the conclusion. □

5.5. **Tensored adjoint pairs and changes of $\mathscr{V}$, $\mathscr{D}$, and $\mathscr{M}$.** We are interested in model categories that have approximations as presheaf categories, so we naturally want to consider situations where, in addition to the adjunction (5.10) between $\mathscr{V}$ and $\mathscr{W}$, we have a $\mathscr{V}$-category $\mathscr{M}$, a $\mathscr{W}$-category $\mathscr{N}$, and an adjunction

$$\mathscr{M} \underset{\mathbb{K}}{\overset{\mathbb{J}}{\rightleftarrows}} \mathscr{N} \tag{5.20}$$

that is suitably compatible with (5.10). In view of our standing assumption that $\mathbb{T}$ is strong symmetric monoidal and therefore $\mathbb{U}$ is lax symmetric monoidal, the following definition seems reasonable. It covers the situations of most interest to us, but the notion of "adjoint module" introduced by Dugger and Shipley [6, §§3,4] gives the appropriate generalization in which it is only assumed that $\mathbb{U}$ is lax symmetric monoidal. Recall the isomorphisms of (1.11).

Definition 5.21. The adjunction $(\mathbb{J},\mathbb{K})$ is tensored over the adjunction $(\mathbb{T},\mathbb{U})$ if there is a natural isomorphism

$$\mathbb{J}X \odot \mathbb{T}V \cong \mathbb{J}(X \odot V) \tag{5.22}$$

such that the following coherence diagrams of isomorphisms commute for $X \in \mathscr{M}$ and $V, V' \in \mathscr{V}$.

$$\begin{array}{ccc} \mathbb{J}X & \longrightarrow & \mathbb{J}(X \odot \mathbf{I}_{\mathscr{V}}) \\ \downarrow & & \uparrow \\ \mathbb{J}X \odot \mathbf{I}_{\mathscr{W}} & \longrightarrow & \mathbb{J}X \odot \mathbb{T}\mathbf{I}_{\mathscr{V}} \end{array}$$

$$\begin{array}{ccccc} (\mathbb{J}X \odot \mathbb{T}V) \odot \mathbb{T}V' & \longrightarrow & \mathbb{J}(X \odot V) \odot \mathbb{T}V' & \longrightarrow & \mathbb{J}((X \odot V) \odot V') \\ \downarrow & & & & \downarrow \\ \mathbb{J}X \odot (\mathbb{T}V \otimes \mathbb{T}V') & \longrightarrow & \mathbb{J}X \odot \mathbb{T}(V \otimes V') & \longrightarrow & \mathbb{J}(X \odot (V \otimes V')). \end{array}$$

The definition implies an enriched version of the adjunction $(\mathbb{J}, \mathbb{K})$.

Lemma 5.23. *If* $(\mathbb{J}, \mathbb{K})$ *is tensored over* $(\mathbb{T}, \mathbb{U})$*, then there is a natural isomorphism*

$$\mathbb{U}\underline{\mathscr{N}}(\mathbb{J}X, Y) \cong \underline{\mathscr{M}}(X, \mathbb{K}Y)$$

in $\mathscr{V}$*, where* $X \in \mathscr{M}$ *and* $Y \in \mathscr{N}$*.*

Proof. For $V \in \mathscr{V}$, we have the sequence of natural isomorphisms

$$\begin{aligned} \mathscr{V}(V, \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}X, Y) &\cong \mathscr{W}(\mathbb{T}V, \underline{\mathscr{N}}(\mathbb{J}X, Y)) \\ &\cong \mathscr{N}(\mathbb{J}X \odot \mathbb{T}V, Y) \\ &\cong \mathscr{N}(\mathbb{J}(X \odot V), Y) \\ &\cong \mathscr{M}(X \odot V, \mathbb{K}Y) \\ &\cong \mathscr{V}(V, \underline{\mathscr{M}}(X, \mathbb{K}Y). \end{aligned}$$

The conclusion follows from the Yoneda lemma. □

We are interested in comparing presheaf categories $\mathscr{V}^{\mathscr{D}}$ and $\mathscr{W}^{\mathscr{E}}$ where $\mathscr{D}$ and $\mathscr{E}$ are full categories of bifibrant objects that correspond under a Quillen equivalence between $\mathscr{M}$ and $\mathscr{N}$. In the context of §5.4, we can change $\mathscr{V}$ to $\mathscr{V}_+$. The following results then combine with Remark 4.16 and Theorem 5.18 to give such a comparison.

Theorem 5.24. *Let* $(\mathbb{J}, \mathbb{K})$ *be tensored over* $(\mathbb{T}, \mathbb{U})$*, where* $(\mathbb{J}, \mathbb{K})$ *is a Quillen equivalence. Let* $\mathscr{E}$ *be a small full* $\mathscr{W}$*-subcategory of bifibrant objects of* $\mathscr{N}$*. Then* $\mathbb{U}\mathscr{E}$ *is quasi-equivalent to the small full* $\mathscr{V}$*-subcategory* $\mathscr{D}$ *of* $\mathscr{M}$ *with bifibrant objects the* $\mathbb{Q}\mathbb{K}Y$ *for* $Y \in \mathscr{E}$*, where* $\mathbb{Q}$ *is a cofibrant approximation functor in* $\mathscr{M}$*.*

Proof. We define a $(\mathbb{U}\mathscr{E}, \mathscr{D})$-bimodule $\mathscr{F}$. Let $X, Y, Z \in \mathscr{E}$. Define

$$\mathscr{F}(X, Y) = \mathscr{M}(\mathbb{Q}\mathbb{K}X, \mathbb{K}Y).$$

The right action of $\mathscr{D}$ is given by composition

$$\mathscr{M}(\mathbb{Q}\mathbb{K}Y, \mathbb{K}Z) \otimes \mathscr{M}(\mathbb{Q}\mathbb{K}X, \mathbb{Q}\mathbb{K}Y) \longrightarrow \mathscr{M}(\mathbb{Q}\mathbb{K}X, \mathbb{K}Z).$$

The counit $\mathbb{J}\mathbb{K} \longrightarrow \mathrm{Id}$ of the adjunction gives a natural map

$$\mathbb{U}\underline{\mathscr{N}}(X, Y) \longrightarrow \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}\mathbb{K}X, Y) \cong \underline{\mathscr{M}}(\mathbb{K}X, \mathbb{K}Y).$$

