PERMUTATIVE G-CATEGORIES IN EQUIVARIANT INFINITE LOOP SPACE THEORY

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ABSTRACT. We explain what genuine permutative G-categories are and, more generally, what E_∞ G-categories are. We give examples showing how they arise, and we show that they serve as input for an equivariant infinite loop space machine. As a first application, we prove the equivariant Barratt-Priddy-Quillen theorem as a statement about genuine G-spectra and use it to give a new, categorical, proof of the tom Dieck splitting theorem for suspension G-spectra. Other examples are geared towards equivariant algebraic K-theory.

CONTENTS

Introduction

Let G be a finite group. We shall develop equivariant infinite loop space theory in a series of papers.[1] In this one, we explain what genuine permutative G-categories are and what E_∞ G-categories are, and we explain how to construct a genuine G-spectrum $\mathbb{K}_G\mathscr{A}$ from a genuine permutative G-category $\mathscr{A}$ or, more generally, from an E_∞ G-category. We use this theory to prove the equivariant Barratt-Priddy-Quillen theorem and to reprove the tom Dieck splitting theorem for suspension G-spectra, starting from categorical data. We also explain how to construct a natural comparison map $\mathbb{K}_G\mathscr{A} \wedge \mathbb{K}_G\mathscr{B} \longrightarrow \mathbb{K}_G(\mathscr{A} \times \mathscr{B})$, that being a small step towards a multiplicative extension of the additive recognition principle for G-spectra.

This material allows us to complete the proofs from equivariant infinite loop space theory promised in [11], where we described the category of G-spectra as an easily understood category of spectral presheaves. We use the second author's approach [21] to infinite loop space theory in this paper, but we could instead use Segal's [35]. In one sequel [2], we will give a multiplicative extension of the theory here that will show how to pass from multicategorical input to genuine G-spectra output. That is needed, for example, in applications to equivariant algebraic K-theory. Elsewhere we plan to study equivariant infinite loop space theory model theoretically and to give a highly structured comparison of the May and Segal approaches to infinite loop space theory, equivariantly and non-equivariantly.

After recalling preliminaries about equivariant universal bundles and equivariant E_∞ operads in §1, we give operadic definitions of naive and genuine permutative G-categories in §2. The latter are defined in terms of a *particular* E_∞ operad $\mathscr{O}_G$ of G-categories. We define an E_∞ G-category to be an algebra over *any* E_∞ operad $\mathscr{P}_G$ of G-categories. Of course, this is analogous to the distinction between

[1] At a later date, these may be collated into fewer documents.

topological monoids, which are algebras over a particular A_∞-operad, and A_∞ spaces, which are algebras over any A_∞ operad.

In the brief and parenthetical §2.5, we point out how these ideas and our prequel with Merling [13] specialize to give a starting point for equivariant algebraic K-theory [7, 10, 16, 32]. We give an alternative and equivalent starting point in the case of G-rings R in §8.2.

We give a precise description of the G-fixed E_∞ categories of free $\mathscr{O}_G$-categories in §3. This is a categorical precursor of the equivariant Barratt-Priddy-Quillen (BPQ) theorem, which we prove in §5.1, and the tom Dieck splitting theorem for suspension G-spectra, which we reprove in §5.2. The proofs are based on use of an equivariant infinite loop space machine that is reviewed in §4. We show how to construct the map $\mathbb{K}_G\mathscr{A} \wedge \mathbb{K}_G\mathscr{B} \longrightarrow \mathbb{K}_G(\mathscr{A} \times \mathscr{B})$ and prove some naturality properties of the BPQ theorem and the tom Dieck splitting in §6. These results complete the proofs promised in [11], as we clarify in §6.3.

Changing focus, in in §7 and §8 we give three interrelated examples of E_∞ G-operads, denoted $\mathscr{P}_G$, $\mathscr{Q}_G$, and $\mathscr{R}_G$, and give examples of their algebras. This approach to examples is more intuitive than the approach based on genuine permutative G-categories, and it has some technical advantages. It is new and illuminating even nonequivariantly. It gives a more intuitive categorical hold on the BPQ theorem than does the treatment starting from genuine permutative G-categories, as we explain in §9.3. It also gives a new starting point for multiplicative infinite loop space theory, both equivariantly and nonequivariantly, but that is work in progress.

We emphasize that this paper is full of open ends. Given that much of equivariant infinite loop space theory has been at least partially understood for three decades, it is surprising how very underdeveloped the subject remains.

Notational preliminaries. A dichotomy between Hom objects with G-actions, denoted using a subscript G, and Hom objects of equivariant morphisms, often denoted using a G in front, is omnipresent. We start with an underlying category $\mathscr{V}$. A G-object X in $\mathscr{V}$ can be defined to be a group homomorphism $G \longrightarrow \mathrm{Aut}\, X$. We have the category $\mathscr{V}_G$ of G-objects in $\mathscr{V}$ and all morphisms in $\mathscr{V}$ between them, with G acting by conjugation. We also have the category $G\mathscr{V}$ of G-objects in $\mathscr{V}$ and G-maps in $\mathscr{V}$. Since objects are fixed by G, $G\mathscr{V}$ is in fact the G-fixed category $(\mathscr{V}_G)^G$, although we shall not use that notation. However, we shall use the notations $G\mathscr{V}(X,Y)$ and $\mathscr{V}_G(X,Y)^G$ interchangeably for the hom object in $\mathscr{V}$ of G-morphisms in $\mathscr{V}$ between G-objects X and Y.

Our most frequently used choices of $\mathscr{V}$ are $\mathscr{C}at$ and $\mathscr{U}$, the category of (small) topological categories and the category of unbased (compactly generated) spaces. We let $\mathscr{T}$ denote the category of based spaces. We assume once and for all that the base points $*$ of all given based G-spaces X (or spaces X when $G = e$) are nondegenerate. This means that $* \longrightarrow X$ is a cofibration (satisfies the G-HEP). Then $* \longrightarrow X^H$ is a cofibration for all $H \subset G$.

For brevity of notation, we shall often write $|-|$ for the composite classifying space functor $B = |N-|$ from topological categories through simplicial spaces to spaces. It works equally well to construct G-spaces from topological G-categories (categories on which G acts continuously). We assume that the reader is familiar with operads (as originally defined in [21]) and especially with the fact that operads can be defined in any symmmetic monoidal category $\mathscr{V}$. Brief modernized expositions are given in [26, 27]. Since it is product-preserving, the functor $|-|$

takes operads in $\mathscr{C}at$ or in $G\mathscr{C}at$ to operads in $\mathscr{U}$ or in $G\mathscr{U}$, and it takes algebras over operads $\mathscr{C}$ in $\mathscr{C}at$ or in $G\mathscr{C}at$ to algebras over $|\mathscr{C}|$ in $\mathscr{U}$ or in $G\mathscr{U}$.

To avoid proliferation of letters, we shall write $\mathbb{P}_G$ for the monad on based G-categories constructed from an operad $\mathscr{P}_G$ of G-categories. We shall write $\mathbf{P}_G$ for the monad on based G-spaces constructed from the operad $|\mathscr{P}_G|$ of G-spaces. More generally, for an operad $\mathscr{C}_G$ of unbased G-spaces, we write $\mathbf{C}_G$ for the associated monad on based G-spaces.

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1. Preliminaries on universal equivariant bundles and operads

We describe an elementary categorical functor $\mathscr{C}at_G(\tilde{G}, -)$ and use it to define a certain operad $\mathscr{O}_G$ of G-categories. The $\mathscr{O}_G$-algebras will be the genuine permutative G-categories. With Merling, we have explored this functor in detail in the context of equivariant bundle theory [13], and we refer the reader there for proofs.

We shall say nothing about equivariant bundle theory here, except to note the following parallel. In [21], an operad $\mathscr{C}$ of spaces was defined to be an E_∞ operad if $\mathscr{C}(j)$ is a free contractible Σ_j-space. Effectively, $\mathscr{C}(j)$ is then a universal principal Σ_j-bundle. Equivariantly, we have an analogous picture: E_∞ operads $\mathscr{C}_G$ of G-spaces are defined so that the $\mathscr{C}_G(j)$ are universal principal (G, Σ_j)-bundles. That dictates the appropriate homotopical properties of the $\mathscr{C}_G(j)$, and it is only those homotopical properties and not their bundle theoretic consequences that concern us in the theory of operads. The bundle theory implicitly tells us which homotopical properties are relevant to equivariant infinite loop space theory.

1.1. **Chaotic topological categories.** For (small) categories $\mathscr{A}$ and $\mathscr{B}$, we let $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ denote the category whose objects are the functors $\mathscr{A} \longrightarrow \mathscr{B}$ and whose morphisms are the natural transformations between them. When $\mathscr{B}$ is a right Π-category, its Π-action induces a right Π-action on $\mathscr{C}at(\mathscr{A}, \mathscr{B})$, which we take for granted and do not indicate notationally. When a group G acts from the left on $\mathscr{A}$ and $\mathscr{B}$, we let $\mathscr{C}at_G(\mathscr{A}, \mathscr{B})$ denote $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ with its left G-action by conjugation on objects and morphisms. Then $G\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is alternative notation for the G-fixed category $\mathscr{C}at_G(\mathscr{A}, \mathscr{B})^G$ of G-functors and G-natural transformations. We have the standard (G-equivariant) adjunction

$$\mathscr{C}at_G(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathscr{C}at_G(\mathscr{A}, \mathscr{C}at_G(\mathscr{B}, \mathscr{C})). \tag{1.1}$$

Definition 1.2. For a space X, the chaotic (topological) category $\tilde{X}$ has object space X, morphism space $X \times X$, and structure maps I, S, T, and C given by $I(x) = (x, x)$, $S(y, x) = x$, $T(y, x) = y$, and $C((z, y), (y, x)) = (z, x)$. For any point $* \in X$, the map $\eta \colon X \longrightarrow X \times X$ specified by $\eta(x) = (*, x)$ is a continuous natural isomorphism from the identity functor to the trivial functor $\tilde{X} \longrightarrow *$, hence $\tilde{X}$ is equivalent to $*$. When $X = G$ is a topological group, $\tilde{G}$ is isomorphic to the translation category of G, but the isomorphism encodes information about the group action; see [13, 1.7]. We say that a topological category with object space X is chaotic if it is isomorphic to $\tilde{X}$.

Definition 1.3. For a topological group Π, let $\mathbf{\Pi}$ denote Π regarded as a category with a single object and observe that $\mathbf{\Pi}$ is isomorphic to the orbit category $\tilde{\Pi}/\Pi$, where Π acts from the right on $\tilde{\Pi}$ via right multiplication on objects and diagonal right multiplication on morphisms. The resulting functor $p\colon \tilde{\Pi} \longrightarrow \mathbf{\Pi}$ is given by the trivial map $\Pi \longrightarrow *$ of object spaces and the map $p\colon \Pi\times\Pi \longrightarrow \Pi\times\Pi/\Pi \cong \Pi$ on morphism spaces specified by $p(\tau,\sigma) = \tau\sigma^{-1}$.

Theorem 1.4. [13, 2.7] *For a G-space X and a topological group Π, regarded as a G-trivial G-space, the functor $p\colon \tilde{\Pi} \longrightarrow \mathbf{\Pi}$ induces an isomorphism of topological G-categories*

$$\xi\colon \mathscr{C}at_G(\tilde{X},\tilde{\Pi})/\Pi \longrightarrow \mathscr{C}at_G(\tilde{X},\mathbf{\Pi}).$$

Therefore, passing to G-fixed point categories,

$$(\mathscr{C}at_G(\tilde{X},\tilde{\Pi})/\Pi)^G \cong \mathscr{C}at_G(\tilde{X},\mathbf{\Pi})^G \cong \mathscr{C}at(\tilde{X}/G,\mathbf{\Pi}).$$

The last isomorphism is clear since G acts trivially on Π. Situations where G is allowed to act non-trivially on Π are of considerable interest, as we shall see, but will only appear peripherally in this paper until §2.5. The paper [13] works throughout in that more general context. The previous result will not be used directly, but it is the key underpinning for the results of the next section.

1.2. **The functor $\mathscr{C}at_G(\tilde{G},-)$.** The functor $\mathscr{C}at_G(\tilde{G},-)$ from G-categories to G-categories is a right adjoint (1.1), hence it preserves limits and in particular products. The projection $\tilde{G} \longrightarrow *$, where $*$ is the trivial G-category, induces a natural map

$$\iota\colon \mathscr{A} = \mathscr{C}at_G(*,\mathscr{A}) \longrightarrow \mathscr{C}at_G(\tilde{G},\mathscr{A}). \tag{1.5}$$

The map ι is not an equivalence of G-categories in general [13, 4.19] but the functor $\mathscr{C}at_G(\tilde{G},-)$ is idempotent in the sense that the following result holds.

Lemma 1.6. *For any G-category $\mathscr{A}$,*

$$\iota\colon \mathscr{C}at_G(\tilde{G},\mathscr{A}) \longrightarrow \mathscr{C}at_G(\tilde{G},\mathscr{C}at_G(\tilde{G},\mathscr{A}))$$

is an equivalence of G-categories.

Proof. This follows from the adjunction (1.1) using that the diagonal $\tilde{G} \longrightarrow \tilde{G}\times\tilde{G}$ is a G-equivalence with inverse given by either projection and that the specialization of ι here is induced by the first projection. □

Assumption 1.7. We assume for now that Π is either a compact Lie group or a discrete group and that G is a discrete group. Later both Π and G will be finite.

Lemma 1.8. [13, 3.7] *Let Λ be a subgroup of $\Pi\times G$. The Λ-fixed category $\mathscr{C}at_G(\tilde{G},\tilde{\Pi})^\Lambda$ is empty if $\Lambda\cap\Pi \neq e$. If $\Lambda\cap\Pi = e$, let $H = q(\Lambda)$, where $q\colon \Pi\times G \longrightarrow G$ is the projection, and define $\alpha\colon H \longrightarrow \Pi$ by letting $\alpha(h)$ be the unique element of Π such that $(\alpha(h),h)\in\Lambda$. Then α is a homomorphism, $\Lambda = \Lambda_\alpha \equiv \{(\alpha(h),h)|h\in H\}$, and $\mathscr{C}at_G(\tilde{G},\tilde{\Pi})^\Lambda$ is nonempty and chaotic.*

Definition 1.9. Let $\mathscr{R}(H,\Pi)$ be the category with objects the homomorphisms $\alpha\colon H \longrightarrow \Pi$ and morphisms $\sigma\colon \alpha \longrightarrow \beta$ the conjugacy relations $\beta = \sigma\alpha\sigma^{-1}$, where $\sigma\in\Pi$. Let $\mathscr{R}^-(H,\Pi)$ be the category with objects the anti-homomorphisms $\alpha\colon H \longrightarrow \Pi$ and morphisms $\sigma\colon \alpha \longrightarrow \beta$ the conjugacy relations $\beta = \sigma\alpha\sigma^{-1}$.

These categories are isomorphic [13, 4.13], but $\mathscr{R}^-(H,\Pi)$ appears naturally.

Theorem 1.10. [13, 4.14, 4.18] *The G-fixed category $\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})^G$ is isomorphic to $\mathscr{R}^-(G,\Pi)$ and therefore to $\mathscr{R}(G,\Pi)$. For $H\subset G$, $\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})^H$ is equivalent to $\mathscr{C}at_H(\tilde{H},\mathbf{\Pi})^H$ and therefore to $\mathscr{R}(H,\Pi)$. In turn, $\mathscr{R}(H,\Pi)$ is equivalent to the coproduct of the categories $\mathbf{Aut}(\alpha)$, where the coproduct runs over $[\alpha]\in H^1(H;\Pi)$.*

Lemma 1.11. [13, §§4.3, 4.4] *The set $H^1(H;\Pi)$ is the set of Π-conjugacy classes of homomorphisms $\alpha\colon H\longrightarrow\Pi$, and it is in bijective correspondence with the set of Π-conjugacy classes of subgroups Λ of $\Pi\times G$ such that $\Lambda\cap\Pi=e$ and $q(\Lambda)=H$. The group $\mathrm{Aut}(\alpha)$ is the centralizer Π^α of α in Π, namely*

$$\Pi^\alpha=\{\sigma\in\Pi|\alpha(h)=\sigma\alpha(h)\sigma^{-1}\ \textit{for}\ h\in H\},$$

and it is equal to the intersection $\Pi\cap N_{\Pi\times G}\Lambda_\alpha$.

Definition 1.12. Define $E(G,\Pi)=|\mathscr{C}at_G(\tilde{G},\tilde{\Pi})|$ and $B(G,\Pi)=|\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})|$. Let $p\colon E(G,\Pi)\longrightarrow B(G,\Pi)$ be induced by the passage to orbits functor $\tilde{\Pi}\longrightarrow\mathbf{\Pi}$.

Theorem 1.13. [13, 3.11] *The map $p\colon E(G,\Pi)\longrightarrow B(G,\Pi)$ is a universal principal (G,Π)-bundle.*

Theorem 1.14. [13, 4.23, 4.24] *For a subgroup H of G,*

$$B(G,\Pi)^H\simeq\coprod B(\Pi^\alpha),$$

where the union runs over $[\alpha]\in H^1(H;\Pi)$.

1.3. **Equivariant E_∞ operads.** Since operads make sense in any symmetric monoidal category, we have operads of categories, spaces, G-categories, and G-spaces. Operads in $G\mathscr{U}$ were first used in [18, VII]. Although we are only interested in finite groups G in this paper, the following definition makes sense for any topological group G and is of interest in at least the generality of compact Lie groups.

Definition 1.15. An E_∞ operad $\mathscr{C}_G$ of G-spaces is an operad in the cartesian monoidal category $G\mathscr{U}$ such that $\mathscr{C}_G(0)$ is a contractible G-space and the $(\Sigma_j\times G)$-space $\mathscr{C}_G(j)$ is a universal principal (G,Σ_j)-bundle for each $j\geq 1$. Equivalently, for a subgroup Λ of $\Sigma_j\times G$, the Λ-fixed point space $\mathscr{C}_G(j)^\Lambda$ is contractible if $\Lambda\cap\Sigma_j=\{e\}$ and is empty otherwise. We say that $\mathscr{C}_G$ is reduced if $\mathscr{C}_G(0)$ is a point. An E_∞ operad $\mathscr{P}_G$ of (topological) G-categories is an operad in the cartesian monoidal category $G\mathscr{C}at$ such that $|\mathscr{P}_G|$ is an E_∞ operad of G-spaces. We say that $\mathscr{P}_G$ is reduced if $\mathscr{P}_G(0)$ is the trivial category. In practice, the $\mathscr{P}_G(j)$ are groupoids, hence are contractible as categories.

