Introduction: On Some Properties of Motivic Cohomology

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In the 1960's, Alexander Grothendieck proposed the idea of a theory of motives. This theory should serve to unify the various geometric cohomology theories. To any variety X, one should be able to associate its motive $M(X)$, which carries all of the relevant cohomological information of X. Grothendieck was able, through the use of "correspondences", to define motives for smooth projective varieties–these are known as "pure" motives. For general X, one expects rather to get "mixed" motives. The philosophy of mixed motives, together with the general yoga of "weights", was due largely to Deligne (cf. [Del3]).

In the early 80's, Beilinson ([Beĭ]) and Lichtenbaum ([Lic]) conjectured the existence of "motivic" complexes of sheaves in the Zariski (resp. étale) topology subject to certain axioms. The cohomology theory obtained by taking hypercohomology with respect to these complexes, dubbed "motivic" cohomology, is thus subject to some axioms. Among these desired axioms are an Atiyah-Hirzebruch-type relation to algebraic K-theory as well as a relation to Milnor K-theory. Shortly thereafter, Bloch ([Blo]) defined his higher Chow groups and showed they satisfied many of the desired properties. In the 90's Suslin and Voevodsky introduced complexes of sheaves satisfying (most of) Beilinson's axioms and proved an identification with Bloch's higher Chow groups ([MVW], [Voe1]). Moreover, Morel and Voevodsky ([MV]) introduced a motivic, or $\mathbb{A}^1$, homotopy theory for varieties. The above motivic cohomology groups are represented in the Morel-Voevodsky homotopy category by maps into appropriate Eilenberg-Mac Lane objects, in analogy with the classical result in topology.

In this thesis, we study various properties of motivic cohomology. Any invertible function f on a variety X determines a cohomology class $[f] \in \mathrm{H}^{1,1}(X)$. A well-known result in K-theory, transported into motivic cohomology, says that $[f] \cdot [1 - f] = 0$; this is called the Steinberg relation. We study higher iterated products of these elements, i.e., Massey products. We show that these Massey products arise naturally in the construction of the motive associated to

$\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$. In this chapter, we are working with $\mathbb{Q}$-coefficients. In the next chapter, we establish a Poincaré duality isomorphism for smooth projective varieties (working with $\mathbb{Z}$-coefficients). Finally, in the last chapter we discuss Steenrod operations in motivic cohomology (with coefficients in $\mathbb{F}_p$). These were previously constructed by Voevodsky ([Voe3]). A more detailed synopsis follows.

Chapters 1 and 2 provide background. In Chapter 1 we introduce the motivic homotopy category of Morel and Voevosky. We also introduce several variants of motivic cohomology. Chapter 2 is intended mainly as background for Chapter 3. It includes a discussion of aspects of the theory of motives and in particular of mixed Tate motives.

In Chapter 3 we study the motive associated to the pronilpotent completion of $\mathbb{Q}[\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})]$. Recall that the Quillen-Sullivan theory of rational homotopy types states that for a nilpotent space, one can recover the rational homotopy type from the rational cochain algebra. If one starts with a non-nilpotent space, one recovers only the so-called nilpotent completion of the rational homotopy type. One should expect group rings of fundamental groups to be motivic since in topology one has the identification

$$\mathbb{Q}[\pi_1(X, x)] \cong \mathrm{H}_0(\Omega_x X; \mathbb{Q}),$$

where $\Omega_x X$ is the space of loops in X based at x. Let $X = \mathbb{P}^1_k - \{0, 1, \infty\}$, let $x \in X(k)$, and let $I \subseteq \mathbb{Q}[\pi_1(X, x)]$ denote the augmentation ideal. It is known ([Del1],[DG],[EL]), at least if k is a number field, how to consider $\mathbb{Q}[\pi_1(X, x)]/I^{n+1}$ as a mixed Tate motive, for $n \geq 0$. We follow Kriz and May ([KM], §IV) in defining rational mixed Tate motives to be a certain subcategory of the derived category $\mathscr{D}_{\mathcal{A}}$ of the Bloch cycle dga $\mathcal{A} = \mathcal{A}(\operatorname{Spec} k)$.