The left action of $\mathbb{U}\mathscr{E}$ is given by the composite

$$\mathbb{U}\underline{\mathscr{N}}(Y, Z) \otimes \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X, \mathbb{K}Y) \longrightarrow \underline{\mathscr{M}}(\mathbb{K}Y, \mathbb{K}Z) \otimes \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X, \mathbb{K}Y) \longrightarrow \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X, \mathbb{K}Z).$$

Using the coherence diagrams in Definition 5.21, a lengthy but routine check shows that the diagrams that are required to commute in §4.2 do in fact commute. Define $\zeta_X\colon \mathbf{I} \longrightarrow \mathscr{F}(\mathbf{X},\mathbf{X})$ to be the composite

$$\mathbf{I} \longrightarrow \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}\mathbf{X},\mathbb{Q}\mathbb{K}\mathbf{X}) \longrightarrow \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}\mathbf{X},\mathbb{K}\mathbf{X})$$

induced by the weak equivalence $\mathbb{Q}\mathbb{K}X \longrightarrow \mathbb{K}X$. By the naturality square

$$\begin{array}{ccc} \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}\mathbb{K}X,Y) & \xrightarrow{\cong} & \underline{\mathscr{M}}(\mathbb{K}X,\mathbb{K}Y) \\ \downarrow & & \downarrow \\ \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}\mathbb{Q}\mathbb{K}X,Y) & \xrightarrow[\cong]{} & \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X,\mathbb{K}Y) \end{array}$$

the map

$$(\zeta_X)^*\colon \mathbb{U}\underline{\mathscr{N}}(X,Y) \longrightarrow \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X,\mathbb{K}Y)$$

is the composite

$$\mathbb{U}\underline{\mathscr{N}}(X,Y) \longrightarrow \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}\mathbb{K}X,Y) \longrightarrow \mathbb{U}\underline{\mathscr{N}}(\mathbb{J}\mathbb{Q}\mathbb{K}X,Y) \cong \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X,\mathbb{K}Y).$$

Since $(\mathbb{J},\mathbb{K})$ is a Quillen equivalence, the composite $\mathbb{J}\mathbb{Q}\mathbb{K}X \longrightarrow \mathbb{J}\mathbb{K}X \longrightarrow X$ is a weak equivalence, hence $(\zeta_X)^*$ is a weak equivalence by Lemma 4.13. The map

$$(\zeta_Y)_*\colon \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X,\mathbb{Q}\mathbb{K}Y) \longrightarrow \underline{\mathscr{M}}(\mathbb{Q}\mathbb{K}X,\mathbb{K}Y)$$

is also a weak equivalence by Lemma 4.13. □

Corollary 5.25. *With the hypotheses of the theorem, let $\mathscr{D}$ be a small full $\mathscr{V}$-subcategory of bifibrant objects of $\mathscr{M}$. Then $\mathscr{D}$ is quasi-equivalent to $\mathbb{U}\mathscr{E}$, where $\mathscr{E}$ is the small full $\mathscr{W}$-subcategory of $\mathscr{N}$ with bifibrant objects the $\mathbb{R}\mathbb{J}X$ for $X \in \mathscr{D}$, where $\mathbb{R}$ is a fibrant approximation functor in $\mathscr{N}$.*

Proof. By the theorem, $\mathbb{U}\mathscr{E}$ is quasi-equivalent to $\mathscr{D}'$, where $\mathscr{D}'$ is the full $\mathscr{V}$-subcategory of $\mathscr{M}$ with objects the $\mathbb{Q}\mathbb{K}\mathbb{R}\mathbb{J}X$, and of course $\mathbb{Q}\mathbb{K}\mathbb{R}\mathbb{J}X$ is weakly equivalent to X. The conclusion follows from Corollary 4.15. □

5.6. **Weakly unital $\mathscr{V}$-categories and presheaves.** In the sequel [10], we shall encounter a topologically motivated variant of presheaf categories. Despite the results of the previous subsection, which show how to compare full enriched subcategories $\mathscr{D}$ of categories $\mathscr{M}$ with differing enriching categories $\mathscr{V}$, when seeking simplified equivalents of full subcategories of $\mathscr{V}$-categories $\mathscr{M}$, the choice of $\mathscr{V}$ can significantly effect the mathematics, and we shall sometimes have to work with a $\mathscr{V}$ in which $\mathbf{I}$ is not cofibrant. We shall encounter domains $\mathscr{D}$ for presheaf categories in which $\mathscr{D}$ is not quite a category since a cofibrant approximation $\mathbb{Q}\mathbf{I}$ rather than $\mathbf{I}$ itself demands to be treated as if it were a unit object. The examples start with a given $\mathscr{M}$ but are not full $\mathscr{V}$-subcategories of $\mathscr{M}$. Retaining our standing assumptions on $\mathscr{V}$, we conceptualize the situation with the following definitions. We fix a weak equivalence $\gamma\colon Q\mathbf{I} \longrightarrow \mathbf{I}$, not necessarily a fibration.

Definition 5.26. Fix a $\mathscr{V}$-model category $\mathscr{M}$ and a set $\mathbb{O} = \{d\}$ of objects of $\mathscr{M}$. A weakly unital $\mathscr{V}$-category $\mathscr{D}$ with object set $\mathbb{O}$ consists of objects $\mathscr{D}(d,e)$ of $\mathscr{V}$ for $d,e \in \mathbb{O}$, an associative pairing $\mathscr{D}(d,e) \otimes \mathscr{D}(c,d) \longrightarrow \mathscr{D}(c,e)$, and, for each $d \in \mathbb{O}$, a map $\eta_d\colon \mathbb{Q}\mathbf{I} \longrightarrow \mathscr{D}(d,d)$ and a weak equivalence $\xi_d\colon d \longrightarrow d$ that induces weak equivalences

$$\xi_d^*\colon \mathscr{D}(d,e) \longrightarrow \mathscr{D}(d,e) \quad \text{and} \quad \xi_{d*}\colon \mathscr{D}(c,d) \longrightarrow \mathscr{D}(c,d)$$

for all $c, e \in \mathbb{O}$. The following unit diagrams must commute.

$$\begin{array}{ccc} \mathscr{D}(d,e)\otimes Q\mathbf{I} & \xrightarrow{\mathrm{id}\otimes\eta_d} & \mathscr{D}(d,e)\otimes\mathscr{D}(d,d) \\ {\scriptstyle \xi_d^*\otimes\gamma}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathscr{D}(d,e)\otimes\mathbf{I} & \xrightarrow[\cong]{} & \mathscr{D}(d,e). \end{array} \quad\text{and}\quad \begin{array}{ccc} \mathbb{Q}\mathbf{I}\otimes\mathscr{D}(c,d) & \xrightarrow{\eta_d\otimes\mathrm{id}} & \mathscr{D}(d,d)\otimes\mathscr{D}(c,d) \\ {\scriptstyle \gamma\otimes\xi_{d*}}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathbf{I}\otimes\mathscr{D}(c,d) & \xrightarrow[\cong]{} & \mathscr{D}(c,d). \end{array}$$