As is usual in equivariant bundle theory, we think of G as acting from the left and Σ_j as acting from the right on the spaces $\mathscr{C}_G(j)$ and categories $\mathscr{P}_G(j)$. These actions must commute and so define an action of $\Sigma_j\times G$.

Remark 1.16. We will encounter one naturally occuring operad that is not reduced. When an operad $\mathscr{C}$ acts on a space X via maps θ_i and we choose points $c_i\in\mathscr{C}(i)$, we have a map $\theta_0\colon\mathscr{C}(0)\longrightarrow X$ and the relation

$$\theta_2(c_2;\theta_0(c_0),\theta_1(c_1,x))=\theta_1(\gamma(c_2;c_0,c_1),x)$$

for $x\in X$. When the $\mathscr{C}(i)$ are connected, this says that $\theta_0(c_0)$ is a unit element for the product determined by c_2. Reduced operads give a single unit element. The original definition [21, 1.1] required operads to be reduced.

Lemma 1.17. *If $\mathscr{C}_G$ is an E_∞ operad of G-spaces, then $\mathscr{C} = (\mathscr{C}_G)^G$ is an E_∞ operad of spaces. If Y is a $\mathscr{C}_G$-space, then Y^G is a $\mathscr{C}$-space. Similarly, if $\mathscr{P}_G$ is an E_∞ operad of G-categories, then $\mathscr{P} = (\mathscr{P}_G)^G$ is an E_∞ operad of categories. If $\mathscr{A}$ is a $\mathscr{P}_G$-category, then $\mathscr{A}^G$ is a $\mathscr{P}$-category.*

Proof. $(\mathscr{C}_G)^G$ is an operad since the fixed point functor commutes with products, and it is an E_∞ operad since the space $\mathscr{C}_G(j)^G$ is contractible and Σ_j-free. The categorical analogue works the same way. □

2. Categorical philosophy: what is a permutative G-category?

2.1. **Naive permutative G-categories.** We have a notion of a monoidal category $\mathscr{A}$ internal to a cartesian monoidal category $\mathscr{V}$. It is a category internal to $\mathscr{V}$ together with a coherently associative and unital product functor $\mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}$. It is strict monoidal if the product is strictly associative and unital. It is symmetric monoidal if it has an equivariant symmetry isomorphism $\gamma\colon \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A} \times \mathscr{A}$ satisfying the usual coherence properties.

A permutative category is a symmetric strict monoidal category.[2] Taking $\mathscr{V}$ to be $\mathscr{U}$, these are the topological permutative categories. Taking $\mathscr{V}$ to be $G\mathscr{U}$, these are the *naive* topological permutative G-categories.

Nonequivariantly, there is a standard E_∞ operad of spaces that is obtained by applying the classifying space functor to an E_∞ operad $\mathscr{O}$ of categories. The following definition goes back to Barratt and Eccles, thought of simplicially [3], and to [22], thought of categorically.

Definition 2.1. We define an E_∞ operad $\mathscr{O}$ of categories. Let $\mathscr{O}(j) = \tilde{\Sigma}_j$. Since Σ_j acts freely and $\tilde{\Sigma}_j$ is chaotic, the classifying space $|\mathscr{O}(j)|$ is Σ_j-free and contractible, as required of an E_∞ operad. The structure maps

$$\gamma\colon \tilde{\Sigma}_k \times \tilde{\Sigma}_{j_1} \times \cdots \times \tilde{\Sigma}_{j_k} \longrightarrow \tilde{\Sigma}_j,$$

where $j = j_1 + \cdots + j_k$, are dictated on objects by the definition of an operad. If we view the object sets of the $\mathscr{O}(j)$ as discrete categories (identity morphisms only), then they form the associativity operad $\mathscr{M}$.

We can define $\mathscr{M}$-algebras and $\mathscr{O}$-algebras in $\mathscr{C}at$ or in $G\mathscr{C}at$. In the latter case, we regard $\mathscr{M}$ and $\mathscr{O}$ as operads with trivial G-action. The following result characterizes naive permutative G-categories operadically. The proof is easy [22].

Proposition 2.2. *The category of strict monoidal G-categories and strict monoidal G-functors is isomorphic to the category of $\mathscr{M}$-algebras in $G\mathscr{C}at$. The category of naive permutative G-categories and strict symmetric monoidal G-functors is isomorphic to the category of $\mathscr{O}$-algebras in $G\mathscr{C}at$.*

The term "naive" is appropriate since naive permutative G-categories give rise to naive G-spectra on application of an infinite loop space machine. Genuine permutative G-categories need more structure, especially precursors of transfer maps, to give rise to genuine G-spectra. Nonequivariantly, there is no distinction.

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[2] In interesting examples, the product cannot be strictly commutative.

2.2. **Genuine permutative G-categories.**

Definition 2.3. Let $\mathscr{O}_G = \mathscr{C}at_G(\tilde{G}, \mathscr{O})$ be the (reduced) operad of G-categories whose j^{th} G-category is $\mathscr{O}_G(j) = \mathscr{C}at_G(\tilde{G}, \mathscr{O}(j))$, where $\mathscr{O}(j) = \tilde{\Sigma}_j$ is viewed as a G-category with trivial G-action and is given its usual right Σ_j-action. The unit in $\mathscr{O}_G(1)$ is the unique functor from $\tilde{G}$ to the trivial category $\mathscr{O}(1) = \mathscr{O}_G(1)$. The structure maps γ of $\mathscr{O}_G$ are induced from those of $\mathscr{O}$, using that the functor $\mathscr{C}at(\tilde{G}, -)$ preserves products. By Theorem 1.13, $\mathscr{O}_G$ is an E_∞ operad of G-categories. The natural map $\iota\colon \mathscr{A} \longrightarrow \mathscr{C}at_G(\tilde{G}, \mathscr{A})$ of (1.5) induces an inclusion $\iota\colon \mathscr{O} \longrightarrow \mathscr{O}_G$ of operads of G-categories, where G acts trivially on $\mathscr{O}$.

Definition 2.4. A *genuine* permutative G-category is an $\mathscr{O}_G$-algebra in $G\mathscr{C}at$.

We generally call these $\mathscr{O}_G$-categories. We have an immediate source of examples. Let ι^* be the functor from genuine permutative G-categories to naive permutative G-categories that is obtained by restricting actions by $\mathscr{O}_G$ to its suboperad $\mathscr{O}$.

Proposition 2.5. *The action of $\mathscr{O}$ on a naive permutative G-category $\mathscr{B}$ induces an action of $\mathscr{O}_G$ on $\mathscr{C}at_G(\tilde{G}, \mathscr{B})$, hence $\mathscr{C}at_G(\tilde{G}, -)$ restricts to a functor, denoted $\mathscr{G}$, from naive permutative G-categories to genuine permutative G-categories. The natural map ι of (1.5) restricts to a natural map $\mathscr{B} \longrightarrow \iota^*\mathscr{G}\mathscr{B}$ of naive permutative G-categories, and ι is an equivalence when $\mathscr{B} = \iota^*\mathscr{G}\mathscr{A}$ for an $\mathscr{O}$-category $\mathscr{A}$.*

Proof. Since the functor $\mathscr{C}at_G(\tilde{G}, -)$ preserves products and ι is induced by the projection $\tilde{G} \longrightarrow \tilde{e} = *$, the first two statements are clear by inspection. The last statement holds by Lemma 1.6. □

One might hope that $(\mathscr{G}, \iota^*)$ is an adjoint pair, but the left adjoint to ι^* can be described as a coend $\mathscr{O}_G \otimes_{\mathscr{O}} \mathscr{B}$. That maps naturally to $\mathscr{G}$, but the map is not an isomorphism.

The $\mathscr{O}_G$-categories of interest in this paper are of the form $\mathscr{G}\mathscr{A}$ for a naive permutative G-category $\mathscr{A}$. In fact, we do not know how to construct other examples.

Remark 2.6. As noted before, the map $\iota\colon \mathscr{A} \longrightarrow \iota^*\mathscr{G}\mathscr{A}$ is not a G-equivalence in general [13, 4.19], although it is so when $\mathscr{A} = \iota^*\mathscr{G}\mathscr{B}$. We shall relate ι^* to the forgetful functor i^* from genuine G-spectra to naive G-spectra in §4.4 below.

2.3. **E_∞ G-categories.** We can generalize the notion of a genuine permutative G-category by allowing the use of E_∞ operads other than $\mathscr{O}_G$. In fact, thinking as algebraic topologists rather than category theorists, there is no need to give the particular E_∞ operad $\mathscr{O}_G$ a privileged role.

Definition 2.7. An E_∞ G-category $\mathscr{A}$ is a G-category together with an action of some E_∞ operad $\mathscr{P}_G$ of G-categories. The classifying space $B\mathscr{A} = |\mathscr{A}|$ is then a $|\mathscr{P}_G|$-space and thus an E_∞ G-space.

We may think of E_∞ G-categories as generalized kinds of genuine permutative G-categories. The point of the generalization is that we have interesting examples of E_∞ operads of G-categories with easily recognizable algebras. We shall later define E_∞ operads $\mathscr{P}_G$, $\mathscr{Q}_G$, and $\mathscr{R}_G$ that are interrelated in a way that illuminates the study of multiplicative structures.

Observe that $\mathscr{O}_G$-algebras, like nonequivariant permutative categories, have a canonical product, whereas E_∞ G-categories over other operads do not. The general philosophy of operad theory is that algebras over an operad $\mathscr{C}$ in any suitable category $\mathscr{V}$ have j-fold operations parametrized by the objects $\mathscr{C}(j)$. Homotopical properties of $\mathscr{C}$ relate these operations. In general, in an E_∞ space, there is no preferred choice of a product on its underlying H-space, and none is relevant to the applications; E_∞ categories work similarly.

Up to homotopy, any two choices of E_∞ operads give rise to equivalent categories of E_∞ G-spaces. To see that, we apply the trick from [21] of using products of operads to transport operadic algebras from one E_∞-operad to another. The product of operads $\mathscr{C}$ and $\mathscr{D}$ in any cartesian monoidal category $\mathscr{V}$ is given by

$$(\mathscr{C} \times \mathscr{D})(j) = \mathscr{C}(j) \times \mathscr{D}(j),$$

with the evident permutations and structure maps. With the choices of $\mathscr{V}$ of interest to us, the product of E_∞ operads is an E_∞ operad. The projections

$$\mathscr{C} \longleftarrow \mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{D}$$

allow us to construct $(\mathscr{C} \times \mathscr{D})$-algebras in $\mathscr{V}$ from either $\mathscr{C}$-algebras or $\mathscr{D}$-algebras in $\mathscr{V}$, by pullback of action maps along the projections.

More generally, for any map $\mu\colon \mathscr{C} \longrightarrow \mathscr{D}$ of operads in $\mathscr{V}$, the pullback functor μ^* from $\mathscr{D}$-algebras to $\mathscr{C}$-algebras has a left adjoint pushforward functor $\mu_!$ from $\mathscr{C}$-algebras to $\mathscr{D}$-algebras. One can work out a homotopical comparison model categorically[3]. Pragmatically, use of the two-sided bar construction as in [21, 28] gives all that is needed. One redefines $\mu_! X = B(\mathbb{D}, \mathbb{C}, X)$, where $\mathbb{C}$ and $\mathbb{D}$ are the monads whose algebras are the $\mathscr{C}$-algebras and $\mathscr{D}$-algebras.[4] In spaces, or equally well G-spaces, μ^* and $\mu_!$ give inverse equivalences of homotopy categories between $\mathscr{C}$-algebras and $\mathscr{D}$-algebras when $\mathscr{C}$ and $\mathscr{D}$ are E_∞-operads.

Starting with operads in $\mathscr{C}at$ or in $G\mathscr{C}at$ we can first apply the classifying space functor and then apply this trick. The conclusion is that all E_∞ categories and E_∞ G-categories give equivalent inputs for infinite loop space machines. In particular, for example, letting $\mathbf{O}_G$, $\mathbf{P}_G$, and $\mathbf{O_G} \times \mathbf{P_G}$ denote the monads in the category of G-spaces whose algebras are $|\mathscr{P}_G|$-algebras, $|\mathscr{O}_G|$-algebras, and $|\mathscr{O}_G \times \mathscr{P}_G|$-algebras, we see that after passage to classifying spaces, every $\mathbf{P}_G$-algebra Y determines an $\mathbf{O}_G$-algebra $X = B(\mathbf{O}_G, \mathbf{O_G} \times \mathbf{P_G}, Y)$ such that X and Y are weakly equivalent as $(\mathbf{O_G} \times \mathbf{P_G})$-algebras (and conversely). This says that for purposes of equivariant infinite loop space theory, $\mathscr{P}_G$ and $\mathscr{O}_G$ can be used interchangeably, regardless of how their algebras compare categorically.

2.4. **What are genuine symmetric monoidal G-categories?** This philosophy allows us to do the mathematics that we care about without worrying about the categorical underpinnings. Arguing with product operads as above, if $\mathscr{A}$ is an E_∞ G-category, then $\mathscr{A}^G$ is an E_∞ category and is therefore weakly equivalent (in the homotopical sense) to an $\mathscr{O}$-algebra and thus to a permutative category. That allows us to work with the nonequivariant fixed point categories that we are after just as if they were permutative categories.

However, thinking as category theorists, there is a very significant practical gap in this philosophy. In principle, the definition of a genuine permutative G-category as an $\mathscr{O}_G$-algebra requires the use of all of the $\mathscr{O}_G(j)$. In contrast, Proposition 2.2 tells us that we can recognize $\mathscr{O}$-algebras as naive permutative G-categories, which

[3] See [1]; we plan to give a model theoretical development elsewhere.

[4] Of course, this is an abuse of notation, since $\mu_!$ here is really a derived functor.

are defined much more simply using only details about that part of the structure given by use of the $\mathscr{O}(j)$ for $j \leq 3$. We do not have a comparably simple algebraic way of recognizing $\mathscr{O}_G$-algebras when we see them.

As we have said, nonequivariant permutative categories are the same thing as symmetric strict monoidal categories. Symmetric monoidal categories are what we most usually encounter "in nature", and we then either notice naturally occurring equivalent permutative categories or rigidify to construct equivalent permutative categories. This works in precisely the same way for naive symmetric monoidal and naive permutative G-categories.

However, we do not currently have a fully satisfactory definition of genuine symmetric monoidal G-categories. There is a known but not explicit solution to this problem in general categorical terms. It runs as follows. There is a well understood 2-monad in the 2-category $\mathscr{C}at$ whose algebras are the permutative categories [15]. We can construct analogous 2-monads in $G\mathscr{C}at$ whose algebras are the naive and genuine permutative G-categories. There is a well-defined notion of a pseudo-algebra over such a monad T, and there is a coherence theorem saying that any pseudo-algebra over T is pseudo-equivalent to a strict algebra [17, 4.4]. In principle, this solves the problem: a pseudo-algebra over the 2-monad in $G\mathscr{C}at$ that defines genuine permutative G-categories gives a reasonable notion of a genuine symmetric monoidal G-category. Another solution might be in terms of a tree operad expressing free symmetric monoidal G-categories. We want something more concrete, but we have not pursued the question.

2.5. **Equivariant algebraic K-theory.** The most interesting non-equivariant permutative categories are given by categories $\mathscr{A} = \coprod \mathbf{\Pi_n}$, where $\{\Pi_n | n \geq 0\}$ is a sequence of groups (regarded as categories with a single object) and where the permutative structure is given by an associative and unital system of pairings $\Pi_m \times \Pi_n \longrightarrow \Pi_{m+n}$. Then the pairings give the classifying space $B\mathscr{A} = \coprod B\Pi_n$ a structure of topological monoid, and one definition of the algebraic K-groups of $\mathscr{A}$ is the homotopy groups of the space $\Omega B(B\mathscr{A})$.

Equivariantly, it is sensible to replace the spaces $B\Pi_n$ by the classifying G-spaces $B(G, \Pi_n)$ and proceed by analogy. This definition of equivariant algebraic K-groups was introduced and studied calculationally in [10]. It is the equivariant analogue of Quillen's original definition in terms of the plus construction. With essentially the same level of generality, the analogue of Quillen's definition in terms of the Q-construction has been studied by Dress and Kuku [7, 16]. Shimada [37] has given an equivariant version of Quillen's "plus = Q" theorem in this context.

Regarding $\mathscr{A}$ as a G-trivial naive permutative G-category, we have that the classifying G-space of the genuine permutative G-category $\mathscr{G}\mathscr{A}$ is the disjoint union of classifying spaces $B(G, \Pi_n)$. Just as nonequivariantly, the functor ΩB can be replaced by the zeroth space functor $\Omega^\infty_G \mathbb{E}_G$ of an infinite loop G-space machine $\mathbb{E}_G$. The underlying equivariant homotopy type is unchanged. Therefore, we may redefine the algebraic K-groups to be the homotopy groups of the genuine G-spectrum $\mathbb{K}_G\mathscr{A} \equiv \mathbb{E}_G B\mathscr{G}\mathscr{A}$. Essentially the same definition is implicit in Shimakawa [36], who focused on an equivariant version of Segal's infinite loop space machine.

Applying the functor $\mathscr{G}$ to naive permutative G-categories $\mathscr{A}$ with non-trivial G-actions gives more general input for equivariant algebraic K-theory than has been studied in the literature. This allows for G-actions on the groups Π_n, and we then replace $B(G, \Pi_n)$ by classifying G-spaces $B(G, (\Pi_n)_G)$ for the $(G, (\Pi_n)_G)$-bundles associated to the split extensions $\Pi_n \rtimes G$. Such classifying spaces are studied in [13]. Alternative but equivalent constructions of the associated G-spectra $\mathbb{K}_G\mathscr{A}$ are given §4.5 and §8.2 below. The resulting generalization of equivariant algebraic K-theory will be studied in [32].