Actually, it will be more convenient to consider homotopy classes of paths rather than loops. Thus let $a, b \in X(k)$ be rational points of X. Let ${}_bP_a$ denote the set of homotopy classes of paths from a to b and note that it is a $\pi(X, b) - \pi_1(X, a)$-bimodule. Deligne and Goncharov ([DG],§3), following Beilinson, show that (the dual of) $\mathbb{Q}[{}_bP_a]/I^{n+1}$ is given by applying $\mathcal{A}(-)$ to the cobar construction on X and truncating:

$$\mathcal{A}(X^n) \to \mathcal{A}(a \times X^{n-1}) \oplus \mathcal{A}(\Delta_X \times X^{n-2}) \oplus \cdots \oplus \mathcal{A}(X^{n-1} \times b) \to \cdots \to \mathcal{A}^{\oplus^{n+1}}.$$

The maps in this complex are given by restriction with the approriate sign. We consider the question of producing an explicit cellular approximation to the totalization of this complex.

There are canonical choices for cellular approximations of each of the terms in the complex:

$$
\begin{array}{ccccccc}
\mathcal{A}(X^n) & \rightarrow & \mathcal{A}(a \times X^{n-1}) \oplus \mathcal{A}(\Delta_X \times X^{n-2}) \oplus \cdots \oplus \mathcal{A}(X^{n-1} \times b) & \rightarrow & \cdots & \rightarrow & \mathcal{A}^{\oplus n+1} \\
\uparrow \sim & & \uparrow \sim & & & & \| \\
C_n & \dashrightarrow & C_{n-1} & \dashrightarrow & \cdots & \dashrightarrow & C_0
\end{array}
$$

but the (horizontal) maps only have approximates up to homotopy. We are thus left with the problem of totalizing something which is only a complex up to homotopy. It is well-known that the obstruction to this totalization is captured by compositional Massey products (see Appendix B). Our main result in this chapter is a description of these compositional Massey products in terms of Massey products $\langle [f], [f], [1-f], \ldots, [f] \rangle$ in $\mathrm{H}^{2,*}(\operatorname{Spec} k; \mathbb{Q})$. This allows for an explicit description of the motive in question. We also include a discussion of these Massey products, including means of finding bounding cochains. The same Massey products, but in Deligne cohomology, were studied by T. Wenger in his thesis [Wen].

Chapter 4 discusses Atiyah duality in the motivic stable homotopy category and Poincaré duality in motivic cohomology. Classical Atiyah duality states that (the suspension spectrum of) a closed, oriented manifold is dualizable in the stable homotopy category with dual given by the Thom complex of the stable normal bundle. Poincaré duality is an easy corollary of this result. It is natural to ask for an analogue of this in the Morel-Voevodsky motivic stable homotopy category, stating that smooth projective varieties are dualizable.

A sketch of a proof of Atiyah duality due to Morel appeared in [Kah] and was fleshed out by Hu and Kriz in the appendix of [Hu]. On the other hand, one can also obtain Atiyah duality once one has set up a categorical framework for base change for motivic homotopy categories over a base scheme. This framework is claimed by Voevodsky in [Del2] and established by Ayoub in his thesis ([Ayo]). Röndigs then describes in [Rön], using some of the categorical considerations in [FHM], how Atiyah duality and Poincaré duality follow.

We give a proof of Atiyah duality for smooth projective varieties. It borrows much from the proof in [Hu], but the difference is that an explicit Poincaré-Thom map constructed by Voevodsky in [Voe2] is used. At present, our argument is complete only when the ground field contains an algebraically closed field and is not of characteristic 2, but we hope to remedy this in the future.

Finally, chapter 5 contains a discussion of Steenrod operations in motivic cohomology with $\mathbb{F}_p$ coefficients. Several constructions have appeared by now in the literature ([Bro], [Nie], [Voe3]). Our construction is more homotopical in nature.