A weakly unital $\mathscr{D}$-presheaf is a $\mathscr{V}$-functor $X\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{V}$ defined as usual, except that the unital property requires commutativity of the following diagrams for $d \in \mathbb{O}$.

$$\begin{array}{ccc} \mathbb{Q}\mathbf{I} & \xrightarrow{\eta_d} & \mathscr{D}(d,d) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle X} \\ \mathbf{I} & \xrightarrow[\xi_d^*]{} & \underline{\mathscr{V}}(X_d, X_d). \end{array}$$

Here the bottom arrow is adjoint to the map $X(\xi_d)\colon X_d \longrightarrow X_d$. We write $\mathscr{V}^{\mathscr{D}}$ for the category of weakly unital presheaves. The morphisms are the $\mathscr{V}$-natural transformations, the definition of which requires no change.

Remark 5.27. A $\mathscr{V}$-category $\mathscr{D}$ may be viewed as a weakly unital $\mathscr{V}$-category $\mathscr{D}'$ by taking $\eta_d = \eta \circ \gamma$, where $\eta\colon \mathbf{I} \longrightarrow \mathscr{D}(d,d)$ is the given unit, and taking $\xi_d = \mathrm{id}$. Then any $\mathscr{D}$-presheaf can be viewed as a $\mathscr{D}'$-presheaf. In principle, $\mathscr{D}'$-presheaves are slightly more general, since it is possible for the last diagram to commute even though the composites

$$\mathbf{I} \xrightarrow{\eta} \mathscr{D}(d,d) \xrightarrow{X} \underline{\mathscr{V}}(X_d, X_d)$$

are not the canonical unit maps η. However, this cannot happen if γ is an epimorphism in $\mathscr{V}$, in which case the categories $\mathscr{V}^{\mathscr{D}}$ and $\mathscr{V}^{\mathscr{D}'}$ are identical.

Virtually everything that we have proven when $\mathbf{I}$ is not cofibrant applies with minor changes to weakly unital presheaf categories.

6. Equivariant model and presheaf categories

6.1. **Enriched model categories of G-spaces.** Here is a key motivating example from [7, 22, 29]. Since it is specified topologically, we use topological enrichments, but the usual suspicious switch to simplicial enrichments can be applied.

Example 6.1. Let G be a topological group and let $\mathscr{F}$ be a family of closed subgroups, so that $\mathscr{F}$ is closed under passage to subconjugate groups. We take $\mathscr{V}$ to be the cartesian monoidal category $\mathscr{U}$ of compactly generated spaces with its standard Quillen model structure. We take $\mathscr{M}$ to be the $\mathscr{U}$-category $G\mathscr{U}$ of G-spaces. The enriched hom $\underline{\mathscr{M}}(M,N)$ is then denoted $G\mathscr{U}(M,N)$ and is just the space of G-maps $M \longrightarrow N$.

Of course, $G\mathscr{U}$ can also be viewed as a closed cartesian monoidal category, with G acting diagonally on cartesian products. Its internal hom G-space $\underline{G\mathscr{U}}(M,N)$ is the space of all maps $X \longrightarrow Y$ with G acting by conjugation. As is typical in equivariant situations, there is a reinterpretation in terms of double enrichment. We can define a $G\mathscr{U}$-category $\mathscr{U}_G$ whose objects are again the G-spaces but whose enriched hom G-spaces are $\underline{\mathscr{U}}_G(M,N) = \underline{G\mathscr{U}}(M,N)$. From this perspective, $G\mathscr{U}$ is the underlying $\mathscr{U}$-category of this $G\mathscr{U}$-category and can be identified with the

G-fixed point category $(\mathscr{U}_G)^G$. Indeed, the objects of $G\mathscr{U}$ and $(\mathscr{U}_G)^G$ are the G-spaces ("G-fixed spaces") and

$$G\mathscr{U}(*, \underline{\mathscr{U}}_G(M,N) \cong (\underline{\mathscr{U}}_G(M,N))^G = G\mathscr{U}(M,N).$$

We regard $G\mathscr{U}(M,N)$ as the space rather than just the set of G-maps $M \longrightarrow N$. When we enrich a category in topological spaces, as here, it seems reasonable to use the same notation for the set and the space of maps, since the latter is just given by a topology on the given set of maps.

Take $\mathscr{D}$ to be $\mathscr{O}_{\mathscr{F}}$, the full $\mathscr{U}$-subcategory of $G\mathscr{U}$ whose objects are the orbit G-spaces G/H with $H \in \mathscr{F}$. The most important example is $\mathscr{F} = \mathscr{A}\ell\ell$, the set of all subgroups of G, in which case we write $\mathscr{O}_G$ for the orbit category. For a G-space M, $G\mathscr{U}(G/H, M)$ can be identified with the fixed point space M^H, so that $\mathbb{U}(M)$ is the presheaf of fixed point spaces M^H for $H \in \mathscr{F}$. The $\mathscr{O}_{\mathscr{F}}$-equivalences on $G\mathscr{U}$ are the $\mathscr{F}$-equivalences, namely the G-maps $f\colon M \longrightarrow N$ such that $f^H\colon M^H \longrightarrow N^H$ is a weak equivalence for $H \in \mathscr{F}$. By the G-Whitehead theorem, a weak $\mathscr{O}_G$-equivalence between G-CW complexes is a G-homotopy equivalence. When $\mathscr{F} = \{e\}$, a weak $\mathscr{F}$-equivalence is just a G-map which is a nonequivariant weak equivalence, giving a naive version of equivariant homotopy theory. Similarly, the $\mathscr{O}_{\mathscr{F}}$-fibrations are the $\mathscr{F}$-fibrations, namely the G-maps f such that each f^H is a (Serre) fibration.