3. The free $|\mathscr{O}_G|$-space generated by a G-space X

3.1. **The monads $\mathbb{O}_G$ and $\mathbf{O}_G$ associated to $\mathscr{O}_G$.** Recall that $\mathscr{O}_G$ is reduced. In fact, both $\mathscr{O}_G(0)$ and $\mathscr{O}_G(1)$ are trivial categories. As discussed for spaces in [28, §4], there are two monads on G-categories whose algebras are the genuine permutative G-categories. The unit object of an $\mathscr{O}_G$-category can be preassigned, resulting in a monad $\mathbb{O}_G$ on based G-categories, or it can be viewed as part of the $\mathscr{O}_G$-algebra structure, resulting in a monad $\mathbb{O}_{G+}$ on unbased G-categories. Just as in [28], these monads are related by

$$\mathbb{O}_G(\mathscr{A}_+) \cong \mathbb{O}_{G+}\mathscr{A},$$

where $\mathscr{A}_+ = \mathscr{A} \amalg *$ is obtained from an unbased G-category $\mathscr{A}$ by adjoining a disjoint copy of the trivial G-category $*$. Explicitly,

$$\mathbb{O}_G(\mathscr{A}_+) = \coprod_{j\geq 0} \mathscr{O}_G(j) \times_{\Sigma_j} \mathscr{A}^j. \tag{3.1}$$

The term with $j = 0$ is $*$ and accounts for the copy of $*$ on the left. The unit $\eta\colon \mathscr{A} \longrightarrow \mathbb{O}_G(\mathscr{A}_+)$ identifies $\mathscr{A}$ with the term with $j = 1$. The product $\mu\colon \mathbb{O}_G\mathbb{O}_G\mathscr{A}_+ \longrightarrow \mathbb{O}_G\mathscr{A}_+$ is induced by the operad structure maps γ. We are only concerned with based G-categories that can be written in the form $\mathscr{A}_+$.

Since we are concerned with the precise point-set relationship between an infinite loop space machine defined on G-categories and suspension G-spectra, it is useful to think of (unbased) G-spaces X as categories. Thus we also let X denote the topological G-category with object and morphism G-space X and with S, T, I, and C all given by the identity map $X \longrightarrow X$; this makes sense for C since $X \times_X X$ can be identified with X. The classifying G-space $|X|$ can be identified with X.

By specialization of (3.1), we have an identification of (topological) categories

$$\mathbb{O}_G(X_+) = \coprod_{j\geq 0} \mathscr{O}_G(j) \times_{\Sigma_j} X^j. \tag{3.2}$$

The following illuminating result gives another description of $\mathbb{O}_G(X_+)$.

Proposition 3.3. *For G-spaces X, there is a natural isomorphism of genuine permutative G-categories*

$$\mathbb{O}_G(X_+) = \coprod_j \mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j) \times_{\Sigma_j} X^j \longrightarrow \coprod_j \mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j \times_{\Sigma_j} X^j) = \mathscr{G}\mathbb{O}(X_+).$$

Proof. For each j and for $(\Sigma_j \times G)$-spaces Y, such as $Y = X^j$, we construct a natural isomorphism of $(\Sigma_j \times G)$-categories

$$\mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j) \times Y \longrightarrow \mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j \times Y).$$

Here Y is viewed as the constant $(\Sigma_j \times G)$-category at Y. The target is

$$\mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j) \times \mathscr{C}at_G(\tilde{G},Y).$$

Since there is a map between any two objects of $\tilde{G}$ but the only maps in Y are identity maps $i_y\colon y \longrightarrow y$ for $y \in Y$, the only functors $\tilde{G} \longrightarrow Y$ are the constant functors c_y at $y \in Y$ and the only natural transformations between them are the identity transformations $\mathrm{id}_y\colon c_y \longrightarrow c_y$. Sending y to c_y on objects and i_y to id_y on morphisms specifies an identification of $(\Sigma_j \times G)$-categories $Y \longrightarrow \mathscr{C}at_G(\tilde{G}, Y)$. The product of the identity functor on $\mathscr{C}at_G(\tilde{G}, \tilde{\Sigma}_j)$ and this identification gives the desired natural equivalence. With $Y = X^j$, passage to orbits over Σ_j gives the j^{th} component of the claimed isomorphism of G-categories. It is an isomorphism of $\mathscr{G}$-categories since on both sides the action maps are induced by the structure maps of the operad $\mathscr{O}$. □

Recall that we write $\mathbf{O}_G$ for the monad on based G-spaces associated to the operad $|\mathscr{O}_G|$. Thus $\mathbf{O}_G(X_+)$ is the free $|\mathscr{O}_G|$-space generated by the G-space X.

Proposition 3.4. *For G-spaces X, there is a natural isomorphism*

$$\mathbf{O}_G(X_+) = \coprod_{j\geq 0} |\mathscr{O}_G(j)| \times_{\Sigma_j} X^j \cong |\mathbb{O}_G X_+|.$$

Proof. For a $(\Sigma_j \times G)$-space Y viewed as a G-category, NY can be identified with the constant simplicial spaces Y_* with $Y_q = Y$. The nerve functor N does not commute with passage to orbits in general, but arguing as in [13, §2.3] we see that

$$N(\mathscr{O}_G(j) \times_{\Sigma_j} Y) \cong (N\mathscr{O}_G(j)) \times_{\Sigma_j} Y_* = N(\mathscr{O}_G(j) \times_{\Sigma_j} NY)$$

Therefore the classifying space functor commutes with coproducts, products, and the passage to orbits that we see here. □

3.2. **The identification of $(\mathbb{O}_G X_+)^G$.** From here on, with a brief exception in §4.4, we assume that G is finite. The functor $|-|$ commutes with passage to G-fixed points, and we shall prove the following identification. Let $\mathbb{O}$ denote the monad on nonequivariant based categories associated to the operad $\mathscr{O}$ that defines permutative categories.

Theorem 3.5. *For G-spaces X, there is a natural equivalence of $\mathscr{O}$-categories*

$$\mathbb{O}_G(X_+)^G \simeq \prod_{(H)} \mathbb{O}(\widetilde{WH} \times_{WH} X^H)_+,$$

where (H) runs over the conjugacy classes of subgroups of G and $WH = NH/H$.

We are regarding $\mathscr{O}$ as a suboperad of $(\mathscr{O}_G)^G$, and the identification of categories will make clear that the identification preserves the action by $\mathscr{O}$. Of course,

$$\mathbb{O}_G(X_+)^G = \coprod_{j\geq 0} (\mathscr{O}_G(j) \times_{\Sigma_j} X^j)^G \tag{3.6}$$

and

$$\mathbb{O}(\widetilde{WH} \times_{WH} X^H)_+ = \coprod_{k\geq 0} \tilde{\Sigma}_k \times_{\Sigma_k} (\widetilde{WH} \times_{WH} X^H)^k. \tag{3.7}$$

We shall prove Theorem 3.5 by identifying both (3.6) and (3.7) with a small (but not skeletal) model $\mathscr{F}_G(X)^G$ for the category of finite G-sets over X and their isomorphisms over X. We give the relevant definitions and describe these identifications here, and we fill in the easy proofs in §3.3 and §3.4.

A homomorphism $\alpha\colon G \longrightarrow \Sigma_j$ is equivalent to the left action of G on the set $\mathbf{j} = \{1, \cdots, j\}$ specified by $g \cdot i = \alpha(g)(i)$ for $i \in \mathbf{j}$. Similarly, an anti-homomorphism $\alpha\colon G \longrightarrow \Sigma_j$ is equivalent to the right action of G on $\mathbf{j}$ specified by $i \cdot g = \alpha(g)(i)$

or, equivalently, the left action specified by $g \cdot i = \alpha(g^{-1})(i)$; of course, if we set $\alpha^{-1}(g) = \alpha(g)^{-1}$, then α^{-1} is a homomorphism. We focus on homomorphisms and left actions, and we denote such G-spaces by $(\mathbf{j}, \alpha)$. When we say that A is a finite G-set, we agree to mean that $A = (\mathbf{j}, \alpha)$ for a given homomorphism $\alpha \colon G \longrightarrow \Sigma_j$. That convention has the effect of fixing a small groupoid $G\mathscr{F}$ equivalent to the groupoid of all finite G-sets and isomorphisms of finite G-sets. By a j-pointed G-set, we mean a G-set with j elements.

Definition 3.8. Let X be a G-space and $j \geq 0$.

(i) Let $\mathscr{F}_G(j)$ be the G-groupoid whose objects are the j-pointed G-sets A and whose morphisms $\sigma \colon A \longrightarrow B$ are the bijections, with G acting by conjugation. Then $\mathscr{F}_G(j)^G$ is the category with the same objects and with morphisms the isomorphisms of G-sets $\sigma \colon A \longrightarrow B$.

(ii) Let $\mathscr{F}_G(j, X)$ be the G-groupoid whose objects are the maps (not G-maps) $p \colon A \longrightarrow X$ and whose morphisms $f \colon p \longrightarrow q$, $q \colon B \longrightarrow X$, are the bijections $f \colon A \longrightarrow B$ such that $q \circ f = p$; G acts by conjugation on all maps p, q, and f. We view $\mathscr{F}_G(j, X)^G$ as the category of j-pointed G-sets over X and isomorphisms of j-pointed G-sets over X.

(iii) Let $\mathscr{F}_G = \coprod_{j \geq 0} \mathscr{F}_G(j)$ and $\mathscr{F}_G(X) = \coprod_{j \geq 0} \mathscr{F}_G(j, X)$.

(iv) Let $\mathscr{F}_G^{\mathscr{O}}(j)$ be the full G-subcategory of G-fixed objects of $\mathscr{O}_G(j)/\Sigma_j$ and let $\mathscr{F}_G^{\mathscr{O}}(j, X)$ be the full G-subcategory of G-fixed objects of $\mathscr{O}_G(j) \times_{\Sigma_j} X^j$. Then $\mathscr{F}_G^{\mathscr{O}}(j)^G = (\mathscr{O}_G(j)/\Sigma_j)^G$ and $\mathscr{F}_G^{\mathscr{O}}(j, X)^G = (\mathscr{O}_G(j) \times_{\Sigma_j} X^j)^G$.

In §3.3, we prove that the right side of (3.6) can be identified with $\mathscr{F}_G(X)^G$.

Theorem 3.9. *There is a natural isomorphism of permutative categories*

$$(\mathbb{O}_G(X_+))^G = \coprod_{j \geq 0} \mathscr{F}_G^{\mathscr{O}}(j, X)^G \cong \coprod_{j \geq 0} \mathscr{F}_G(j, X)^G = \mathscr{F}_G(X)^G.$$

In §3.4, we prove that the right side of (3.7) can be identified with $\mathscr{F}_G(X)^G$. At least implicitly, this identification of categories (not G-categories) has been known since the 1970's; see for example Nishida [33, App. A].

Theorem 3.10. *There is a natural equivalence of categories*

$$\prod_{(H)} \coprod_{k \geq 0} \tilde{\Sigma}_k \times_{\Sigma_k} (\widetilde{WH} \times_{WH} X^H)^k \longrightarrow \coprod_{j \geq 0} \mathscr{F}_G(j, X)^G = \mathscr{F}_G(X)^G.$$

These two results prove Theorem 3.5.

Remark 3.11. With our specification of finite G-sets as $A = (\mathbf{j}, \alpha)$, the disjoint union of A and $B = (\mathbf{k}, \beta)$ is obtained via the obvious identification of $\mathbf{j} \coprod \mathbf{k}$ with $\mathbf{j} + \mathbf{k}$. The disjoint union of finite G-sets over a G-space X gives $\mathscr{F}_G(X)$ a structure of naive permutative G-category. By Theorem 3.9, its fixed point category $\mathscr{F}_G(X)^G$ is an $\mathscr{O}$-category equivalent to $(\mathbb{O}_G(X_+))^G$. One might think that $\mathscr{F}_G(X)$ is a genuine permutative G-category equivalent to the free $\mathscr{O}_G$-category $\mathscr{O}_G(X_+)$. However, its H-fixed subcategory for $H \neq G$ is not equivalent to $\mathscr{F}_H(X)^H$, and there does not seem to be an action of $\mathscr{O}_G$ (or any other E_∞ G-operad) on $\mathscr{F}_G(X)$.

To compare with [11], we offer some alternative notations.

Definition 3.12. For an unbased G-space X, let $\mathscr{E}_G(X) = \mathscr{E}_G^{\mathscr{O}}(X) = \mathbb{O}_G(X_+)$. It is a genuine permutative G-category, and its H-fixed subcategory $\mathscr{E}_G(X)^H$ is equivalent to $\mathscr{E}_H(X)^H$ and therefore to $\mathscr{F}_H(X)^H$.

Remark 3.13. In [11], we gave a considerably more intuitive definition of a G-category $\mathscr{E}_G(X)$. It will reappear in §9 and will be given the alternative notation $\mathscr{E}_G^{\mathscr{P}}(X)$. It is acted on by an E_∞ operad $\mathscr{P}_G$ of G-categories, and, again, its fixed point category $\mathscr{E}_G(X)^H$ is equivalent to $\mathscr{E}_H(X)^H$ and therefore to $\mathscr{F}_H(X)^H$.

3.3. **The proof of Theorem 3.9.** We first use Theorem 1.10 to identify (3.6) when X is a point. Since we are comparing several equivalent categories, we describe the relevant categories implicit in our operad $\mathscr{O}_G$ in their simplest forms up to isomorphism, although that is perhaps redundant; [13, §§2.1, 2.2, 4.1, 4.2] gives details. The formula for the G-action in (ii) ensures that $(g\alpha)(e) = e$.

Proposition 3.14. *(i) The objects of the $(\Sigma_j \times G)$-category $\mathscr{O}_G(j)$ are the functions $\phi\colon G \longrightarrow \Sigma_j$. There is a unique morphism $\iota\colon \phi \longrightarrow \psi$ for each pair of objects (ψ, ϕ); that is, $\mathscr{O}_G(j)$ is chaotic. The (left) action of G on $\mathscr{O}_G(j)$ is given by $(g\phi)(h) = \phi(g^{-1}h)$ on objects and the corresponding diagonal action on morphisms. The (right) action of Σ_j is given by $(\phi\sigma)(h) = \phi(h)\sigma$ on objects and the corresponding diagonal action on morphisms.*

(ii) The objects of the G-category $\mathscr{O}_G(j)/\Sigma_j$ are the functions $\alpha\colon G \longrightarrow \Sigma_j$ such that $\alpha(e) = e$. The morphisms $\sigma\colon \alpha \longrightarrow \beta$ are the elements $\sigma \in \Sigma_j$, thought of as the functions $G \longrightarrow \Sigma_j$ specified by $\sigma(h) = \beta(h)\sigma\alpha(h)^{-1}$. The composite of σ with $\tau\colon \beta \longrightarrow \gamma$ is $\tau\sigma\colon \alpha \longrightarrow \gamma$. The action of G is given on objects by

$$(g\alpha)(h) = \alpha(g^{-1}h)\alpha(g^{-1})^{-1}$$

and is given on morphisms by $g(\sigma\colon \alpha \longrightarrow \beta) = \sigma\colon g\alpha \longrightarrow g\beta$.

(iii) For $\Lambda \subset \Sigma_j \times G$, $\mathscr{O}_G(j)^\Lambda$ is empty if $\Lambda \cap \Sigma_j \neq e$. It is a nonempty and hence chaotic subcategory of $\mathscr{O}_G(j)$ if $\Lambda \cap \Sigma_j = e$.

(iv) The category $(\mathscr{O}_G(j)/\Sigma_j))^G$ is isomorphic to $\mathscr{R}^-(G, \Sigma_j)$; its objects are the anti-homomorphisms $\alpha\colon G \longrightarrow \Sigma_j$ and its morphisms $\sigma\colon \alpha \longrightarrow \beta$ are the conjugacy relations $\beta = \sigma\alpha\sigma^{-1}$, $\sigma \in \Sigma_j$. For $H \subset G$, restriction of functions gives an equivalence of categories $(\mathscr{O}_G(j)/\Sigma_j))^H \longrightarrow (\mathscr{O}_H(j)/\Sigma_j))^H$.

Now return to a general G-space X. To prove Theorem 3.9, it suffices to prove that $(\mathscr{O}_G(j) \times_{\Sigma_j} X^j)^G$ is isomorphic to $\mathscr{F}_G(j, X)^G$ for all j. Passage to orbits here means that for $\phi \in \mathscr{O}_G(j)$, $y \in X^j$, and $\sigma \in \Sigma_j$ (thought of as acting on the left on $\mathbf{j}$ and therefore on j-tuples of elements of X), $(\phi\sigma, y) = (\phi, \sigma y)$ in $\mathscr{O}_G(j) \times_{\Sigma_j} X^j$. Observe that an object $(\phi, z_1, \cdots, z_j) \in \mathscr{O}_G(j) \times_{\Sigma_j} X^j$ has a unique representative in the same orbit under Σ_j of the form $(\alpha, x_1, \cdots, x_j)$ where $\alpha(e) = e$. It is obtained by replacing ϕ by $\phi\tau$, where $\tau = \phi(e)^{-1}$, and replacing z_i by $x_i = z_{\tau(i)}$.