We begin by recalling briefly Milgram's construction ([Gra], Ch. 27) of the Steenrod operations in topology. Let X be a space and suppose given $\alpha \in$

$H^{2n}(X;\mathbb{Z}/p)$ (we will take p to be odd; there is a similar story if $p=2$). We can think of α as represented in the homotopy category by a map

$$X \xrightarrow{\alpha} K(\mathbb{Z}/p, 2n).$$

The p-fold external power $\alpha \times \cdots \times \alpha$ of α is then given by

$$X^p \xrightarrow{\alpha\times\cdots\times\alpha} K(\mathbb{Z}/p, 2n)^n \xrightarrow{\mu} K(\mathbb{Z}/p, 2pn).$$

Since the cohomology ring $H^*(X;\mathbb{Z}/p)$ is graded-commutative, the above map commutes up to homotopy with the action of the symmetric group Σ_p which permutes the factors of X. In fact one can build a Σ_p-equivariant map

$$E\Sigma_p \times X^p \to K(\mathbb{Z}/p, 2pn), \tag{$\star$}$$

where Σ_p is acting trivially on $K(\mathbb{Z}/p, 2pn)$. Milgram's method is to use obstruction theory and an explicit model for $E\Sigma_p$ to build this map. Passing to the quotient yields a map

$$E\Sigma_p \times_{\Sigma_p} X^p \to K(\mathbb{Z}/p, 2pn).$$

Finally, pulling back along the diagonal of X^p gives a map

$$B\Sigma_p \times X \cong E\Sigma_p \times_{\Sigma_p} X \xrightarrow{P(\alpha)} K(\mathbb{Z}/p, 2pn).$$

Using the calculation

$$H^*(B\Sigma_p;\mathbb{F}_p) \cong \mathbb{F}_p[u] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(v)$$

with $|u| = 2(p-1)$ and $|v| = 2(p-1)-1$, we have that $P(\alpha)$ is a polynomial in u and v with coefficients in $H^*(X;\mathbb{Z}/p)$. One defines the Steenrod operations $P^i(\alpha)$ in terms of the coefficients of this polynomial $P(\alpha)$.

A small reduction is given by the observation that it suffices to produce the map $P(\alpha)$ for α of the form $\iota_n \in H^n(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$. In addition, an alternative method of producing the map ($\star$) is to show that the space of all maps

$$K(\mathbb{Z}/p, n)^p \to K(\mathbb{Z}/p, pn)$$

of the desired form is contractible. This was done by Segal in [Seg]; this allowed him to put a multiplicative infinite loop space structure on $\prod_{n>0} K(\mathbb{Z}, 2n)$, thus making it into an E_∞-ring space.

It turns out that much of the above discussion has a direct analogue in the Morel-Voevodsky $\mathbb{A}^1$-homotopy category of smooth schemes over a perfect base field k. There are objects $K(A,m,n)$ which represent the motivic cohomology groups $\mathrm{H}^{m,n}(-;A)$ and which will play exactly the same role as above in producing the operations. We will mimic Segal's proof to show that a certain space of maps is contractible; since Segal's argument relies on the fact that $\Omega K(A,n) \simeq K(A,n-1)$, the argument in the motivic setting will involve Voevodsky's Cancellation Theorem ([Voe4]). Really, what we will show is that there is an acyclic operad that acts on the product $\prod_n K(\mathbb{Z},2n,n)$.

There is a classifying object $B_{gm}\Sigma_p$ in the $\mathbb{A}^1$-homotopy category whose motivic cohomology has been computed ([Voe3], [DI]): if $\mathbb{M}_p = \mathrm{H}^{*,*}(\mathrm{Spec}(k);\mathbb{F}_p)$ denotes the bigraded motivic cohomology ring of a point, then the formula is given by

$$\mathrm{H}^{*,*}(B_{gm}\Sigma_p;\mathbb{F}_p) \cong \mathbb{M}_p[u] \otimes_{\mathbb{M}_p} \Lambda_{\mathbb{M}_p}(v)$$

with $|u| = (2(p-1), p-1)$ and $|v| = (2(p-1)-1, p-1)$ if $p \neq \mathrm{char}(k)$ and p is odd (when $p=2$, the formula is more complicated, and this in turn complicates the Cartan formula and Adem relations). The operad action will produce for us a map

$$B\Sigma_p \times K(\mathbb{Z}/p, 2n, n) \to K(\mathbb{Z}/p, 2pn, pn),$$

but unfortunately the classifying space $B\Sigma_p$ that appears in the domain is the more naive object rather than the geometric object mentioned above. This, in turn, means that the cohomology operations that are produced by this method do not agree dimensionally with those of Voevodsky. Nevertheless, we have some hope that a modification of our construction will produce the correct operations.

We should also mention that the recipe for constructing Steenrod operations described here does, when carried out in the classical setting, give a construction of the classical Steenrod operations. Moreover, it gives conceptual proofs of the Cartan formula and Adem relations.

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