Since $\mathbb{T}X$ is the G-space $X_{G/e}$, it is easy to check that $\mathscr{O}_{\mathscr{F}}$ is $G\mathscr{U}$-good in the sense of Remark 2.14. Moreover, the maps (3.5) are the evident homeomorphisms

$$M^H \times V \cong (M \times V)^H,$$

and the H-fixed point functors preserve the colimits relevant to the construction of cell complexes. It follows that $\mathscr{O}_{\mathscr{F}}$ is $(G\mathscr{U}, \mathscr{U})$-good in the sense of Definition 3.9 and that $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is an isomorphism in $\mathscr{U}^{\mathscr{O}_{\mathscr{F}}}$ for all cofibrant X. We conclude from Theorem 3.10 that $G\mathscr{U}$ has a $\mathscr{U}$-enriched "$\mathscr{F}$-model structure" under which it is Quillen equivalent to the $\mathscr{U}$-enriched diagram category $\mathscr{U}^{\mathscr{O}_{\mathscr{F}}}$. Of course, unless G is discrete, the topological enrichment of $\mathscr{O}_{\mathscr{F}}$ is essential.

Using smash products instead of cartesian products, this example works just as well using the categories $\mathscr{T}$ and $G\mathscr{T}$ of based spaces and based G-spaces.

We can combine Examples 6.1 and 5.5 to show that good symmetric monoidal categories of G-spectra are enriched over $\mathscr{T}$, but we are more interested in the enrichment over a well-chosen category $\mathscr{S}$ of spectra. We shall return to G-spectra in the sequel [10], but we consider equivariant examples in general here.

6.2. **Equivariant contexts.** There are a number of generalized versions of Example 6.1 in different contexts. We consider three categorical contexts in this subsection and the next, ignoring both model categorical structures and presheaves. Let $\mathscr{V}$ be a closed symmetric monoidal category and let $\mathscr{M}$ be a $\mathscr{V}$-category. We consider the following three equivariant contexts, each of which gives a category $G\mathscr{M}$ of (left) G-objects in $\mathscr{M}$ whose morphisms are the evident G-maps.

(A) G is a discrete group. An action of G on an object $M \in \mathscr{M}$ is a homomorphism $G \longrightarrow \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the discrete automorphism group of isomorphisms $M \longrightarrow M$ in $\mathscr{M}$. In this context, $\mathscr{V}$ and the enrichment on $\mathscr{M}$ do not enter into the definition of a G-action. (For that purpose, $\mathscr{V} = Set$.)

(B) G is a topological group and $\mathscr{M}$ is enriched in $\mathscr{U}$. An action of G on an object $M \in \mathscr{M}$ is a continuous homomorphism $G \longrightarrow \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is the topological group of homeomorphisms $M \longrightarrow M$.

(C) $\mathscr{V}$ is cartesian monoidal and G is a group object in $\mathscr{V}$. An action of G on an object $M \in \mathscr{M}$ is a homomorphism $G \longrightarrow \underline{\mathscr{M}}(M, M)$ of monoids in $\mathscr{V}$.

Context (B) is the special case $\mathscr{V} = \mathscr{U}$ of context (C). Another example of context (C) is $\mathscr{V} = sSet$, in which case G is a simplicial group, which can act on objects in any simplicially enriched category $\mathscr{M}$. In context (C), the reader may want to assume that $\mathscr{V}$ is concrete, meaning that it has a faithful underlying set functor. That allows subgroups and conjugate groups to be interpreted concretely. Otherwise, to be precise, we should understand a subgroup to be an isomorphism class of monomorphisms $\iota\colon H \longrightarrow G$ of groups in $\mathscr{V}$. The assumption is not necessary, but the added categorical care would be digressive.

We consider categories of G-objects in $\mathscr{V}$-categories $\mathscr{M}$ in the rest of the paper, and we assume throughout that we are in one of the contexts (A), (B), or (C). We shall make much use of the construction $\mathbf{I}[-]$ of Remark 2.15, and when we do so we assume that $\mathbf{I}$ is cofibrant in $\mathscr{V}$.

Remark 6.2. We are especially interested in doubly enriched situations, where we start with an adjunction $\mathscr{V} \rightleftarrows \mathscr{W}$ satisfying the hypotheses of Proposition 5.1 and a bicomplete $\mathscr{W}$-category $\mathscr{M}$. For example, $\mathscr{V}$ could be spaces and $\mathscr{W}$ could be spectra, or $\mathscr{V}$ could be k-modules and $\mathscr{W}$ could be chain complexes of k-modules for some ring k.

For $M, N \in G\mathscr{M}$, the enriched hom $\underline{\mathscr{M}}(M, N)$ in $\mathscr{V}$ can be given a G-action, formalizing conjugation, and we then denote it by $\underline{\mathscr{M}}_G(M, N)$. To see this in context (C), observe that an action by G on an object $V \in \mathscr{V}$ is given in adjoint form by a map $G \times V \longrightarrow V$. Let $\Delta\colon G \longrightarrow G \times G$ be the diagonal and $\chi\colon G \longrightarrow G$ be the inverse map. The conjugation action is the evident composite

$$\begin{array}{c}
G \times \underline{\mathscr{M}}(M, N) \\
\Big\downarrow{\scriptstyle (\mathrm{id} \times \chi)\circ\Delta} \\
G \times G \times \underline{\mathscr{M}}(M, N) \\
\Big\downarrow{\scriptstyle \cong} \\
G \times \underline{\mathscr{M}}(M, N) \times G \\
\Big\downarrow \\
\underline{\mathscr{N}}(N, N) \times \underline{\mathscr{M}}(M, N) \times \underline{\mathscr{M}}(M, M) \\
\Big\downarrow \\
\underline{\mathscr{M}}(M, N).
\end{array}$$

Formally, we define a category $\mathscr{M}_G$ enriched over $G\mathscr{V}$ with the same objects as $G\mathscr{M}$ and with hom objects the $\underline{\mathscr{M}}(M, N)$. Taking $\mathscr{M} = \mathscr{V}$, this makes $G\mathscr{V}$ into a closed symmetric monoidal category.

Moreover, we have a strong symmetric monoidal functor $\varepsilon^*\colon \mathscr{V} \to G\mathscr{V}$ which assigns to any V the object V with trivial G-action. This functor is left adjoint to a G-fixed point functor $(-)^G\colon G\mathscr{V} \to \mathscr{V}$, constructed as an equalizer. Applying Proposition 5.1 to $\mathscr{M}_G$ produces a $\mathscr{V}$-category $G\mathscr{M}$, the G-fixed category of $\mathscr{M}_G$. This is bitensored over $\mathscr{V}$. For $M \in G\mathscr{M}$ and $V \in G\mathscr{V}$, the tensor $M \odot V$ and cotensor $\Phi(V,M)$ in $\mathscr{M}_G$ are the tensor and cotensor in $\mathscr{M}$ endowed with actions of G induced by the actions of G on V and M. The bitensor adjunctions

$$(6.3)\qquad \underline{G\mathscr{M}}(M\odot V,N)\cong \underline{G\mathscr{V}}(V,\underline{\mathscr{M}}_G(M,N))\cong \underline{G\mathscr{M}}(M,\Phi(V,N))$$

in $\mathscr{V}$ arise from the enriched versions, which are given by isomorphisms

$$(6.4)\qquad \underline{\mathscr{M}}_G(M\odot V,N)\cong \underline{\mathscr{V}}_G(V,\underline{\mathscr{M}}_G(M,N))\cong \underline{\mathscr{M}}_G(M,\Phi(V,N))$$

in $G\mathscr{V}$.