Lemma 3.15. *An object $(\alpha, y) \in \mathscr{O}_G(j) \times_{\Sigma_j} X^j$, where $\alpha(e) = e$ and $y \in X^j$, is G-fixed if and only if $\alpha\colon G \longrightarrow \Sigma_j$ is an anti-homomorphism and $\alpha(g^{-1})y = gy$ for all $g \in G$.*

Proof. Assume that $(\alpha, y) = (g\alpha, gy)$ for all $g \in G$. Then each $g\alpha$ must be in the same Σ_j-orbit as α, where α is regarded as an object of $\mathscr{O}_G(j)$ and not $\mathscr{O}_G(j)/\Sigma_j$, so that $(g\alpha)(h) = \alpha(g^{-1}h)$. Then $(g\alpha)(h) = \alpha(h)\sigma$ for all $h \in G$ and some $\sigma \in \Pi$. Taking $h = e$ shows that $\sigma = \alpha(g^{-1})$. The resulting formula $\alpha(g^{-1}h) = \alpha(h)\alpha(g^{-1})$ implies that α is an anti-homomorphism. Now

$$(\alpha, y) = (g\alpha, gy) = (\alpha\alpha(g^{-1}), gy) = (\alpha, \alpha(g)gy),$$

which means that $\alpha(g)gy = y$ and thus $gy = \alpha(g^{-1})y$. $\square$

Use α^{-1} to define a left action of G on $\mathbf{j}$ and define $p\colon \mathbf{j} \longrightarrow X$ by $p(i) = x_i$. Then the lemma shows that the G-fixed elements (α, y) are in bijective correspondence with the maps of G-sets $p\colon A \longrightarrow X$, where A is a j-pointed G-set. Using Proposition 3.14(iv), we see similarly that maps $f\colon A \longrightarrow B$ of j-pointed G-sets over X correspond bijectively to morphisms in $(\mathscr{O}_G(j) \times_{\Sigma_j} X^j)^G$. These bijections specify the required isomorphism between $\mathscr{F}_G(j, X)^G$ and $(\mathscr{O}_G(j) \times_{\Sigma_j} X^j)^G$.

3.4. **The proof of Theorem 3.10.** This decomposition is best proven by a simple thought exercise. Every finite G-set A decomposes non-uniquely as a disjoint union of orbits G/H, and orbits G/H and G/J are isomorphic if and only if H and J are conjugate. Choose one H in each conjugacy class. Then A decomposes uniquely as the disjoint union of the G-sets A_H, where A_H is the set of elements of A with isotropy group conjugate to H. This decomposes the category $G\mathscr{F} \equiv (\mathscr{F}_G)^G$ as the product over H of the categories $G\mathscr{F}(H)$ of finite G-sets all of whose isotropy groups are conjugate to H.

In turn, $G\mathscr{F}(H)$ decomposes uniquely as the coproduct over $k \geq 0$ of the categories $G\mathscr{F}(H,k)$ whose objects are isomorphic to the disjoint union, denoted kG/H, of k copies of G/H. Up to isomorphism, kG/H is the only object of $G\mathscr{F}(H,k)$. The automorphism group of the G-set G/H is WH, hence the automorphism group of kG/H is the wreath product $\Sigma_k \int WH$. Viewed as a category with a single object, we may identify this group with the category $\tilde{\Sigma}_k \times_{\Sigma_k} (\mathbf{WH})^k$. This proves the following result.

Proposition 3.16. *The category $G\mathscr{F}$ is equivalent to the category*

$$\prod_{(H)} \coprod_{k \geq 0} \tilde{\Sigma}_k \times_{\Sigma_k} (\mathbf{WH})^k.$$

The displayed category is a skeleton of $G\mathscr{F}$. As written, its objects are sets of numbers $\{k_H\}$, one for each (H), but they are thought of as the finite G-sets $\coprod_H k_H G/H$. Its morphism groups specify the automorphisms of these objects. On objects, the equivalence sends a finite G-set A to the unique finite G-set of the form $\coprod_{(H)} kG/H$ in the same isomorphism class as A. Via chosen isomorphisms, this specifies the inverse equivalence to the inclusion of the chosen skeleton in $G\mathscr{F}$.

We parametrize this equivalence to obtain a description of the category $G\mathscr{F}(X)$ of finite G-sets over X. Given any H and k, a k-tuple of elements $\{x_1, \cdots, x_k\}$ of X^H determines the G-map $p\colon kG/H \longrightarrow X$ that sends eH in the ith copy of G/H to x_i, and it is clear that every finite G-set A over X is isomorphic to one of this form. Similarly, for a finite G-set $q\colon B \longrightarrow X$ over X and an isomorphism $f\colon A \longrightarrow B$, f is an isomorphism over X from q to $p = q \circ f$, and every isomorphism over X can be constructed in this fashion. Since we may as well choose A and B to be in our chosen skeleton of $G\mathscr{F}$, this argument proves Theorem 3.10.

4. The equivariant recognition principle

The equivariant recognition principle shows how to recognize (genuine) G-spectra in terms of category or space level information. It comes in various versions. Here we use two modernized variants of the machine from [21]. As in [11], we let $\mathscr{S}$, $\mathscr{S}p$, and $\mathscr{Z}$ denote the categories of orthogonal spectra [20], Lewis-May spectra [18], and EKMM S-modules [9]. Similarly, we let $G\mathscr{S}$, $G\mathscr{S}p$ and $G\mathscr{Z}$ denote the corresponding categories of genuine G-spectra from [19], [18], and again [19]. Each has its advantages. We start with a machine that lands in $G\mathscr{S}$. That is the choice preferred in [11]. Its sphere G-spectrum S_G is cofibrant, and so are the suspension G-spectra $\Sigma_G X$ of cofibrant based G-spaces X. A key advantage of this machine is that it gives the quickest route to the modicum of information about pairings that we require. A key disadvantage is that the machine we construct does not naturally land in fibrant G-spectra. We then give a variant machine that lands in $G\mathscr{S}p$ or $G\mathscr{Z}$, where every object is fibrant, and give a comparison that illuminates homotopical properties of the first machine via its comparison with the second.

4.1. **The equivariant infinite loop space machine: $G\mathscr{S}$ version.** In brief, we have a functor $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{S}}$ that assigns an orthogonal G-spectrum $\mathbb{E}_G Y$ to a G-space Y with an action by some chosen E_∞ operad $\mathscr{C}_G$ of G-spaces. We quickly summarize what we need, much of which is contained in the paper [5] of Costenoble and Waner.[5] We must start with a family of operads $\mathscr{K}_V$, one for each finite dimensional real inner product space, as well as the given operad $\mathscr{C}_G$.

Scholium 4.1. We must use the Steiner operads $\mathscr{K}_V$ and not the little discs operads $\mathscr{D}_V$, which was the choice in [5]. As explained in [28, §3], for inclusions $V \subset W$ of inner product spaces, there is no map of operads $\mathscr{D}_V \longrightarrow \mathscr{D}_W$ compatible with suspension, and the Steiner operads remedy the defect. We describe the equivariant Steiner operads in the appendix, §10, since they do not appear in the literature. We let $\mathscr{K}_U$ denote the colimit of the operads $\mathscr{K}_V$ where V runs over the finite dimensional subspaces of a complete G-universe U. It is an E_∞ operad of G-spaces.

We have two minor variants of our machine, both of which make use of the product of operads trick recalled in §2.3; compare [28, §9]. We can first use that trick to convert $\mathscr{C}_G$-spaces to $\mathscr{K}_U$-spaces and then use only Steiner operads, or we can build the machine using the product operads $\mathscr{C}_G \times \mathscr{K}_V$, without first changing the input. There are only minor reasons for preferring one approach over the other, and we prefer the latter approach here. We therefore define $\mathscr{C}_V = \mathscr{C}_G \times \mathscr{K}_V$. Via its projection to $\mathscr{C}_G$, $\mathscr{C}_G$-spaces can be viewed as $\mathscr{C}_V$-spaces for all V. Via its projection to $\mathscr{K}_V$, $\mathscr{C}_V$ also acts on V-fold loop spaces.

Write $\mathbf{C}_V$ for the monad on based G-spaces associated to the operad $\mathscr{C}_V$. The categories of $\mathscr{C}_V$-spaces and $\mathbf{C}_V$-algebras are isomorphic. The unit $\eta\colon \mathrm{Id} \longrightarrow \Omega^V\Sigma^V$ of the monad $\Omega^V\Sigma^V$ and the action of $\mathbf{C}_V$ on the G-spaces $\Omega^V\Sigma^V X$ induce a composite natural map

$$\alpha_V : C_V X \xrightarrow{C_V\eta} C_V\Omega^V\Sigma^V X \longrightarrow \Omega^V\Sigma^V X,$$

and $\alpha_V : C_V \longrightarrow \Omega^V\Sigma^V$ is a map of monads whose adjoint defines a right action of C_V on the functor Σ^V. The following result is due to Rourke and Sanderson [34, Corollary 1]. It generalizes earlier partial results of Hauschild [14] and Caruso and Waner [5, 1.18].

Definition 4.2. A Hopf G-space Y is grouplike if each fixed point Hopf space Y^H is grouplike, that is, each $\pi_0(Y^H)$ is a group. A map $\alpha\colon X \longrightarrow Y$ of Hopf G-spaces is a group completion if Y is grouplike and $\alpha^H : X^H \longrightarrow Y^H$ is a nonequivariant group completion for each $H \subset G$. That is, $\pi_0(Y^H)$ is the Grothendieck group of the monoid $\pi_0(X^H)$ and $H_*(Y;k)$ is the localization $H_*(X;k)[\pi_0(X)^{-1}]$ for any field of coefficients k.

[5] As noted before, we plan to give a model theoretic elaboration elsewhere.

In particular, α is a weak G-equivalence if all X^H are path connected.

Theorem 4.3. *If $V^G \neq 0$, so that V contains a copy of the trivial representation $\mathbb{R}$, then $\alpha_V \colon C_V X \longrightarrow \Omega^V \Sigma^V X$ is a group completion.*

The two-sided monadic bar construction is described in [21, 28], and we have the following construction, as in [19, §8].

Definition 4.4. Let Y be a $\mathscr{C}_G$-space. We define an orthogonal G-spectrum $\mathbb{E}_G Y = \mathbb{E}_G^{\mathscr{I}} Y$. Let

$$\mathbb{E}_G Y(V) = B(\Sigma^V, \mathbf{C}_V, Y).$$

Using the evident action of isometries on $\mathscr{K}_V$ and Σ^V, this defines a G-functor from the G-category $\mathscr{I}_G$ of finite dimensional real inner product spaces and linear isometric isomorphisms to the G-category $\mathscr{T}_G$ of based G-spaces and continuous maps. For $V \subset W$, the structure G-map

$$\sigma \colon \Sigma^{W-V} \mathbb{E}_G Y(V) \longrightarrow \mathbb{E}_G Y(W)$$

is the composite

$$\Sigma^{W-V} B(\Sigma^V, \mathbf{C}_V, Y) \cong B(\Sigma^W, \mathbf{C}_V, Y) \longrightarrow B(\Sigma^W, \mathbf{C}_W, Y).$$

obtained by commuting Σ^{W-V} with geometric realization and using the stabilization map of monads $C_V \longrightarrow C_W$, which is a spacewise closed inclusion.

We have the diagram of $\mathscr{C}_V$-spaces and $\mathscr{C}_V$-maps

$$Y \xleftarrow{\ \varepsilon\ } B(\mathbf{C}_V, \mathbf{C}_V, Y) \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} B(\Omega^V\Sigma^V, \mathbf{C}_V, Y) \xrightarrow{\ \zeta\ } \Omega^V B(\Sigma^V, \mathbf{C}_V, Y).$$

The map ε is a homotopy equivalence for formal reasons explained in [21, 9.8], and it has a natural homotopy inverse ν, although ν is not a $\mathscr{C}_V$-map. When $V^G \neq 0$, $B(\alpha, \mathrm{id}, \mathrm{id})$ is a group completion since α is so, as follows from the nonequivariant version explained in [22, 2.3]. The map ζ is defined and shown to be a weak equivalence by Costenoble and Waner in [5, 5.5], following [21, §12]. Define

$$\eta \colon Y \longrightarrow \Omega^V \mathbb{E}_G Y(V)$$

to be the composite $\zeta \circ B(\alpha, \mathrm{id}, \mathrm{id}) \circ \nu$. Then η is a natural group completion of Hopf G-spaces when $V^G \neq 0$. It is therefore a weak equivalence when Y is grouplike. Moreover, the following diagram commutes for $V \subset W$, where $\tilde{\sigma}$ is adjoint to σ.

$$\begin{array}{ccc} & Y & \\ {}^{\eta_V}\swarrow & & \searrow^{\eta_W} \\ \Omega^V \mathbb{E}_G Y(V) & \xrightarrow[\Omega^V \tilde{\sigma}]{} & \Omega^W \mathbb{E}_G Y(W) \end{array}$$

Therefore $\Omega^V \tilde{\sigma}$ is a weak equivalence if $V^G \neq 0$. If we replace $\mathbb{E}Y$ by a fibrant approximation $R\mathbb{E}Y$, there results a group completion $\eta \colon Y \longrightarrow (R\mathbb{E}Y)_0$. It is harmless to pretend that the map η above with $V = 0$ is itself a group completion, but we shall use the category $\mathscr{S}p$ to give a more structured way to think about this.

4.2. **The equivariant infinite loop space machine: $G\mathscr{S}p$ version.** Since we are especially interested in suspension G-spectra, it is natural to want an infinite loop space machine that lands in the category $G\mathscr{S}p$ of G-spectra. While $G\mathscr{S}p$ is not symmetric monoidal under its smash product, it has the compensating and incompatible virtue that the suspension G-functor Σ^∞_G from the category $G\mathscr{T}$ of based G-spaces to $G\mathscr{S}p$ is a left adjoint with a right adjoint infinite loop 0^{th} G-space functor Ω^∞_G. We summarize the construction of an equivariant infinite loop space machine $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{S}p}$ that lands in $G\mathscr{S}p$. Formally, the equivariant theory works in the same way as the nonequivariant theory, and we follow the summary in [28, §9].

Again let $\mathscr{C}_G$ be an E_∞ operad of G-spaces and now write $\mathscr{C}_U = \mathscr{C}_G \times \mathscr{K}_U$. Let Q_G denote the functor $\Omega^\infty_G \Sigma^\infty_G$. For based G-spaces X, we have a natural group completion $\alpha\colon \mathbf{C}_U X \longrightarrow \mathbf{K}_U X \longrightarrow Q_G X$. The map α is a map of monads, and its adjoint gives a right action of the monad $\mathbf{C}_U$ on the functor Σ^∞_G. For a $\mathscr{C}_G$-space Y, the G-spectrum $\mathbb{E}_G Y$ is the bar construction

$$\mathbb{E}_G Y = B(\Sigma^\infty_G, \mathbf{C}_U, Y).$$

For every Y, $\mathbb{E}_G Y$ is connective in the sense that the negative homotopy groups of its fixed point spectra $(\mathbb{E}_G Y)^H$ are zero.

Here we have the diagram of $\mathscr{C}_U$-spaces and $\mathscr{C}_U$-maps

$$Y \xleftarrow{\ \varepsilon\ } B(\mathbf{C}_U, \mathbf{C}_U, Y) \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} B(Q_G, \mathbf{C}_U, Y) \xrightarrow{\ \zeta\ } \Omega^\infty_G B(\Sigma^\infty_G, \mathbf{C}_U, Y).$$

The map ε is a homotopy equivalence [21, 9.8] with a natural homotopy inverse ν, which is not a $\mathscr{C}_U$-map. Again, $B(\alpha, \mathrm{id}, \mathrm{id})$ is a group completion since α is so, and ζ is a weak equivalence [5, §5]. Defining $\eta\colon Y \longrightarrow \Omega^\infty_G \mathbb{E}_G Y$ to be the composite $\zeta \circ B(\alpha, \mathrm{id}, \mathrm{id}) \circ \nu$, it follows that η is a natural group completion and thus a weak equivalence when Y is grouplike. Here, of course, there is no need for fibrant approximation.

We compare the $\mathscr{S}$ and $\mathscr{S}p$ machines $\mathbb{E}_G^{\mathscr{S}}$ and $\mathbb{E}_G^{\mathscr{S}p}$ by transporting both of them to $\mathscr{Z}$, following [19]. As discussed in [19, IV§4] with slightly different notations, there is a (noncommutative) diagram of Quillen equivalences

$$\begin{array}{ccc} G\mathscr{P} & \underset{\ell}{\overset{L}{\rightleftarrows}} & G\mathscr{S}p \\ \mathbb{U}\uparrow\downarrow\mathbb{P} & & \mathbb{V}\uparrow\downarrow\mathbb{F} \\ G\mathscr{S} & \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} & G\mathscr{Z}. \end{array}$$

Here $G\mathscr{P}$ is the category of coordinate-free G-prespectra. The left adjoint $\mathbb{N}$ is strong symmetric monoidal, and the unit map $\eta\colon X \longrightarrow \mathbb{N}^{\#}\mathbb{N}X$ is a weak equivalence for all cofibrant orthogonal G-spectra X. It is a fibrant approximation in the positive stable model structure on $G\mathscr{S}$, but we prefer to think in terms of the stable model structure on $G\mathscr{S}$ (see [19, III§§4,5]).

We can compare machines using the diagram. In fact, by an elementary inspection of definitions, we see the following result, which is essentially a reinterpretation of the original construction of [21] that becomes visible as soon as one introduces orthogonal spectra.

Lemma 4.5. *The functor $\mathbb{E}_G^{\mathscr{S}p}$ from $\mathscr{C}_G$-spaces to the category $G\mathscr{S}p$ is naturally isomorphic to the composite functor $L \circ \mathbb{U} \circ \mathbb{E}_G^{\mathscr{S}}$.*

As explained in [19, IV§5], there is a monad $\mathbb{L}$ on $G\mathscr{S}p$ and a category $G\mathscr{S}p[\mathbb{L}]$ of $\mathbb{L}$-algebras. The left adjoint $\mathbb{F}$ in the diagram is the composite of left adjoints

$$\mathbb{L}\colon G\mathscr{S}p \longrightarrow G\mathscr{S}p[\mathbb{L}] \quad \text{and} \quad \mathbb{J}\colon G\mathscr{S}p[\mathbb{L}] \longrightarrow G\mathscr{Z}.$$

The functor $L \circ \mathbb{U}\colon G\mathscr{S} \longrightarrow G\mathscr{S}p$ lands naturally in $G\mathscr{S}p[\mathbb{L}]$, so that we can define

$$\mathbb{M} = \mathbb{J} \circ L \circ \mathbb{U}\colon G\mathscr{S} \longrightarrow G\mathscr{Z}.$$

By [19, IV.5.2 and IV.5.4], $\mathbb{M}$ is lax symmetric monoidal and there is a natural lax symmetric monoidal map $\alpha\colon \mathbb{N}X \longrightarrow \mathbb{M}X$ that is a weak equivalence when X is cofibrant. Effectively, we have two infinite loop space machines landing in $G\mathscr{Z}$, namely $\mathbb{N} \circ \mathbb{E}_G^{\mathscr{S}}$ and $\mathbb{J} \circ \mathbb{E}_G^{\mathscr{S}p}$. In view of the lemma, the latter is isomorphic to $\mathbb{M} \circ \mathbb{E}_G^{\mathscr{S}}$, hence

$$\alpha\colon \mathbb{N} \circ \mathbb{E}_G^{\mathscr{S}} \longrightarrow \mathbb{M} \circ \mathbb{E}_G^{\mathscr{S}} \cong \mathbb{J} \circ \mathbb{E}_G^{\mathscr{S}p}$$

compares the two machines, showing that they are equivalent for all practical purposes. Homotopically, these categorical distinctions are irrelevant, and we can use whichever machine we prefer, deducing properties of one from the other.