The essential starting point for enriched equivariant homotopy theory is an understanding of fixed point objects M^H and orbit objects M/H in $\mathscr{M}$ for objects $M\in G\mathscr{M}$ and subgroups H of G.[4] We also need induction and coinduction functors $H\mathscr{M}\longrightarrow G\mathscr{M}$. If we view G as a $\mathscr{V}$-category with a single object, these can be specified as suitable limits and colimits (weighted when G is a group object in $\mathscr{V}$) defined on the subcategory H of G, but we want a better enriched categorical perspective. However, $G\mathscr{M}$ may contain no "orbit objects G/H", and it is not in general a suitable category in which to enrich things.

6.3. **Orbit tensors and fixed point cotensors.** We get around this by constructing "orbit tensors" $V\odot_H M$ for left H-objects $M\in\mathscr{M}$ and right H-objects $V\in\mathscr{V}$ and "fixed point cotensors" $\Phi_H(V,N)$ for left H-objects $M\in\mathscr{M}$ and $V\in\mathscr{V}$. These specialize to give change of group functors that are entirely analogous to those in familiar examples.

When G is discrete and S is a G-set, we have an object $\mathbf{I}[S]$ in $G\mathscr{V}$, constructed as in Remark 2.15 and given the G-action that permutes coproduct summands as G permutes elements of S. When G is a group object in a cartesian monoidal category $\mathscr{V}$ and S is a G-object in $\mathscr{V}$, we agree to interpret $\mathbf{I}[S]$ as S itself. This makes sense since the unit object $\mathbf{I}$ is then just a terminal object. In all of our contexts, a left action of H on M can be viewed as a map $\mathbf{I}[H]\odot M\longrightarrow M$ in $\mathscr{M}$.

Using (1.11) implicitly, define $V\odot_H M$ to be the coequalizer

$$V\odot \mathbf{I}[H]\otimes M \rightrightarrows V\odot M\longrightarrow V\odot_H M.$$

Dually, define $\Phi_H(V,N)$ to be the equalizer

$$\Phi_H(V,N)\longrightarrow \Phi(V,N)\rightrightarrows \Phi(\mathbf{I}[H]\otimes V,N).$$

One of each of the parallel pairs of arrows is induced by the action of H on V and the other is induced by the action of H on M or N.

Observe that the left G-object $\mathbf{I}[G/H]$ in $\mathscr{V}$ can be identified with $\mathbf{I}[G]\otimes_H \mathbf{I}$, where $\mathbf{I}$ is viewed as a trivial left H-object. More generally, for a left H-object M in $\mathscr{M}$, we can use the left action of G on $\mathbf{I}[G]$ to give $\mathbf{I}[G]\odot_H M$ a left G-action. Similarly, we can give $\Phi_H(\mathbf{I}[G],N)$ the left action induced by the right action of G on $\mathbf{I}[G]$. The inclusion $\iota\colon H\longrightarrow G$ induces a $\mathscr{V}$-functor $\iota^*\colon G\mathscr{M}\longrightarrow H\mathscr{M}$. The following result identifies $\mathbf{I}[G]\odot_H -$ and $\Phi_H(\mathbf{I}[G],-)$ as the enriched left and right adjoints, called induction and coinduction, of the functor ι^*.

[4]In [7], the authors start with a simplicially enriched category $\mathscr{N}$ and a set $\mathscr{O}$ of objects, which they call 'orbits', in $\mathscr{N}$. For $O\in\mathscr{O}$ and $N\in\mathscr{N}$, they view the simplicial sets $\mathscr{N}(O,N)$ as analogues of fixed point objects. When $\mathscr{N}=G\,sSet$, their context leads to the simplicial analogue of Example 6.1. However, their general context is not relevant to the equivariant theory discussed here since the fixed point objects N^H are in $\mathscr{N}$ and not $sSet$, so play no role in their theory.

Lemma 6.5. *There are $\mathscr{V}$-adjunctions*

$$\underline{G\mathscr{M}}(\mathbf{I}[G] \odot_H N, M) \cong \underline{H\mathscr{M}}(N, \iota^* M)$$

and

$$\underline{H\mathscr{M}}(\iota^* M, N) \cong \underline{G\mathscr{M}}(M, \Phi_H(\mathbf{I}[G], N)),$$

where $M \in G\mathscr{M}$ and $N \in H\mathscr{M}$.

For $N \in H\mathscr{M}$, such as $N = \iota^* M$, we define the orbit objects N/H and fixed point objects N^H in $\mathscr{M}$ to be

$$N/H = \mathbf{I} \odot_H N \quad \text{and} \quad N^H = \Phi_H(\mathbf{I}, N), \tag{6.6}$$

where $\mathbf{I}$ has trivial H-action. These functors actually take values in $WH\mathscr{M}$, where $WH = NH/H$, but we shall ignore that. The trivial homomorphism $\varepsilon\colon H \longrightarrow \{e\}$ induces a $\mathscr{V}$-functor $\varepsilon^*\colon \mathscr{M} \longrightarrow H\mathscr{M}$ that assigns the trivial H-action to an object $M \in \mathscr{M}$, and we have the expected enriched adjunctions.

Lemma 6.7. *There are $\mathscr{V}$-adjunctions*

$$\underline{H\mathscr{M}}(N, \varepsilon^* L) \cong \underline{\mathscr{M}}(N/H, L)$$

and

$$\underline{H\mathscr{M}}(\varepsilon^* M, N) \cong \underline{\mathscr{M}}(M, N^H),$$

where $N \in H\mathscr{M}$ and $M \in \mathscr{M}$.