4.3. **Some properties of equivariant infinite loop space machines.** Many properties of the infinite loop space machine $\mathbb{E}_G$ follow directly from the group completion property, independent of how the machine is constructed, but it is notationally convenient to work with the machine $\mathbb{E}_G^{\mathscr{S}p}$, for which η is a natural group completion without any bother with fibrant approximation. The results apply equally well to $\mathbb{E}_G^{\mathscr{S}}$. As in May and Thomason [31], the group completion property actually characterizes the machine up to homotopy, but that is not needed here and will be proven elsewhere. We illustrate with the following two results. The first says that, up to weak equivalence, the functor $\mathbb{E}_G$ commutes with products. The second says similarly that it commutes with passage to fixed points.

Theorem 4.6. *Let X and Y be $\mathscr{C}_G$-spaces. Then the map*

$$\mathbb{E}_G(X \times Y) \longrightarrow \mathbb{E}_G X \times \mathbb{E}_G Y$$

induced by the projections is a weak equivalence of G-spectra.

Proof. We are using that the product of $\mathscr{C}_G$-spaces is a $\mathscr{C}_G$-space, the proof of which uses that the category of operads is cartesian monoidal. Working in $G\mathscr{S}p$, the functor Ω_G^∞ commutes with products and passage to fixed points and we have the commutative diagram

$$\begin{array}{ccc} & (X\times Y)^H \cong X^H \times Y^H & \\ \swarrow^{\eta^H} & & \searrow^{\eta^H\times\eta^H} \\ (\Omega_G^\infty \mathbb{E}_G(X\times Y))^H & \longrightarrow & (\Omega_G^\infty \mathbb{E}_G X)^H \times (\Omega_G^\infty \mathbb{E}_G Y)^H. \end{array}$$

Since the product of group completions is a group completion, the diagonal arrows are both group completions. Therefore the horizontal arrow is a weak equivalence. Since our spectra are connective, the conclusion follows. □

Theorem 4.7. *For $\mathscr{C}_G$-spaces Y, there is a natural weak equivalence of spectra*

$$\mathbb{E}(Y^G) \longrightarrow (\mathbb{E}_G Y)^G.$$

Proof. For based G-spaces X, we have natural inclusions $\mathbf{C}_{U^G}(X^G) \longrightarrow (\mathbf{C}_U X)^G$ and $\Sigma^\infty(X^G) \longrightarrow (\Sigma_G^\infty X)^G$. For G-spectra E, we have a natural isomorphism $\Omega^\infty(E^G) \cong (\Omega_G^\infty E)^G$. There results a natural map of spectra

$$\mathbb{E}(Y^G) = B(\Sigma^\infty, \mathbf{C}_{U^G}, Y^G) \longrightarrow (B(\Sigma_G^\infty, \mathbf{C}_U, Y))^G = (\mathbb{E}_G Y)^G$$

and a natural map of spaces under Y^G

$$\begin{array}{ccc} & Y^G & \\ \swarrow & & \searrow \\ \Omega^\infty \mathbb{E}(Y^G) & \longrightarrow & (\Omega_G^\infty \mathbb{E}_G Y)^G. \end{array}$$

Since the diagonal arrows are both group completions, the horizontal arrow must be a weak equivalence. Since our spectra are connective, the map $\mathbb{E}(Y^G) \longrightarrow (\mathbb{E}_G Y)^G$ of spectra must also be a weak equivalence. □

4.4. **The recognition principle for naive G-spectra.** We elaborate on Theorem 4.7. The functor $\mathbb{E} = \mathbb{E}_e$ in that result is the nonequivariant infinite loop space machine, which is defined using the nonequivariant Steiner operad $\mathscr{K} = \mathscr{K}_{\mathbb{R}^\infty}$. In Theorem 4.7, we thought of $\mathbb{R}^\infty$ as U^G. There is also a recognition principle for naive G-spectra, which are just spectra with G-actions. Again we can use either the category $\mathscr{S}$ of orthogonal spectra or the category $\mathscr{S}p$ of Lewis-May spectra, comparing the two by mapping to the category $\mathscr{Z}$ of EKMM S-modules, but letting G act on objects in all three. Naive G-spectra represent $\mathbb{Z}$-graded cohomology theories, whereas genuine G-spectra represent $RO(G)$-graded cohomology theories. The machine $\mathbb{E}$ is obtained by regarding $\mathscr{C} = (\mathscr{C}_G)^G$ as an operad of G-spaces with trivial G-action. We continue to write $\mathbb{E}$ for this construction since it is exactly the same construction as the nonequivariant one, but applied to G-spaces with an action by the E_∞ operad $\mathscr{C}$ of G-spaces.

It is worth emphasizing that when working with naive G-spectra, there is no need to restrict to finite groups. We can just as well work with general discrete groups G. The machine $\mathbb{E}$ enjoys the same properties as the machine $\mathbb{E}_G$, including the group completion property. Working with Lewis-May spectra, the adjunction $(\Sigma^\infty, \Omega^\infty)$ relating spaces and spectra applies just as well to give an adjunction relating G-spaces and naive G-spectra. For based G-spaces X, the map $\alpha\colon CX \longrightarrow \Omega^\infty\Sigma^\infty X$ is a group completion of Hopf G-spaces by the nonequivariant special case since $(CX)^H = C(X^H)$ and $(\Omega^\infty\Sigma^\infty X)^H = \Omega^\infty\Sigma^\infty(X^H)$.

In the language of [18], genuine G-spectra are indexed on a complete G-universe U, namely the sum of countably many copies of all irreducible representations of G; a canonical choice is the sum of countably many copies of the regular representation. Naive G-spectra are indexed on the trivial G-universe $U^G \cong \mathbb{R}^\infty$. The inclusion of universes $\iota\colon U^G \longrightarrow U$ induces a forgetful functor $\iota^*\colon G\mathscr{S} \longrightarrow \mathscr{S}$ from genuine G-spectra to naive G-spectra. It represents the forgetful functor from $RO(G)$-graded cohomology theories to $\mathbb{Z}$-graded cohomology theories.

As we observe in §10, the inclusion of universes $\iota\colon U^G \longrightarrow U$ induces an inclusion of operads of G-spaces $\iota\colon \mathscr{K}_{U^G} \longrightarrow \mathscr{K}_U$ and therefore an inclusion of operads

$$\mathscr{C}_{U^G} = \mathscr{C} \times \mathscr{K}_{U^G} \longrightarrow \mathscr{C}_G \times \mathscr{K}_U = \mathscr{C}_U.$$

The following consistency statement is important since, by definition, the H-fixed point spectrum E^H of a genuine G-spectrum E is $(\iota^*E)^H$, and the homotopy groups of E are $\pi_*^H(E) \equiv \pi_*(E^H)$.

Theorem 4.8. *Let Y be a $\mathscr{C}_G$-space. Then there is a natural weak equivalence of naive G-spectra $\mu\colon \mathbb{E}\iota^*Y \longrightarrow \iota^*\mathbb{E}_GY$.*

Proof. Again, we work with $\mathbb{E}_G^{\mathscr{S}p}$, but the result applies equally well to $\mathbb{E}^{\mathscr{S}}$. It is easy to check from the definitions that, for G-spaces X, we have a natural commutative diagram of G-spaces

$$\begin{array}{ccc} CX & \xrightarrow{\alpha} & \Omega^\infty\Sigma^\infty X \\ \downarrow & & \downarrow \\ C_GX & \xrightarrow[\alpha]{} & \Omega_G^\infty\Sigma_G^\infty X. \end{array}$$

Passing to adjoints, we obtain a natural composite map

$$\Sigma_G^\infty CX \longrightarrow \Sigma_G^\infty C_GX \longrightarrow \Sigma_G^\infty X$$

of genuine G-spectra. It specifies a right action of the monad C on the functor Σ_G^∞ that is compatible with the right action of C_G on Σ_G^∞. Clearly $\Omega^\infty\iota^* = \Omega_G^\infty$, since both are evaluation at $V = 0$, hence their left adjoints Σ_G^∞ and $\iota_*\Sigma^\infty$ are isomorphic. The unit of the adjunction (ι_*, ι^*) gives a natural map of naive G-spectra $\Sigma^\infty X \longrightarrow \iota^*\Sigma_G^\infty X$, and there results a natural map

$$\mu\colon \mathbb{E}\iota^*Y = B(\Sigma^\infty, \mathbf{C}, Y) \longrightarrow B(\iota^*\Sigma_G^\infty, \mathbf{C}_G, Y) \cong \iota^*\mathbb{E}_GY$$

of naive G-spectra. The following diagram commutes by a check of definitions.

$$\begin{array}{ccccccc} Y & \xleftarrow{\varepsilon} & B(\mathbf{C},\mathbf{C},Y) & \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} & B(Q,\mathbf{C},Y) & \xrightarrow{\zeta} & \Omega^\infty B(\Sigma^\infty,\mathbf{C},Y) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega^\infty\mu} \\ Y & \xleftarrow{\varepsilon} & B(\mathbf{C}_G,\mathbf{C}_G,Y) & \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} & B(Q_G,\mathbf{C}_G,Y) & \xrightarrow{\zeta} & \Omega^\infty B(\Sigma_G^\infty,\mathbf{C}_G,Y). \end{array}$$

Replacing the maps ε with their homotopy inverses ν, the horizontal composites are group completions. Therefore $\Omega^\infty\mu$ is a weak equivalence, hence so is μ. □

4.5. **The recognition principle for permutative G-categories.** We may start with any E_∞ operad $\mathscr{P}_G$ of G-categories and apply the classifying space functor to obtain an E_∞ operad $|\mathscr{P}_G|$ of G-spaces. If $\mathscr{P}_G$ acts on a category $\mathscr{A}$, then $|\mathscr{P}_G|$ acts on $|\mathscr{A}| = B\mathscr{A}$. We can replace $|\mathscr{P}_G|$ by its product with the Steiner operads $\mathscr{K}_V$ or with the Steiner operad $\mathscr{K}_U$ and apply the functor $\mathbb{E}_G^{\mathscr{S}}$ or $\mathbb{E}_G^{s\mathscr{S}}$ to obtain a (genuine) associated G-spectrum, which we denote ambiguously by $\mathbb{E}_G(B\mathscr{A})$.

Definition 4.9. Define the (genuine) algebraic K-theory G-spectrum of a $\mathscr{P}_G$-category $\mathscr{A}$ by $\mathbb{K}_G(\mathscr{A}) = \mathbb{E}_G(B\mathscr{A})$.

We might also start with an operad $\mathscr{P}$ of categories such that $|\mathscr{P}|$ is an E_∞ operad of spaces and regard these as G-objects with trivial action. Following up the previous section, we then have the following related but less interesting notion.

Definition 4.10. Define the (naive) algebraic K-theory G-spectrum of a $\mathscr{P}$-category $\mathscr{A}$ by $\mathbb{K}(\mathscr{A}) = \mathbb{E}(B\mathscr{A})$.

Until §7, we restrict attention to the cases $\mathscr{P}_G = \mathscr{O}_G$ and $\mathscr{P} = \mathscr{O}$, recalling that the $\mathscr{O}_G$-categories are the genuine permutative G-categories, the $\mathscr{O}$-categories are the naive permutative G-categories, and the inclusion $\iota\colon \mathscr{O} \longrightarrow \mathscr{O}_G$ induces a forgetful functor ι^* from genuine to naive permutative G-categories. Since the classifying space functor commmutes with products, passage to fixed points, and ι^*, Theorems 4.6, 4.7, and 4.8 have the following immediate corollaries. The second was promised in [11, Thm 2.8].

Theorem 4.11. *Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{O}_G$-categories. Then the map*

$$\mathbb{K}_G(\mathscr{A}\times\mathscr{B}) \longrightarrow \mathbb{K}_G\mathscr{A}\times\mathbb{K}_G\mathscr{B}$$

induced by the projections is a weak equivalence of G-spectra.

Theorem 4.12. *For $\mathscr{O}_G$-categories $\mathscr{A}$, there is a natural weak equivalence of spectra*

$$\mathbb{K}(\mathscr{A}^G) \longrightarrow (\mathbb{K}_G\mathscr{A})^G.$$

Theorem 4.13. *For $\mathscr{O}_G$-categories $\mathscr{A}$, there is a natural weak equivalence of naive G-spectra $\mu\colon \mathbb{K}\iota^*\mathscr{A} \longrightarrow \iota^*\mathbb{K}_G\mathscr{A}$.*

The algebraic K-groups of $\mathscr{A}$ are defined to be the groups

$$K^H_*\mathscr{A} = \pi^H_*(\mathbb{K}\iota^*\mathscr{A}) \cong \pi^H_*(\mathbb{K}_G\mathscr{A}). \tag{4.14}$$

We are particularly interested in examples of the form $\mathscr{A} = \mathscr{G}\mathscr{B}$, where $\mathscr{B}$ is a naive permutative G-category. As noted in Proposition 2.5, we then have a natural map $\iota\colon \mathscr{B} \longrightarrow \iota^*\mathscr{G}\mathscr{B}$ of naive permutative G categories. We can pass to classifying spaces and apply the functor $\mathbb{E}$ to obtain a natural map

$$\mathbb{K}\mathscr{B} \xrightarrow{\mathbb{K}\iota} \mathbb{K}\iota^*\mathscr{G}\mathscr{B} \underset{\simeq}{\xrightarrow{\mu}} \iota^*\mathbb{K}_G\mathscr{G}\mathscr{B}. \tag{4.15}$$

This map is a weak equivalence when $\iota^H\colon \mathscr{B}^H \longrightarrow (\iota^*\mathscr{G}\mathscr{B})^H$ is an equivalence of categories for all $H\subset G$. The following example where this holds is important in equivariant algebraic K-theory.

Example 4.16. Let K be a Galois extension of k with Galois group G and let G act entrywise on $GL(n,K)$ for $n\geq 0$. The disjoint union of the $GL(n,K)$ is a naive permutative G-category $GL(K/k)$ with product given by the block sum of matrices. Write $GL(R)$ for the nonequivariant permutative general linear category of a ring R. As observed in [13, 4.20], Hilbert's Theorem 90 implies that

$$\iota^H\colon GL(K^H) \cong GL(K/k)^H \longrightarrow (\iota^*\mathscr{G}(GL(K/k))^H$$

is an equivalence of categories for all $H\subset G$. This identifies the equivariant algebraic K-groups of the Galois extension with the nonequivariant algebraic K-groups of its fixed fields.

5. The Barratt-Priddy-Quillen and tom Dieck splitting theorems

5.1. **The equivariant Barratt-Priddy-Quillen theorem.**

The equivariant BPQ theorem shows how to model suspension G-spectra in terms of free E_∞ G-categories and G-spaces. It is built tautologically into the equivariant infinite loop space machine in the same way as it is nonequivariantly [22, 2.3(vii)] or [28, §10]. We work with the machine $\mathbb{E}_G = \mathbb{E}^{\mathscr{S}}_G$ in this section in order to focus on actual suspension

G-spectra and not just fibrant replacements, but it is a simple matter to retool so as to work with $\mathbb{E}_G^{\mathscr{L}}$ instead; see §6.2.

Theorem 5.1 (Equivariant Barratt-Priddy-Quillen theorem)**.** *For an E_∞ operad $\mathscr{C}_G$ of G-spaces and based G-spaces Y, there is a natural weak equivalence*

$$\Sigma_G^\infty Y \longrightarrow \mathbb{E}_G \mathbf{C_G} Y.$$

Proof. The nonequivariant Barratt-Quillen theorem is the case $G = e$. For formal reasons explained in [21, 9.9] we have a natural homotopy equivalence of G-spectra

$$\mathbb{E}_G C_G Y = B(\Sigma_G^\infty, \mathbf{C}_G, \mathbf{C}_G Y) \xrightarrow{\varepsilon} \Sigma_G^\infty Y.$$

Its homotopy inverse is the map $\eta\colon \Sigma_G^\infty Y \longrightarrow \mathbb{E}_G Y$ adjoint to the composite map

$$Y \longrightarrow \mathbf{C}_G Y \longrightarrow B(Q_G, \mathbf{C}_G, \mathbf{C}_G Y) \xrightarrow{\zeta} \Omega_G^\infty B(\Sigma_G^\infty, \mathbf{C}_G, \mathbf{C}_G Y) = \Omega_G^\infty \mathbb{E}_G \mathbf{C}_G Y$$

of G-spaces, where the first two maps are natural inclusions. □

In fact, with the model theoretic modernization of the original version of the theory that is given nonequivariantly in [1][6] one can redefine the restriction of $\mathbb{E}_G$ to cofibrant $\mathscr{C}_G$-spaces Y to be

$$\mathbb{E}_G Y = \Sigma_G^\infty \otimes_{\mathbf{C}_G} Y,$$

where $\otimes_{\mathbf{C}_G}$ is the evident coequalizer. With that reinterpretation, $\mathbb{E}_G C_G Y$ is actually isomorphic to $\Sigma_G^\infty Y$ when Y is a G-CW complex.

The nonequivariant statement is often restricted to the case $Y = S^0$. Then $\mathbf{C}S^0$ is the disjoint union of operadic models for the classifying spaces $B\Sigma_j$. Similarly, $\mathbf{C_G}S^0$ is the disjoint union of operadic models for the classifying G-spaces $B(G, \Sigma_j)$.

Taking $Y = X_+$ for an unbased G-space X and using (3.2), we can rewrite the BPQ theorem categorically.

Theorem 5.2. *For unbased G-spaces X, there is a natural weak equivalence*

$$\Sigma_G^\infty X_+ \longrightarrow \mathbb{K}_G \mathbb{O}_G(X_+).$$

5.2. **The tom Dieck splitting theorem.** The G-fixed point spectra of suspension G-spectra have a well-known splitting. It is due to tom Dieck [6] on the level of homotopy groups and was lifted to the spectrum level in [18, §V.11]. The tom Dieck splitting actually works for all compact Lie groups G, but we have nothing helpful to add in that generality. Our group G is always finite. In that case, we have already given the ingredients for a new categorical proof, as we now explain.

Theorem 5.3. *For a based G-space Y,*

$$(\Sigma_G^\infty Y)^G \simeq \bigvee_{(H)} \Sigma^\infty (EWH_+ \wedge_{WH} Y^H).$$

The wedge runs over the conjugacy classes of subgroups H of G, and $WH = NH/H$.

Theorem 5.3 and the evident natural identifications

$$EWH_+ \wedge_{WH} X_+^H \cong (EWH \times_{WH} X^H)_+ \tag{5.4}$$

imply the following version for unbased G-spaces X.

[6]We plan an equivariant generalization elsewhere.

Theorem 5.5. *For an unbased G-space X,*

$$(\Sigma_G^\infty X_+)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH \times_{WH} X^H)_+.$$

Conversely, we can easily deduce Theorem 5.3 from Theorem 5.5. Viewing S^0 as $\{1\}_+$ with trivial G action, our standing assumption that basepoints are nondegenerate gives a based G-cofibration $S^0 \longrightarrow Y_+$ that sends 1 to the basepoint of Y, and $Y = Y_+/S^0$. The functors appearing in Theorem 5.5 preserve cofiber sequences, and the identifications (5.4) imply identifications

$$(5.6) \qquad (EWH \times_{WH} Y^H)_+/(EWH \times_{WH} \{1\})_+ \cong EWH_+ \wedge_{WH} Y^H.$$

Therefore Theorem 5.5 implies Theorem 5.3.

We explain these splittings in terms of the equivariant BPQ theorem. We begin in the based setting. The nonequivariant case $G = e$ of the BPQ theorem relates to the equivariant case through Theorem 4.7. Explicitly, Theorems 4.7 and 5.1 give a pair of weak equivalences

$$(5.7) \qquad (\Sigma_G^\infty Y)^G \longrightarrow (\mathbb{E}_G \mathbf{C}_G Y)^G \longleftarrow \mathbb{E}((\mathbf{C}_G Y)^G).$$

Since the functor Σ^∞ commutes with wedges, the nonequivariant BPQ theorem gives a weak equivalence

$$(5.8) \qquad \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} Y^H) \longrightarrow \mathbb{E}\mathbf{C}(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H).$$

If we could prove that there is a natural weak equivalence of $\mathscr{C}$-spaces

$$(\mathbf{C}_G Y)^G \simeq \mathbf{C}(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H),$$

that would imply a natural weak equivalence

$$(5.9) \qquad \mathbb{E}((\mathbf{C}_G Y)^G) \simeq \mathbb{E}\mathbf{C}(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H).$$

and complete the proof of Theorem 5.3. However, the combinatorial study of the behavior of C on wedges is complicated by the obvious fact that wedges do not commute with products.

We use the following consequence of Theorem 3.5 and the relationship between wedges and products of spectra to get around this. Recall that $\mathbf{O}_G$ is the monad on based G-spaces obtained from the operad $|\mathscr{O}_G|$ of G-spaces.

Theorem 5.10. *For unbased G-spaces X, there is a natural equivalence of $|\mathscr{O}|$-spaces*

$$(\mathbf{O}_G X_+)^G \simeq \prod_{(H)} \mathbf{O}(EWH \times_{WH} X^H)_+,$$

where (H) runs over the conjugacy classes of subgroups of G and $WH = NH/H$.

Proof. Remembering that $|\tilde{G}| = EG$, we see that the classifying space of the category $\widetilde{WH} \times_{WH} X^H$ can be identified with $EWH \times_{WH} X^H$. The commutation relations between $|-|$ and the constituent functors used to construct the monads $\mathbf{O}_G$ on G-spaces and $\mathbb{O}_G$ on G-categories make the identification clear. □

Remark 5.11. Of course, we can and must replace $\mathscr{O}_G$ and $\mathscr{O}$ by their products with the equivariant and nonequivariant Steiner operad to fit into the infinite loop space machine. There is no harm in doing so since if we denote the product operads by $\mathscr{C}_G$ and $\mathscr{C}$, as before, the projections $\mathscr{C}_G \longrightarrow \mathscr{O}_G$ and $\mathscr{C} \longrightarrow \mathscr{O}$ induce weak equivalences of monads that fit into a commutative diagram

$$\begin{array}{ccc} (\mathbf{C_G} X_+)^G & \xrightarrow{\simeq} & \prod_{(H)} \mathbf{C}(EWH \times_{WH} X^H)_+ \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathbf{O}_G X_+)^G & \xrightarrow{\simeq} & \prod_{(H)} \mathbf{O}(EWH \times_{WH} X^H)_+. \end{array}$$

The functor Σ^∞_G commutes with wedges, and the natural map of G-spectra

$$E \vee F \longrightarrow E \times F$$

is a weak equivalence. Theorems 4.6 and 5.1 have the following implication. We state it equivariantly, but we shall apply its nonequivariant special case.

Proposition 5.12. *For based G-spaces X and Y, the natural map*

$$\mathbb{E}_G \mathbf{C}_G(X \vee Y) \longrightarrow \mathbb{E}_G(\mathbf{C}_G X \times \mathbf{C}_G Y)$$

is a weak equivalence of spectra.

Proof. The following diagram commutes by the universal property of products.

$$\begin{array}{ccc} \Sigma^\infty_G(X \vee Y) & \longrightarrow & \mathbb{E}_G \mathbf{C}_G(X \vee Y) \\ \downarrow \cong & & \downarrow \\ \Sigma^\infty_G X \vee \Sigma^\infty_G Y & & \mathbb{E}_G(\mathbf{C}_G X \times \mathbf{C}_G Y) \\ \downarrow & & \downarrow \\ \Sigma^\infty_G X \times \Sigma^\infty_G Y & \longrightarrow & \mathbb{E}_G \mathbf{C}_G X \times \mathbb{E}_G \mathbf{C}_G Y. \end{array}$$

All arrows except the upper right vertical one are weak equivalences, hence that arrow is also a weak equivalence. □

For any E_∞ operad $\mathscr{C}$, we therefore have a weak equivalence

$$\mathbb{E}\mathbf{C}\Big(\bigvee_{(H)} (EWH_+ \wedge_{WH} Y^H)\Big) \longrightarrow \mathbb{E} \prod_{(H)} \mathbf{C}(EWH_+ \wedge_{WH} Y^H). \qquad (5.13)$$

Together with (5.13), Theorem 5.10 and Remark 5.11 give a weak equivalence (5.9) in the case $Y = X_+$. Together with (5.7) and (5.8), this completes the proof of Theorem 5.5, and Theorem 5.3 follows.

6. Pairings and the G-category $\mathscr{E}_G$

We use orthogonal G-spectra and the machine $\mathbb{E}_G = \mathbb{E}^{\mathscr{L}}_G$ in this section. With this machine, we can use an elementary treatment of pairings to construct a version of the G-category $\mathscr{E}_G$ that we promised in [11] and so complete the proofs of the results claimed in that paper.

6.1. **Pairings of G-operads and pairings of G-spectra.** We recall the following definition from [24, 1.4]. It applies equally well equivariantly. We write it elementwise, but written diagrammatically it applies to operads in any symmetric monoidal category. Write $\mathbf{j} = \{1, \cdots, j\}$ and let

$$\otimes\colon \Sigma_j \times \Sigma_k \longrightarrow \Sigma_{jk}$$

be the homomorphism obtained by identifying $\mathbf{j} \times \mathbf{k}$ with $\mathbf{jk}$ by ordering the set of jk elements (q, r), $1 \le q \le j$ and $1 \le r \le k$, lexicographically. For nonnegative integers h_q and i_r, let

$$\delta\colon (\coprod_{(q,r)} (\mathbf{h}_q \times \mathbf{i}_r) \longrightarrow (\coprod_q \mathbf{h}_q) \times (\coprod_r \mathbf{i}_r)$$

be the distributivity isomorphism viewed as a permutation (via block and lexicographic identifications of the source and target sets with the appropriate set $\mathbf{n}$).

Definition 6.1. Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be operads in a symmetric monoidal category $\mathscr{V}$ (with product denoted $\boxtimes$). A pairing of operads

$$\otimes\colon (\mathscr{C}, \mathscr{D}) \longrightarrow \mathscr{E}$$

consists of maps

$$\otimes\colon \mathscr{C}(j) \boxtimes \mathscr{D}(k) \longrightarrow \mathscr{E}(jk)$$

in $\mathscr{V}$ for $j \ge 0$ and $k \ge 0$ such that the diagramatic versions of the following properties hold. Let $c \in \mathscr{C}(j)$ and $d \in \mathscr{D}(k)$.

(i) If $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$, then

$$c\mu \otimes d\nu = (c \otimes d)(\mu \otimes \nu)$$

(ii) With $j = k = 1$, $\mathrm{id} \otimes \mathrm{id} = \mathrm{id}$.

(iii) If $c_q \in \mathscr{C}(h_q)$ for $1 \le q \le j$ and $d_r \in \mathscr{D}(i_r)$ for $1 \le r \le k$, then

$$\gamma(c \otimes d; \times_{(q,r)} c_q \otimes d_r)\delta = (\gamma(c; \times_q c_q) \otimes \gamma(d; \times_r d_r).$$

A permutative operad $\mathscr{D}$ is an operad equipped with a unital and associative pairing $(\mathscr{D}, \mathscr{D}) \longrightarrow \mathscr{D}$ which is commutative in the sense that $c \otimes d = (d \otimes c)\tau(j,k)$, where $\tau(j,k)$ is the permutation of the set $\mathbf{jk}$ that maps it from its lexicographic identification with $\mathbf{j} \times \mathbf{k}$ to its lexicographic identification with $\mathbf{k} \times \mathbf{j}$.

Restricting to spaces or G-spaces, with $\boxtimes = \wedge$, and passing to monads on based spaces or based G-spaces, we have the following observation.

Lemma 6.2. *A pairing $\otimes\colon (\mathscr{C}_G, \mathscr{D}_G) \longrightarrow \mathscr{E}_G$ of operads of G-spaces induces a natural map*

$$\otimes\colon \mathbf{C}_G X \wedge \mathbf{D}_G Y \longrightarrow \mathbf{E}_G(X \wedge Y)$$

such that the following diagrams commute.

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\eta \wedge \eta} & \mathbf{C}_G X \wedge \mathbf{D}_G Y \\ & \searrow^{\eta} & \downarrow{\scriptstyle \otimes} \\ & & \mathbf{E}_G(X \wedge Y) \end{array}$$

$$\begin{array}{ccccc}
\mathbf{C}_G\mathbf{C}_G X \wedge \mathbf{D}_G\mathbf{D}_G Y & & \xrightarrow{\mu\wedge\mu} & & \mathbf{C}_G X \wedge \mathbf{D}_G Y \\
\downarrow{\scriptstyle\otimes} & & & & \downarrow{\scriptstyle\otimes} \\
\mathbf{E}_G(\mathbf{C}_G X \wedge \mathbf{D}_G Y) & \xrightarrow[\mathbf{E}_G\otimes]{} & \mathbf{E}_G\mathbf{E}_G(X\wedge Y) & \xrightarrow[\mu]{} & \mathbf{E}_G(X\wedge Y)
\end{array}$$

Proof. If $x = (x_1, \cdots, x_j) \in X^j$ and $y = (y_1, \cdots, y_k) \in Y^k$, then

$$(c, x) \otimes (d, y) = (c \otimes d, z),$$

where $z \in (X \wedge Y)^{jk}$ is the jk-tuple $(x_a \wedge y_b)$, ordered lexicographically. The diagrams are checked by chases from the definition. □

There is a general notion of a pairing of a $\mathscr{C}_G$-space X and a $\mathscr{D}_G$-space Y to an $\mathscr{E}_G$-space Z, obtained by specialization of [24, 1.1], but we shall not need that for the applications here. Suffice it to say that the pairing $\mathbf{C}_G X \wedge \mathbf{D}_G Y \longrightarrow \mathbf{E}_G(X \wedge Y)$ is an example, the only one relevant to us in this paper. We have two obvious examples of pairings of operads and a hybrid. The first is from [23, p. 248] and starts with a pairing of operads in $\mathscr{C}at$.

Example 6.3. The unique functors $\otimes\colon \tilde{\Sigma}_j \times \tilde{\Sigma}_k \longrightarrow \tilde{\Sigma}_{jk}$ given on objects by $\otimes\colon \Sigma_j \times \Sigma_k \longrightarrow \Sigma_{jk}$ specify a pairing $(\mathscr{O}, \mathscr{O}) \longrightarrow \mathscr{O}$ such that $\mathscr{O}$ is a permutative operad in $\mathscr{C}at$. Since the functor $\mathscr{C}at_G(\tilde{G}, -)$ preserves products, there results a pairing $(\mathscr{O}_G, \mathscr{O}_G) \longrightarrow \mathscr{O}_G$ that makes $\mathscr{O}_G$ a permutative operad in $G\mathscr{C}at$. Applying the classifying space functor, there result pairings

$$\otimes\colon (|\mathscr{O}|, |\mathscr{O}|) \longrightarrow |\mathscr{O}| \quad \text{and} \quad \otimes\colon (|\mathscr{O}_G|, |\mathscr{O}_G|) \longrightarrow |\mathscr{O}_G|$$

that make $|\mathscr{O}|$ and $|\mathscr{O}_G|$ permutative operads of spaces and G-spaces.

Details of the next example are given in §10.2.

Example 6.4. For finite dimensional real inner product spaces V and W, there is a unital, associative, and commutative system of pairings $\otimes\colon (\mathscr{K}_V, \mathscr{K}_W) \longrightarrow \mathscr{K}_{V\oplus W}$ of Steiner operads of G-spaces.

Example 6.5. If $\otimes\colon (\mathscr{C}, \mathscr{D}) \longrightarrow \mathscr{E}$ and $\otimes\colon (\mathscr{C}', \mathscr{D}') \longrightarrow \mathscr{E}'$ are pairings of operads, then the maps

$$\begin{array}{c}
\mathscr{C}(j) \times \mathscr{C}'(j) \times \mathscr{D}(k) \times \mathscr{D}'(k) \\
\downarrow{\scriptstyle \mathrm{id}\times t\times \mathrm{id}} \\
\mathscr{C}(j) \times \mathscr{D}(j) \times \mathscr{C}'(k) \times \mathscr{D}'(k) \\
\downarrow{\scriptstyle \otimes\times\otimes} \\
\mathscr{E}(jk) \times \mathscr{E}'(jk)
\end{array}$$

specify a pairing of operads $(\mathscr{C} \times \mathscr{C}', \mathscr{D} \times \mathscr{D}') \longrightarrow \mathscr{E} \times \mathscr{E}'$.

These examples can be combined by use of the following observation.

Lemma 6.6. *Let $\mathscr{C}_G$ be a permutative category of G-spaces and let $\mathscr{C}_V = \mathscr{C}_G \times \mathscr{K}_V$. Then there is a unital, associative, and commutative system of pairings of operads*

$$\otimes\colon (\mathscr{C}_V, \mathscr{C}_W) \longrightarrow \mathscr{C}_{V\oplus W}.$$

Passing to monads, we obtain a system of pairings

$$\otimes\colon \mathbf{C}_V X \wedge \mathbf{C}_W Y \longrightarrow \mathbf{C}_{V\oplus W}(X\wedge Y).$$

We can compose with $\mathbf{C}_{V\oplus W} f$ for any map $f\colon X\wedge Y \longrightarrow Z$ of based G-spaces.

We use orthogonal G-spectra and the machine $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{I}}$ to put things together and obtain the following result. If $\mathscr{C}_G$ acts on X and Y, then, using the diagonal on the spaces $\mathscr{C}_G(j)$, it acts on $X\times Y$. However, this action does not descend to an action of $\mathscr{C}_G$ on $X\wedge Y$. The general notion of a pairing $(X,Y)\longrightarrow Z$ of $\mathscr{C}_G$-spaces can be used to generalize the following result.

Proposition 6.7. *Let $\mathscr{C}_G$ be a permutative operad of G-spaces, such as $|\mathscr{O}_G|$, and let X and Y be based G-spaces. There is a natural associative system of pairings*

$$\mathbb{E}_G\mathbf{C}_G(X)(V)\wedge \mathbb{E}_G\mathbf{C}_G(Y)(W) \longrightarrow \mathbb{E}_G\mathbf{C}_G(X\wedge Y)(V\oplus W)$$

Proof. The bar construction $B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_G X)$ is the geometric realization of a simplicial G-space with q-simplices $\Sigma^V \mathbf{C}_V^q \mathbf{C}_G X$. Using the pairing above inductively, along with the projection $\mathbf{C}_V \longrightarrow \mathbf{C}_G$, we obtain G-maps

$$\Sigma^V \mathbf{C}_V^q \mathbf{C}_G X \wedge \Sigma^W \mathbf{C}_W^q \mathbf{C}_G X \longrightarrow \Sigma^{V\oplus W}\mathbf{C}_{V\oplus W}^q \mathbf{C}_G(X\wedge Y)$$

These commute with faces and degeneracies, and since geometric realization commutes with products they induce the claimed maps

$$B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_G X)\wedge B(\Sigma^W, \mathbf{C}_W, \mathbf{C}_G X) \longrightarrow B(\Sigma^{V\oplus W}, \mathbf{C}_{V\oplus W}, \mathbf{C}_G(X\wedge Y)) \quad \square$$

The pairings of the proposition give a map

$$\mathbb{E}_G\mathbf{C}_G(X)\,\bar{\wedge}\,\mathbb{E}_G\mathbf{C}_G(Y) \longrightarrow \mathbb{E}_G\mathbf{C}_G(X\wedge Y)\circ\oplus$$

of functors $\mathscr{I}_G\times\mathscr{I}_G \longrightarrow \mathscr{T}_G$, where $\bar{\wedge}$ denotes the external smash product. Taking $\mathscr{C}_G = |\mathscr{O}_G|$, we obtain the following result directly from the definition of the smash product of orthogonal G-spectra [19, §§I.2, II.3].