As in familiar examples, for $M \in G\mathscr{M}$ we have the natural isomorphism

$$\mathbf{I}[G] \odot_H \varepsilon^* M \cong \mathbf{I}[G/H] \odot M \tag{6.8}$$

in $G\mathscr{M}$, where the diagonal G-action is used on the right. Composing adjunctions and omitting ι^* from the notations, we obtain

$$\underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot M, N) \cong \underline{\mathscr{M}}(M, N^H), \tag{6.9}$$

where M,$N \in G\mathscr{M}$. When $\mathscr{M} = \mathscr{V}$ and $M = \mathbf{I}$, this specializes to give

$$\underline{G\mathscr{V}}(\mathbf{I}[G/H], V) \cong V^H.$$

A further comparison of definitions gives the following expected identifications.

$$M/H \cong \mathbf{I}[G/H] \odot_G M \quad \text{and} \quad M^H \cong \Phi_G(\mathbf{I}[G/H], M). \tag{6.10}$$

6.4. **Equivariant model categories and presheaf categories in $\mathscr{M}$.** We return to model category theory and work in one of our contexts (A), (B), or (C). Remember that (B) is the special case $\mathscr{V} = \mathscr{U}$ of (C). We shall see that context (A), discrete groups and general enriching categories, behaves quite differently from context (C), group objects in cartesian monoidal categories. In context (C), everything works exactly the same way as in Example 6.1, so we focus primarily on context (A), which is especially interesting in algebraic contexts. Let $\mathscr{F}$ be a family of subgroups of G. As in Example 6.1, the most important example is $\mathscr{F} = \mathscr{A}\ell\ell$, which leads to genuine equivariant homotopy theory, but the example $\mathscr{F} = \{e\}$, which leads to naive equivariant homotopy theory, is also of interest.

Definition 6.11. A G-map $f\colon M \longrightarrow N$ between objects of $G\mathscr{M}$ is an $\mathscr{F}$-equivalence or $\mathscr{F}$-fibration if $f^H\colon M^H \longrightarrow N^H$ is a weak equivalence or fibration in $\mathscr{M}$ for all $H \in \mathscr{F}$; f is an $\mathscr{F}$-cofibration if it satisfies the LLP with respect to all acyclic $\mathscr{F}$-fibrations. Define $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and $\mathscr{F}\mathcal{J}_{\mathscr{M}}$ to be the sets of maps obtained by applying the functors $\mathbf{I}[G/H] \otimes (-)$ to the maps in $\mathcal{I}_{\mathscr{M}}$ and $\mathcal{J}_{\mathscr{M}}$, where $H \in \mathscr{F}$.

Specializing Theorem 1.20, we obtain the following result.

Theorem 6.12. *When the sets $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and $\mathscr{F}\mathcal{J}_{\mathscr{M}}$ admit the small object argument and every relative $\mathscr{F}\mathcal{J}_{\mathscr{M}}$-cell complex is an $\mathscr{F}$-equivalence, $G\mathscr{M}$ is a cofibrantly generated model category with generating cofibrations $\mathscr{F}\mathcal{I}_{\mathscr{M}}$ and generating acyclic cofibrations $\mathscr{F}\mathcal{J}_{\mathscr{M}}$.*

Here (ii) of Theorem 1.20 follows formally from (6.9), and (6.9) also reduces the small object argument to a question about colimits of (transfinite) sequences in $\mathscr{M}$ that are obtained by passing to H-fixed points from relative cell complexes in $G\mathscr{M}$. The goodness condition (i), which we have restated explicitly, will hold provided that passage to H-fixed points from a relative $\mathscr{F}\mathcal{J}_{\mathscr{M}}$-cell complex gives a weak equivalence in $\mathscr{M}$.

We can compare the model structures on $G\mathscr{M}$ of Theorem 6.12 to model categories of presheaves in $\mathscr{M}$, generalizing Example 6.1. We agree to view a discrete group as a category with a single object and to view a group object in a cartesian monoidal category $\mathscr{V}$, such as $\mathscr{U}$, as a $\mathscr{V}$-category with a single object.

When G is discrete, the category $\mathbf{I}[G]$, as defined in Remark 2.15, is the $\mathscr{V}$ analogue of the group ring of G. Indeed, when k is a commutative ring and $\mathscr{V}$ is the category of k-modules or the category $\mathscr{M}_k$ of chain complexes over k, it is precisely the group ring $k[G]$, regarded as a $\mathscr{V}$-category with a single object. As in §6.2, when G is a group object in $\mathscr{V}$ we write $G = \mathbf{I}[G]$ for uniformity of notation. Remember that we assume that $\mathbf{I}$ is cofibrant.

In both cases, we have the $\mathscr{V}$-category $\mathscr{M}^{\mathbf{I}[G]}$ of $\mathscr{V}$-enriched presheaves in $\mathscr{M}$. Its underlying category is the ordinary category of unenriched presheaves, which is just the ordinary category $G\mathscr{M}$ of G-objects in $\mathscr{M}$:[5]

$$\mathscr{M}_0^{\mathbf{I}[G]} \cong G\mathscr{M}. \tag{6.13}$$

Evaluation at the single object $*$, ev_*, forgets the G-action. Its left adjoint, F_*, sends an object $M \in \mathscr{M}$ to the free G-object $\mathbf{I}[G] \odot M$ in $G\mathscr{M}$. Since $\mathbf{I}[G]$ is $\mathscr{V}$-Quillen (in the sense of Definition 2.11), Theorem 2.12 applies to give $\mathscr{M}^{\mathbf{I}[G]}$ a level $\mathscr{V}$-model structure. Clearly, it coincides with the $\{e\}$-model structure on $G\mathscr{M}$ of Theorem 6.12. We regard this as a naive model structure, rather than a truly equivariant one.

For larger families $\mathscr{F}$, such as $\mathscr{A}\ell\ell$, we need orbit categories in order to compare the $\mathscr{F}$-model structure on $G\mathscr{M}$ to a presheaf model category. When G is discrete, we have the usual category $\mathscr{O}_{\mathscr{F}}$ of orbits G/H with $H \in \mathscr{F}$ and G-maps between them, so that $\mathscr{O}_{\mathscr{F}}(G/H, G/K) = (G/K)^H$. This is a subcategory of the category of sets. Here Remark 2.15 gives a $\mathscr{V}$-category $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ with objects $\mathbf{I}[G/H]$ and

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}](\mathbf{I}[G/H], \mathbf{I}[G/K]) = \mathbf{I}[(G/K)^H].$$

When G is a group object in a cartesian monoidal category $\mathscr{V}$, we have the orbit category $\mathscr{O}_{\mathscr{F}}$ of orbits $G/H \in \mathscr{V}$ and G-maps between them. It is the underlying category of a $\mathscr{V}$-category with morphism objects $\underline{G\mathscr{V}}(G/H, G/K)$. For uniformity of notation, we denote this $\mathscr{V}$-category by $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$, which is consistent since $\mathbf{I}$ is a terminal object. In both cases, we have

$$\mathscr{M}_0^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]} \cong \mathscr{M}_0^{\mathscr{O}_{\mathscr{F}}}. \tag{6.14}$$

[5]Strictly speaking, since presheaves are contravariant functors, these are right G-objects unless we use the opposite multiplication on G when regarding it as a category with a single object.