Theorem 6.8. *For based G-spaces X and Y, there is a natural pairing of orthogonal G-spectra*

$$\wedge\colon \mathbb{E}_G\mathbf{O}_G(X)\wedge \mathbb{E}_G\mathbf{O}_G(Y) \longrightarrow \mathbb{E}_G\mathbf{O}_G(X\wedge Y).$$

Recall that, for an unbased G-space X, $\mathbf{O_G(X_+)} = |\mathbb{O}_\mathbf{G}(\mathbf{X}_+)| \equiv \mathbf{B}\mathbb{O}_\mathbf{G}(\mathbf{X}_+)$, where X is viewed as a topological category. Writing $\mathbb{K}_G(\mathscr{A}) = \mathbb{E}_G(B\mathscr{A})$ as before, we have the following equivalent reformulation for unbased G-spaces.

Theorem 6.9. *For unbased G-spaces X and Y, there is a natural pairing of orthogonal G-spectra*

$$\wedge\colon \mathbb{K}_G\mathbb{O}_G(X_+)\wedge \mathbb{K}_G\mathbb{O}_G(Y_+) \longrightarrow \mathbb{K}_G\mathbb{O}_G((X\times Y)_+).$$

Said another way, we start with the pairing of genuine permutative G-categories

$$\mathbb{O}_G(X_+)\times\mathbb{O}_G(Y_+) \longrightarrow \mathbb{O}_G((X\times Y)_+) \tag{6.10}$$

that is given explicitly as the composite

$$\begin{array}{c}\coprod_j(\mathscr{O}_G(j)\times_{\Sigma_j}X^j)\times\coprod_k(\mathscr{O}_G(k)\times_{\Sigma_k}Y^k)\\ \big\downarrow\\ \coprod_{j,k}\mathscr{O}_G(jk)\times_{\Sigma_{jk}}(X\times Y)^{jk}\\ \big\downarrow\\ \coprod_\ell\mathscr{O}_G(\ell)\times_{\Sigma_\ell}(X\times Y)^\ell\end{array}$$

determined by the pairing $(\mathscr{O}_G, \mathscr{O}_G) \longrightarrow \mathscr{O}_G$. Observe that the terms with $j = 0$ or $k = 0$ all map to the term with $\ell = 0$, which is a point, so that the induced map of classifying spaces factors through the smash product. We apply the infinite loop space machine $\mathbb{K}_G = \mathbb{E}_G B$ to this pairing, and we regard the processing of pairings described in this section as part of the machine.

6.2. **The BPQ theorem for the machine $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{L}}$.** The suspension orthogonal G-spectrum $\Sigma_G^\infty Y$ has Vth space $\Sigma^V Y$. It is cofibrant in the stable model structure if Y is a cofibrant based G-space, such as a based G-CW complex. The following variant of Theorem 5.1 holds, and we shall use it to complete the proofs promised in [11, §2.5 and §2.7].

Theorem 6.11 (Equivariant Barratt–Priddy–Quillen theorem)**.** *For an E_∞ operad $\mathscr{C}_G$ of G-spaces, there is a weak equivalence of orthogonal G-spectra*

$$\alpha\colon \Sigma_G^\infty Y \longrightarrow \mathbb{E}_G\mathbf{C_G}Y$$

which is natural on G-maps $Y \longrightarrow Y'$ of based G-spaces.

Proof. Recall that $\mathscr{C}_V = \mathscr{K}_V \times \mathscr{C}_G$. There is an evident orthogonal G-spectrum $\mathbb{D}_G Y$ with Vth space

$$(\mathbb{D}_G Y)(V) = B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_V Y).$$

The projections $\mathbf{C}_V Y \longrightarrow \mathbf{C}_G Y$ induce a weak equivalence $\mathbb{D}_G Y \longrightarrow \mathbb{E}_G\mathbf{C}_G Y$. The functor $\Sigma_G^\infty Y$ is left adjoint to the zeroth space functor. Since $\mathscr{K}_0(j)$ is empty for $j > 1$ and a point for $j = 0$ or $j = 1$ (see §10), $\mathscr{C}_0(j)$ is empty for $j > 1$ and is $\mathscr{C}_G(j)$ for $j = 0$ or $j = 1$. Using the unit id $\in \mathscr{C}(1)$, we obtain a canonical map $Y \longrightarrow (\mathbb{D}_G Y)(0)$, hence a canonical composite map

$$\Sigma_G^\infty Y \longrightarrow \mathbb{D}_G Y \longrightarrow \mathbb{E}_G\mathbf{C}_G Y.$$

The first arrow is a homotopy equivalence with homotopy inverse given by the maps of Vth spaces

$$\varepsilon\colon B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_V Y) \longrightarrow \Sigma^V Y$$

induced by the evident maps of simplicial spaces from the $B_*(\Sigma^V, \mathbf{C}_V, \mathbf{C}_V Y)$ to the constant simplicial spaces at $\Sigma^V Y$. □

Taking $\mathscr{C}_G = \mathscr{O}_G$, the following result gives consistency with our pairings from the previous section.

Theorem 6.12. *The following diagram of G-spectra commutes for based G-spaces X and Y.*

$$\begin{array}{ccc} \Sigma^\infty_G X \wedge \Sigma^\infty_G Y & \xrightarrow{\alpha\wedge\alpha} & \mathbb{E}_G \mathbf{O}_G(X) \wedge \mathbb{E}_G \mathbf{O}_G(Y) \\ {\scriptstyle\wedge}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\wedge} \\ \Sigma^\infty_G (X\wedge Y) & \xrightarrow[\alpha]{} & \mathbb{E}_G \mathbf{O}_G(X\wedge Y) \end{array}$$

Proof. A check of definitions shows that the following diagrams commute, where the left vertical arrow is the evident homeomorphism.

$$\begin{array}{ccc} \Sigma^V X \wedge \Sigma^W Y & \longrightarrow & B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_G X) \wedge B(\Sigma^W, \mathbf{C}_W, \mathbf{C}_G X) \\ \downarrow & & \downarrow \\ \Sigma^{V\oplus W}(X\wedge Y) & \longrightarrow & B(\Sigma^{V\oplus W}, \mathbf{C}_{V\oplus W}, \mathbf{C}_G(X\wedge Y)) \end{array}$$

This gives the relevant diagram on the external level, and passing to the internal smash product gives the conclusion. □

6.3. **The G-bicategory $\mathscr{E}_G = \mathscr{E}_G^{\mathscr{O}}$.** We specialize to finite G-sets A and B instead of general G-spaces X and Y. Returning to the level of categories, recall from Definition 3.12 that we write

$$\mathscr{E}_G(A) = \mathscr{E}_G^{\mathscr{S}}(A) = \mathbb{O}_G(A_+).$$

Again writing $\mathbb{K}_G\mathscr{A} = \mathbb{E}_G B\mathscr{A}$, Theorem 6.12 specializes as follows.

Theorem 6.13. *The following diagram of G-spectra commutes for finite G-sets A and B.*

$$\begin{array}{ccc} \Sigma^\infty_G A_+ \wedge \Sigma^\infty_G B_+ & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G\mathscr{E}_G(A) \wedge \mathbb{K}_G\mathscr{E}_G(B) \\ {\scriptstyle\wedge}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\wedge} \\ \Sigma^\infty_G (A\times B)_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(A\times B). \end{array}$$

For a G-map $f\colon A \longrightarrow B$, we have a map of $\mathscr{O}_G$-categories

$$f_!\colon \mathscr{E}_G(A) \longrightarrow \mathscr{E}_G(B).$$

The naturality statement of Theorem 6.11 specializes to give that the following diagram commutes.

(6.14)
$$\begin{array}{ccc} \Sigma^\infty_G A_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(A) \\ {\scriptstyle\Sigma^\infty_G f_+}\downarrow & & \downarrow{\scriptstyle\mathbb{K}_G f_!} \\ \Sigma^\infty_G B_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(B). \end{array}$$

Observe that we do not have a right adjoint f^* to the "base change" functors $f_!$ for general maps of G-spaces f. However, we do have a version of such a functor for an inclusion of finite G-sets, and then our covariant naturality specializes to a kind of contravariant naturality.

Proposition 6.15. *Let $i\colon A \longrightarrow B$ be an inclusion of finite G-sets and define a G-map $t\colon B_+ \longrightarrow A_+$ by $ti(a) = a$ for $a \in A$ and $t(b) = *$ for $b \notin \mathrm{im}(i)$. Then t induces a map*

$$i^*\colon \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A)$$

of $\mathscr{O}_G$-categories such that the following diagram of G-spectra commutes.

$$\begin{array}{ccc} \Sigma^\infty_G B_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(B) \\ {\scriptstyle \Sigma^\infty_G t}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G i^*} \\ \Sigma^\infty_G A_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(A) \end{array}$$

Proof. Identifying A with $i(A)$ and letting $C = B - i(A)$, we may identify B with the disjoint union $A \amalg C$ and therefore B_+ with $A_+ \vee C_+$. Then t restricts to the identity map on A_+ and to the trivial map to the basepoint on C_+. Since $\mathbb{O}_G(A_+)$ is functorial on all maps of based G-spaces, not just those of the form f_+, the map t induces the required map $t_! = i^*$. □

We now perform a bait and switch. In [11, §2], we described in intuitive terms a G-bicategory $\mathscr{E}_G$ enriched in permutative categories with G-fixed category $G\mathscr{E}$ such that the category of enriched presheaves $\mathbb{K}(G\mathscr{E})^{op} \longrightarrow \mathscr{S}$ is Quillen equivalent to $G\mathscr{S}$. We give definitions here that allow us to prove the results of [11], but the definitions that we start with are less intuitive variants of those of [11, §2]. We shall give alternative definitions that closely follow those of [11] in §9.3, but they depend on use of an E_∞ operad $\mathscr{P}_G$ for which we do not have the elementary theory of pairings of §6.1. We shall indicate an alternative approach to multiplicative structures that should be applicable, but the details are not yet in place. We here prove everything that is needed to make the arguments of [11] rigorous, but from our alternative categorical starting point.

The objects of $\mathscr{E}_G = \mathscr{E}_G^{\mathscr{O}}$ are the finite G-sets A, and they can be thought of as the based $\mathscr{O}_G$-categories $\mathbb{O}_G(A_+) = \mathscr{E}_G(A)$. The $\mathscr{O}_G$-category $\mathscr{E}_G(A, B)$ of morphisms $A \longrightarrow B$ is $\mathbb{O}_G((B\times A)_+) = \mathscr{E}_G(B\times A)$. The unit $\eta\colon *_+ \longrightarrow \mathscr{E}_G(B, B)$ is the map of based $\mathscr{O}_G$-categories that sends the object of $*$ to the object $(\beta, (1, 1), \cdots, (j, j))$ of $\mathscr{O}_G(j) \times (B \times B)^j$ where B is a j-pointed G-set thought of as $\mathbf{j}$ with left action of G specified by β^{-1} for an anti-homomorphism $\beta\colon G \longrightarrow \Sigma_j$ (see Lemma 3.15). Composition is given by the following composite, where the first map is a specialization of (6.10).

$$\begin{array}{c} \mathscr{E}_G(C \times B) \times \mathscr{E}_G(B \times A) \\ \downarrow \\ \mathscr{E}_G(C \times B \times B \times A) \\ \downarrow{\scriptstyle (\mathrm{id}\times\Delta\times\mathrm{id})^*} \\ \mathscr{E}_G(C \times B \times A) \\ \downarrow{\scriptstyle (\mathrm{id}\times\varepsilon\times\mathrm{id})_!} \\ \mathscr{E}_G(C \times A). \end{array}$$

The associativity of composition is an easy diagram chase, starting from the associativity of the pairing of (6.10), as is the verification that composition with the prescribed units gives identity functors. Implicitly, we have given a model for the G-bicategory of spans described in [11, 2.5]. Applying the functor $\mathbb{K}_G$, we obtain the following version of [11, 2.7].

Theorem 6.16. *The composition pairing and unit functors of $\mathscr{E}_G$ induce pairings*

$$\mathbb{K}_G\mathscr{E}_G(B,C) \wedge \mathbb{K}_G\mathscr{E}_G(A,B) \longrightarrow \mathbb{K}_G\mathscr{E}_G(A,C)$$

of (orthogonal) G-spectra and unit maps of G-spectra

$$S_G \longrightarrow \mathbb{K}_G\mathscr{E}_G(B,B)$$

that give us a (skeletally) small $\mathscr{S}_G$-category $\mathbb{K}_G\mathscr{E}_G$.

Using Theorem 4.12, there results an equivalence of $\mathscr{S}$-categories

$$\mathbb{K}(G\mathscr{E}) \longrightarrow (\mathbb{K}_G\mathscr{E}_G)^G,$$

as promised in [11, 2.8]. The main theorem, [11, 2.9], of [11] asserts that $\mathbb{K}_G\mathscr{E}_G$ is equivalent to a certain full $\mathscr{S}_G$-category $\mathscr{B}_G$ of $G\mathscr{S}$. In [11, §2.6], its proof is reduced to the verification that a certain diagram commutes. In [11, §2.7], the commutativity of that diagram is reduced to verifying the naturality properties of the BPQ equivalence α that we have recorded in Theorem 6.13, (6.14), and Proposition 6.15 above. Therefore we have completed the proofs required to validate the results of [11].

7. The E_∞ operads $\mathscr{P}_G$, $\mathscr{Q}_G$, and $\mathscr{R}_G$

The operad $\mathscr{O}_G$ has a privileged conceptual role, but there are other categorical E_∞ G-operads with different good properties. We define three interrelated examples. The objects of the chaotic category $\mathscr{O}_G(j)$ are functions $G \longrightarrow \Sigma_j$. We give analogous chaotic G-categories in which the objects are suitable functions between well chosen infinite G-sets, with G again acting by conjugation. Their main advantage over $\mathscr{O}_G$ is that it is easier to recognize G-categories on which they act.

7.1. **The definitions of $\mathscr{P}_G$ and $\mathscr{Q}_G$.** We start with what we would like to take as a particularly natural choice for the j^{th} category of an E_∞ G-operad. It is described in more detail in [13, §6.1].

Definition 7.1. Let U be a countable ambient G-set that contains countably many copies of each orbit G/H. Let U^j be the product of j copies of U with diagonal action by G, and let ${}^jU$ be the disjoint union of j copies of the G-set U. Here U^0 is a one-point set, sometimes denoted 1, and ${}^0U$ is the empty set, sometimes denoted $\emptyset$ and sometimes denoted 0.

Let $\mathbf{j} = \{1, \cdots, j\}$ with its natural left action by Σ_j, written $\sigma\colon \mathbf{j} \longrightarrow \mathbf{j}$.

Definition 7.2. For $j \geq 0$, let $\tilde{\mathscr{E}}_G(j)$ be the chaotic $\Sigma_j \times G$-category whose objects are the pairs (A, α), where A is a j-element subset of U and $\alpha\colon \mathbf{j} \longrightarrow A$ is a bijection. The group G acts on objects by $g(A,\alpha) = (gA, g\alpha)$, where $(g\alpha)(i) = g \cdot \alpha(i)$. The group Σ_j acts on objects by $(A,\alpha)\sigma = (A, \alpha \circ \sigma)$ for $\sigma \in \Sigma_j$. Since $\tilde{\mathscr{E}}_G(j)$ is chaotic, this determines the actions on morphisms.

Proposition 7.3. [13, 6.3] *For each j, the classifying space $|\tilde{\mathscr{E}}_G(j)|$ is a universal principal (G, Σ_j)-bundle.*

Therefore $\tilde{\mathscr{E}}_G(j)$ satisfies the properties required of the j^{th} category of an E_∞ G-operad. However, these categories as j varies do not form an operad. The problem is a familiar one in topology: these categories are analogous to configuration spaces. In topology, in order to give configuration spaces the structure of an operad, one must fatten them up; examples are the little cubes operads and the little disks operads. Similarly, we must fatten up the above categories to provide enough room for the operad structure. We will say more about the analogy in Remark 9.7.

Definition 7.4. We define a reduced operad $\mathscr{P}_G$ of G-categories. Let $\mathscr{P}_G(j)$ be the chaotic G-category whose set of objects is the set of injective functions ${}^jU \longrightarrow U$. Let G act by conjugation and let Σ_j have the right action induced by its left action on ${}^jU$. Let $\mathrm{id} \in \mathscr{P}_G(1)$ be the identity function $U \longrightarrow U$. Define

$$\gamma\colon \mathscr{P}_G(k) \times \mathscr{P}_G(j_1) \times \cdots \times \mathscr{P}_G(j_k) \longrightarrow \mathscr{P}_G(j),$$

where $j = j_1 + \cdots + j_k$, to be the composite

$$\mathscr{P}_G(k) \times \mathscr{P}_G(j_1) \times \cdots \times \mathscr{P}_G(j_k) \longrightarrow \mathscr{P}_G(k) \times \mathscr{P}_G({}^jU, {}^kU) \longrightarrow \mathscr{P}_G(j)$$

obtained by first taking coproducts of maps and then composing, where $\mathscr{P}({}^jU, {}^kU)$ is the set of injections ${}^jU \longrightarrow {}^kU$. The operad axioms [21, 1.1] are easily verified.

As recalled briefly in §10, there is an E_∞ operad of G-spaces, denoted $\mathscr{L}_G$, whose j^{th}-space is the space of linear isometries $U^j \longrightarrow U$, where U here is a complete G-universe. Remembering that taking sets to the free R-modules they generate gives a coproduct-preserving functor from sets to R-modules, we see that $\mathscr{P}_G$ is a categorical analogue of $\mathscr{L}_G$.