On the right, $\mathscr{O}_{\mathscr{F}}$ and $\mathscr{M}_0^{\mathscr{O}_{\mathscr{F}}}$ are not enriched; the presheaves are diagrams of maps in the underlying category $\mathscr{M}$, and the morphisms are maps of diagrams. On the left, $\mathscr{M}_0^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$ is the underlying category of a $\mathscr{V}$-category of $\mathscr{V}$-enriched presheaves.

We also have the full $\mathscr{V}$-subcategory $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ of $G\mathscr{V}$ whose objects are again the $\mathbf{I}[G/H]$ for $H \in \mathscr{F}$. Its hom objects in $\mathscr{V}$ are

$$\mathscr{V}\mathscr{O}_{\mathscr{F}}(\mathbf{I}[G/H], \mathbf{I}[G/K]) = \underline{G\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K]).$$

In context (C), $\mathscr{V}$ is cartesian monoidal and we use that its unit and terminal objects coincide to see that the $\mathscr{V}$-categories $\mathscr{O}_{\mathscr{F}}$ and $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ can be identified.

In context (A), where $\mathscr{V}$ is a general symmetric monoidal category, the $\mathscr{V}$-categories $\mathscr{O}_{\mathscr{F}}$ and $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ are often quite different. For example, when $\mathscr{V} = \mathscr{M}_k$,

$$k[\mathscr{O}_{\mathscr{F}}](k[G/H], k[G/K])] \cong k[(G/K)^H]$$

is quite different from

$$\underline{G\mathscr{M}_k}(k[G/H], k[G/K]) = \underline{k[G]}(k[G/H], k[G/K]) \cong (k[G/K])^H.$$

We have a $\mathscr{V}$-functor $\delta\colon \mathbf{I}[\mathscr{O}_{\mathscr{F}}] \longrightarrow \mathscr{V}\mathscr{O}_{\mathscr{F}}$. The maps

$$\delta\colon \mathbf{I}[\mathscr{O}_{\mathscr{F}}](I[G/H], I[G/K]) \longrightarrow \underline{\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K])$$

are the adjoints of the evaluation maps

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}](I[G/H], I[G/K]) \otimes I[G/H] \cong \mathbf{I}[\mathscr{O}_{\mathscr{F}}(G/H, G/K) \times G/H] \longrightarrow \mathbf{I}[G/K].$$

In context (C), δ may be viewed as an identification. In context (A), the maps δ of hom objects in $\mathscr{V}$ need not be weak equivalences.

We described the $\mathscr{F}$-model structure on $G\mathscr{M}$ in Theorem 6.12. For comparison, Theorem 2.12 generally gives level model structures on $\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$ and on $\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$, which we also call $\mathscr{F}$-model structures. Note that $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ is $\mathscr{V}$-Quillen (Definition 2.11) since its hom objects are cofibrant by our assumption that $\mathbf{I}$ is cofibrant, so that no extra goodness hypothesis is needed to apply Theorem 2.12 to $\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$. We agree to write X_H for the levels $X_{G/H}$ of a presheaf X in either of these presheaf categories. Using the levelwise model notions of Definition 2.11, we have the following result.

Theorem 6.15. *$\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$ and, if $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ is $\mathscr{M}$-good, $\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$ are cofibrantly generated $\mathscr{V}$-model categories. The sets $\mathscr{O}(G/H)\odot\mathcal{I}_{\mathscr{M}}$ and $\mathscr{O}(G/H)\odot\mathcal{J}_{\mathscr{M}}$ for $H \in \mathscr{F}$ are the generating cofibrations and generating acyclic cofibrations. If $\mathscr{M}$ is proper, so are these presheaf categories. The cofibrations of $\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$ and, if $\mathscr{V}\mathscr{O}_{\mathscr{F}}$ is $\mathscr{M}$-Quillen, the cofibrations of $\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$ are level cofibrations.*

In context (A), the presheaf in $\mathscr{V}$ represented by the object $I[G/H]$ has different interpretations in $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$ and in $\mathscr{V}\mathscr{O}_{\mathscr{F}}$. For $I[G/H]$ viewed as an object of $\mathbf{I}[\mathscr{O}_{\mathscr{F}}]$, the value of this presheaf on $I[G/K]$ is

$$\mathbf{I}[\mathscr{O}_{\mathscr{F}}(G/K, G/H)] = \mathbf{I}[(G/H)^K]. \tag{6.16}$$

For $I[G/H]$ viewed as an object of $\mathscr{V}\mathscr{O}_{\mathscr{F}}$, its value on $I[G/K]$ is

$$\underline{G\mathscr{V}}(\mathbf{I}[(G/K], \mathbf{I}[G/H]) \cong (\mathbf{I}[G/H])^K. \tag{6.17}$$

We agree to let $\mathbb{O}_{\mathscr{F}}(G/H)$ denote the second of these represented presheaves.

Assuming the hypotheses of Theorems 6.12 and 6.15, we have the $\mathscr{F}$-model categories $G\mathscr{M}$, $\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$, and $\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$. In context (C), we may identify the two presheaf categories. In context (A), the $\mathscr{V}$-functor δ induces a Quillen adjunction, not generally an equivalence, between them. It is $\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$ and not $\mathscr{M}^{\mathbf{I}[\mathscr{O}_{\mathscr{F}}]}$ that correctly models the $\mathscr{F}$-model structure on $G\mathscr{M}$.

Theorem 6.18. *There is a Quillen $\mathscr{V}$-adjunction*

$$\mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{M}$$

and it is a Quillen equivalence if the functors $(-)^H$ preserve the cotensors, coproducts, pushouts, and sequential colimits that appear in the construction of cell complexes.

Proof. We display the adjunction on underlying categories; on the enriched level, the adjunction is a comparison of equalizer diagrams. For $N \in G\mathscr{M}$, we define $\mathbb{U}(N)_H = N^H$. For $X \in \mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$, we define $\mathbb{T}X = X_e$. Using the canonical G-maps $G/e \longrightarrow G/H$, we easily check the expected adjunction. Since $\mathbb{U}$ creates the $\mathscr{F}$-equivalences and $\mathscr{F}$-fibrations in $G\mathscr{M}$, $(\mathbb{T},\mathbb{U})$ is clearly a Quillen adjunction, and it is a Quillen equivalence if and only if $\eta\colon X \longrightarrow \mathbb{U}\mathbb{T}X$ is a level equivalence when $X \in \mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$ is cofibrant. For a start, let $X = \mathbb{O}_{\mathscr{F}}(G/H) \odot M$, where $M \in \mathscr{M}$. Evaluated at G/e, this gives $\mathbf{I}[G/H] \odot M$, by (6.17). Now take K-fixed points. The assumption that $(-)^K$ preserves cotensors means that the result is $(\mathbf{I}[G/H])^K \odot M$. This agrees with X_K, and η is an isomorphism. Now the assumed commutation of passage to K-fixed points and the relevant colimits ensures that $\mathbb{U}$ maps relative cell complexes to relative cell complexes bijectively and that η is an isomorphism for any cell complex X, just as for topological spaces in Example 6.1. □

6.5. **Equivariant model categories and presheaf categories in $\mathscr{V}$.** Now that we understand equivariant model categories as presheaf categories in $\mathscr{M}$, we find that we can understand them as presheaf categories in $\mathscr{V}$ whenever we can understand $\mathscr{M}$ itself as a presheaf category in $\mathscr{V}$. That is, if we have an answer to Question 0.2 or Question 0.3 for $\mathscr{M}$, then we have an answer to the corresponding question with $\mathscr{M}$ replaced by $G\mathscr{M}$. This is an immediate application of the observation that a presheaf category in a presheaf category is again a presheaf category.

Proposition 6.19. *Let $\mathscr{D}$ and $\mathscr{E}$ be small $\mathscr{V}$-categories and let $\mathscr{N}$ be any $\mathscr{V}$-category. Then there is a canonical isomorphism of $\mathscr{V}$-categories*

$$(\mathscr{N}^{\mathscr{E}})^{\mathscr{D}} \cong \mathscr{N}^{\mathscr{D}\otimes\mathscr{E}}$$

If we have level $\mathscr{V}$-model structures on all presheaf categories in sight induced by a $\mathscr{V}$-model structure on $\mathscr{N}$, then this is an isomorphism of $\mathscr{V}$-model categories.

Proof. The isomorphism can be written $(X_d)_e = X_{d,e}$ on objects. That is, if X is given as a presheaf of either type, then the equality specifies a presheaf of the other type. It requires just a bit of thought to see how this works on the maps of enriched homs that specify the presheaf structure. One point is that for a presheaf X in $\mathscr{N}^{\mathscr{D}\otimes\mathscr{E}}$, we obtain maps

$$\mathscr{E}(e,e') \cong \mathbf{I} \otimes \mathscr{E}(e,e') \longrightarrow \mathscr{D}(d,d) \otimes \mathscr{E}(e,e') \xrightarrow{X} \mathscr{N}(X_{d,e'}, X_{d,e})$$

which specify a presheaf X_d in $\mathscr{N}^{\mathscr{E}}$. Comparisons of equalizers show that the hom objects in $\mathscr{V}$ between presheaves are also isomorphic. □

Now return to the equivariant context. Suppose we have a $\mathscr{V}$-model category $\mathscr{M}$ and a $\mathscr{V}$-functor $\delta\colon \mathscr{D} \longrightarrow \mathscr{M}$ that gives rise to a Quillen equivalence $\mathscr{M} \longrightarrow \mathscr{V}^{\mathscr{D}}$, as in Question 0.2 or Question 0.3. Retaining the assumptions of the previous subsection, for any family of subgroups $\mathscr{F}$ we also have a Quillen equivalence $G\mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}$. Composing, these give a composite Quillen equivalence

$$G\mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}} \longrightarrow (\mathscr{V}^{\mathscr{D}})^{\mathscr{V}\mathscr{O}_{\mathscr{F}}}.$$

Proposition 6.19 allows us to rewrite this, giving the following general conclusion.

Theorem 6.20. *The $\mathscr{F}$-model category $G\mathscr{M}$ is Quillen equivalent to the presheaf category $\mathscr{V}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}\otimes\mathscr{D}}$.*

The objects of $\mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D}$ are pairs $(\mathbf{I}[G/H], d)$, where $H \in \mathscr{F}$ and $d \in \mathscr{D}$. We have a canonical $\mathscr{V}$-functor $\tau\colon \mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D} \longrightarrow G\mathscr{M}$ that sends an object $(\mathbf{I}[G/H], d)$ to the object $\mathbf{I}[G/H] \odot \delta d$ of $G\mathscr{M}$. The maps of enriched hom objects are given by the tensor bifunctor

$$\odot\colon \underline{G\mathscr{V}}(\mathbf{I}[G/H], \mathbf{I}[G/K]) \otimes \mathscr{D}(d,e) \longrightarrow \underline{G\mathscr{M}}(\mathbf{I}[G/H] \odot \delta d, \mathbf{I}[G/K] \odot \delta e).$$

Let $\mathscr{F}\mathscr{D}$ denote the full $\mathscr{V}$-subcategory of $G\mathscr{M}$ whose objects are the $\mathbf{I}[G/H] \odot \delta d$ with $H \in \mathscr{F}$. Since τ lands in $\mathscr{F}\mathscr{D}$, it specifies a $\mathscr{V}$-functor

$$\tau\colon \mathscr{V}\mathscr{O}_{\mathscr{F}} \otimes \mathscr{D} \longrightarrow \mathscr{F}\mathscr{D}.$$

Even when δ is the inclusion of a full subcategory, it is unclear to us whether or not τ is a weak equivalence. In any case, this is an important example where the domain of the presheaf category that arises most naturally in answering Question 0.2 or Question 0.3 is not a full $\mathscr{V}$-subcategory. We have Quillen adjunctions of $\mathscr{F}$-model categories

$$\mathscr{V}^{\mathscr{V}\mathscr{O}_{\mathscr{F}}\otimes\mathscr{D}} \underset{\tau^*}{\overset{\tau_*}{\rightleftarrows}} \mathscr{V}^{\mathscr{F}\mathscr{D}} \quad \text{and} \quad \mathscr{V}^{\mathscr{F}\mathscr{D}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{M}.$$

A check of definitions using (6.10) shows that the composite Quillen adjunction is the Quillen equivalence of Theorem 6.20, but we do not know whether or not these Quillen adjunctions themselves can also be expected to be Quillen equivalences.

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