There is a parallel definition that uses products instead of coproducts.

Definition 7.5. We define an unreduced operad $\bar{\mathscr{Q}}_G$ of G-categories. Let $\bar{\mathscr{Q}}_G(j)$ be the chaotic G-category whose set of objects is the set of injective functions $U^j \longrightarrow U$. Let G act by conjugation and let Σ_j have the right action induced by its left action on U^j. Let $\mathrm{id} \in \bar{\mathscr{Q}}_G(1)$ be the identity function. Define

$$\gamma\colon \bar{\mathscr{Q}}_G(k) \times \bar{\mathscr{Q}}_G(j_1) \times \cdots \times \bar{\mathscr{Q}}_G(j_k) \longrightarrow \bar{\mathscr{Q}}_G(j),$$

where $j = j_1 + \cdots + j_k$, to be the composite

$$\bar{\mathscr{Q}}_G(k) \times \bar{\mathscr{Q}}_G(j_1) \times \cdots \times \bar{\mathscr{Q}}_G(j_k) \longrightarrow \bar{\mathscr{Q}}_G(k) \times \bar{\mathscr{Q}}_G(U^j, U^k) \longrightarrow \bar{\mathscr{Q}}_G(j)$$

obtained by first taking products of maps and then composing, where $\bar{\mathscr{Q}}_G(U^j, U^k)$ is the set of injections $U^j \longrightarrow U^k$. Again, the operad axioms are easily verified.

Observe that the objects of $\bar{\mathscr{Q}}(0)$ are the injections from the point U^0 into U and can be identified with the set U, whereas $\mathscr{P}_G(0)$ is the trivial category given by the injection of the empty set ${}^0U$ into U. As in Remark 1.16, the objects of the zeroth category give unit objects for operad actions, and it is convenient to restrict attention to a reduced variant of $\bar{\mathscr{Q}}$.

Definition 7.6. Choose a G-fixed point $1 \in U$ (or, equivalently, adjoin a G-fixed basepoint 1 to U) and also write 1 for the single point in U^0. Give U^j, $j \geq 0$, the basepoint whose coordinates are all 1. The reduced variant of $\bar{\mathscr{Q}}$ is the operad $\mathscr{Q}$ of G-categories that is obtained by restricting the objects of the $\bar{\mathscr{Q}}(j)$ to consist only of the basepoint preserving injections $U^j \longrightarrow U$ for all $j \geq 0$.

Remark 7.7. If $\bar{\mathscr{Q}}$ acts on a category $\mathscr{A}$, then $\mathscr{Q}$ acts on $\mathscr{A}$ by restriction of the action. However, $\mathscr{Q}$ can act even though $\bar{\mathscr{Q}}$ does not. This happens when the structure of $\mathscr{A}$ encodes a particular unit object and the operad action conditions fail for other choices of objects in $\mathscr{A}$.

Proposition 7.8. *The classifying spaces $|\mathscr{P}_G(j)|$, $|\bar{\mathscr{Q}}_G(j)|$, and $|\mathscr{Q}_G(j)|$ are universal principal (G, Σ_j)-bundles, hence $\mathscr{P}_G$, $\bar{\mathscr{Q}}_G$, and $\mathscr{Q}_G$ are E_∞ operads.*

Proof. Since the objects of our categories are given by injective functions, Σ_j acts freely on the objects of $\mathscr{P}_G(j)$ and $\mathscr{Q}_G(j)$. Since our categories are chaotic, it suffices to show that if $\Lambda\cap\Sigma_j = \{e\}$, where $\Lambda \subset \Sigma_j \times G$, then the object sets $\mathscr{P}_G(j)^\Lambda$ and $\mathscr{Q}_G(j)^\Lambda$ are nonempty. This means that there are Λ-equivariant injections ${}^jU \longrightarrow U$ and $U^j \longrightarrow U$, and in fact there are Λ-equivariant bijections. We have $\Lambda = \{(\rho(h), h) | h \in H\}$ for a subgroup H of G and a homomorphism $\rho\colon H \longrightarrow \Sigma_j$, and we may regard U as an H-set via the canonical isomorphism $H \cong \Lambda$. Since countably many copies of every orbit of H embed in U, ${}^jU$, and U^j for $j \geq 1$, these sets are all isomorphic as H-sets and therefore as Λ-sets. □

7.2. **The definition of $\mathscr{R}_G$ and its action on $\mathscr{P}_G$.** This section is parenthetical, aimed towards work in progress on a new version of multiplicative infinite loop space theory. The notion of an action of a "multiplicative" operad $\mathscr{G}$ on an "additive" operad $\mathscr{C}$ was defined in [23, VI.1.6], and $(\mathscr{C},\mathscr{G})$ was then said to be an "operad pair". This notion was redefined and discussed in [28, 29]. Expressed in terms of diagrams rather than elements, it makes sense for operads in any cartesian monoidal category, such as the categories of G-categories and G-spaces. As is emphasized in the cited papers, although this notion is the essential starting point for the theory of E_∞ ring spaces, the only interesting nonequivariant example we know is $(\mathscr{C},\mathscr{L})$, where $\mathscr{C}$ is the Steiner operad. As pointed out in the Appendix, this example works equally well equivariantly. The pair of operads $(\mathscr{P}_G, \mathscr{Q}_G)$ very nearly gives another example, but we must shrink $\mathscr{Q}_G$ and drop its unit object to obtain this.

Definition 7.9. Define $\mathscr{R}_G \subset \mathscr{Q}_G$ to be the suboperad such that $\mathscr{R}_G(j)$ is the full subcategory of $\mathscr{Q}_G(j)$ whose objects are the based bijections $U^j \longrightarrow U$. In particular, $\mathscr{R}(0)$ is the empty category, so that the operad $\mathscr{R}_G$ does not encode unit object information. By the proof of Proposition 7.8, for $j \geq 1$, $\mathscr{R}_G(j)$ is again a universal principal (G, Σ_j)-bundle. We view $\mathscr{R}_G$ as a restricted E_∞ operad, namely one without unit objects.

Proposition 7.10. *The restricted operad $\mathscr{R}_G$ acts on the operad $\mathscr{P}_G$.*

Proof. We must specify action maps

$$\lambda\colon \mathscr{R}_G(k) \times \mathscr{P}_G(j_1) \times \cdots \times \mathscr{P}_G(j_k) \longrightarrow \mathscr{P}_G(j),$$

where $j = j_1 \cdots j_k$ and $k \geq 1$. To define them, consider the set of sequences $I = \{i_1, \cdots, i_k\}$, ordered lexicographically, where $1 \leq i_r \leq j_r$ and $1 \leq r \leq k$. For an injection $\phi_r\colon {}^{j_r}U \longrightarrow U$, let $\phi_{i_r}\colon U \longrightarrow U$ denote the restriction of ϕ_r to the i_r^{th} copy of U in ${}^{j_r}U$. Then let

$$\phi_I = \phi_{i_1} \times \cdots \times \phi_{i_k}\colon U^k \longrightarrow U^k.$$

For a bijection $\psi\colon U^k \longrightarrow U$, define

$$\lambda(\psi; \phi_1, \cdots, \phi_k)\colon {}^jU \longrightarrow U$$

to be the injection which restricts on the I^{th} copy of U to the composite

$$U \xrightarrow{\psi^{-1}} U^k \xrightarrow{\phi_I} U^k \xrightarrow{\psi} U.$$

It is tedious but straightforward to verify that all conditions specified in [23, VI.1.6], [29, 4.2] that make sense are satisfied[7]. $\square$

Remark 7.11. When all $j_i = 1$, so that there is only one sequence I, we can define λ more generally, with $\mathscr{Q}(k)$ replacing $\mathscr{R}(k)$, by letting

$$\lambda(\psi; \phi_1, \cdots, \phi_k) \colon U \longrightarrow U$$

be the identity on the complement of the image of the injection $\psi \colon U^k \longrightarrow U$ and

$$\psi(U) \xrightarrow{\psi^{-1}} U^k \xrightarrow{\phi_I} U^k \xrightarrow{\psi} \psi(U)$$

on the image of ψ. Clearly we can replace $\mathscr{P}(1)$ by $\mathscr{Q}(1)$ here.

This allows us to give the following speculative analogue of Definition 2.7. The notion of a $(\mathscr{C},\mathscr{G})$-space was defined in [23, VI.1.10], and an E_∞ ring space is defined to be a $(\mathscr{C},\mathscr{G})$-space, where $\mathscr{C}$ and $\mathscr{G}$ are E_∞ operads of spaces. Briefly, a $(\mathscr{C},\mathscr{G})$-space X is a $\mathscr{C}$-space and a $\mathscr{G}$-space with respective base points 0 and 1 such that 0 is a zero element for the $\mathscr{G}$-action and the action $\mathbf{C}X \longrightarrow X$ is a map of $\mathscr{G}$-spaces with zero, where $\mathbf{C}$ denotes the monad associated to the operad $\mathscr{C}$. Here the action of $\mathscr{G}$ on $\mathscr{C}$ induces an action of $\mathscr{G}$ on the free $\mathscr{C}$-spaces $\mathbf{C}X$, so that $\mathbf{C}$ restricts to a monad in the category of $\mathscr{G}$-spaces. These notions are redefined in the more recent papers [28, 29]. The definitions are formal and apply equally well to spaces, G-spaces, categories, and G-categories.

Definition 7.12. An E_∞ ring G-category $\mathscr{A}$ is a G-category together with an action by the E_∞ operad pair $(\mathscr{P}_G, \mathscr{R}_G)$ such that the multiplicative action extends from the restricted E_∞ operad $\mathscr{R}_G$ to an action of the E_∞ operad $\mathscr{Q}_G$.

The notion of a bipermutative category, or strict symmetric bimonoidal category, was specified in [23, VI.3.3]. With the standard skeletal model, the direct sum and tensor product on the category of finite dimensional free modules over a commutative ring R gives a typical example. Without any categorical justification, we allow ourselves to think of E_∞ ring G-categories as an E_∞ version of genuine operadic bipermutative G-categories even though the latter are as yet undefined.

Our notion of an E_∞ G-category $\mathscr{A}$ implies that $B\mathscr{A}$ is an E_∞ G-space. We would like to say that our notion of an E_∞ ring G-category $\mathscr{A}$ implies that $B\mathscr{A}$ is an E_∞ ring G-space, but that is not quite true. However, we believe there is a way to prove the following conjecture that avoids any of the categorical work of [8, 25, 29]. However, that proof is work in progress.

Conjecture 7.13. *There is an infinite loop space machine that carries E_∞ ring G-categories to E_∞ ring G-spectra.*

8. Examples of E_∞ and E_∞ ring G-categories

We have several interesting examples. We emphasize that these particular constructions are new even when $G = e$. In that case, we may take U to be the set of positive integers, with 1 as basepoint.

We have the notion of a genuine permutative G-category, which comes with a preferred product, and the notion of a $\mathscr{P}_G$-category, which does not. It seems

[7] In fact, with the details of [29, 4.2], the only condition that does not make sense would require $\lambda(1) = \mathrm{id} \in \mathscr{P}_G(1)$, where $\{1\} = \mathscr{R}(0)$, and that condition lacks force since it does not interact with the remaining conditions.

plausible that the latter notion is more general, but to verify that we would have to show how to regard a permutative category as a $\mathscr{P}_G$-algebra. One natural way to do so would be to construct a map of operads $\mathscr{P}_G \longrightarrow \mathscr{O}_G$, but we do not know how to do that. Of course, the equivalence of $\mathscr{P}_G$-categories and $\mathscr{O}_G$-categories shows that genuine permutative categories give a plethora of examples of $\mathscr{P}_G$-algebras up to homotopy. However, the most important examples can easily be displayed directly, without recourse to the theory of permutative categories.

8.1. **The G-category $\mathscr{E}_G = \mathscr{E}_G^{\mathscr{P}}$ of finite sets.** Recall Remark 3.13. Intuitively, we would like to have a genuine permutative G-category whose product is given by disjoint unions of finite sets, with G relating finite sets (not G-sets) by translations. Even nonequivariantly, this is imprecise due to both size issues and the fact that categorical coproducts are not strictly asssociative. We make it precise by taking coproducts of finite subsets of our ambient G-set U, but we must do so without assuming that our given finite subsets are disjoint. We achieve this by using injections ${}^jU \longrightarrow U$ to separate them. We do not have canonical choices for the injections, hence we have assembled them into our categorical E_∞ operad $\mathscr{P}_G$. Recall Definition 7.2 and Proposition 7.3.

Definition 8.1. The G-category $\tilde{\mathscr{E}}_G$ of finite ordered sets is the coproduct over $n \geq 0$ of the G-categories $\tilde{\mathscr{E}}_G(n)$. The G-category $\mathscr{E}_G = \mathscr{E}_G^{\mathscr{P}}$ of finite sets is the coproduct over $n \geq 0$ of the orbit categories $\tilde{\mathscr{E}}_G(n)/\Sigma_n$. By Proposition 7.3, $B\mathscr{E}_G$ is the coproduct over $n \geq 0$ of classifying spaces $B(G, \Sigma_n)$. Explicitly, by [13, 6.5], the objects of $\mathscr{E}_G$ are the finite subsets (not G-subsets) A of U. Its morphisms are the bijections $\alpha\colon A \longrightarrow B$; if A has n points, the morphisms $A \longrightarrow A$ give a copy of the set Σ_n. The group G acts by translation on objects, so that $gA = \{ga | a \in A\}$, and by conjugation on morphisms, so that $g\alpha\colon gA \longrightarrow gB$ is given by $(g\alpha)(g \cdot a) = g \cdot \alpha(a)$.

Proposition 8.2. *The G-categories $\tilde{\mathscr{E}}_G$ and $\mathscr{E}_G$ are $\mathscr{P}_G$-categories, and passage to orbits over symmetric groups defines a map $\tilde{\mathscr{E}}_G \longrightarrow \mathscr{E}_G$ of $\mathscr{P}_G$-categories.*

Proof. Define a G-functor

$$\theta_j\colon \mathscr{P}_G(j) \times \mathscr{E}_G^j \longrightarrow \mathscr{E}_G$$

as follows. On objects, for $\phi \in \mathscr{P}_G(j)$ and $A_i \in \mathcal{O}b\, \mathscr{E}_G$, $1 \leq i \leq j$, define

$$\theta_j(\phi; A_1, \cdots, A_j) = \phi(A_1 \amalg \cdots \amalg A_j),$$

where A_i is viewed as a subset of the i^{th} copy of U in ${}^jU$. For a morphism

$$(\iota; \alpha_1, \cdots, \alpha_j)\colon (\phi; A_1, \cdots, A_j) \longrightarrow (\psi; B_1, \cdots, B_j),$$

where $\iota\colon \phi \longrightarrow \psi$ is the unique morphism, define $\theta_j(\iota; \alpha_1, \cdots, \alpha_j)$ to be the unique bijection that makes the following diagram commute.

$$\begin{array}{ccc} A_1 \amalg \cdots \amalg A_j & \xrightarrow{\ \phi\ } & \phi(A_1 \amalg \cdots \amalg A_j) \\ {\scriptstyle \alpha_1 \amalg \cdots \amalg \alpha_j}\downarrow & & \downarrow{\scriptstyle \theta_j(\iota; \alpha_1, \cdots, \alpha_j)} \\ B_1 \amalg \cdots \amalg B_j & \xrightarrow[\ \psi\ ]{} & \psi(B_1 \amalg \cdots \amalg B_j) \end{array}$$

Then the θ_j specify an action of $\mathscr{P}_G$ on $\mathscr{E}_G$.

Since the $\tilde{\mathscr{E}}_G(n)$ are chaotic, to define an action of $\mathscr{P}_G$ on $\tilde{\mathscr{E}}_G$, we need only specify the required G-functors

$$\tilde{\theta}_j\colon \mathscr{P}_G(j)\times \tilde{\mathscr{E}}_G^j \longrightarrow \tilde{\mathscr{E}}_G$$

on objects. A typical object has the form $(\phi; (A_1,\alpha_1),\cdots,(A_j,\alpha_j))$, $\alpha_i\colon \mathbf{n_i}\longrightarrow A_i$. We have the canonical isomorphism $\mathbf{n_1}\amalg\cdots\amalg\mathbf{n_1}\cong \mathbf{n}$, $n=n_1+\cdots n_j$, and $\tilde{\theta}_j$ sends our typical object to

$$(\phi(A_1\amalg\cdots,\amalg A_j),\phi\circ(\alpha_1\amalg\cdots\amalg\alpha_j).$$

Again, the $\tilde{\theta}_j$ specify an action. The compatibility with passage to orbits is verified by use of canonical orbit representatives for objects A that are obtained by choosing fixed reference maps $\eta_A\colon \mathbf{n}\longrightarrow A$ for each n-point set $A\subset U$; compare [13, Proposition 6.3 and Lemma 6.5]. □

Remark 8.3. If we restrict to the full G-subcategory of $\mathscr{E}_G$ of G-fixed sets A of cardinality n, we obtain an equivalent analogue of the category $\mathscr{F}_G(n)$ of Definition 3.8: these are two small models of the G-category of all G-sets with n elements and the bijections between them, and they have isomorphic skeleta. Thus the restriction of $\mathscr{E}_G$ to its full G-subcategory of G-fixed sets A is an equivalent analogue of $\mathscr{F}_G$. Remember from Remark 3.11 that no E_∞ operad can be expected to act on $\mathscr{F}_G$. The $\mathscr{P}_G$-category $\mathscr{E}_G$ gives a convenient substitute.

8.2. **The G-category $\mathscr{GL}_G(R)$ for a G-ring R.** Let R be a G-ring, that is a ring with an action of G through automorphisms of R. We have analogues of Definitions 7.2 and 8.1 that can be used in equivariant algebraic K-theory. For a set A, let $R[A]$ denote the free R-module on the basis A. Let G act entrywise on the matrix group $GL(n,R)$ and diagonally on R^n. Our conventions on semi-direct products and their universal principal $(G,GL(n,R)_G)$-bundles are in [13], and [13, §6.3] gives more details on the following definitions.

Definition 8.4. We define the chaotic general linear category $\widetilde{\mathscr{GL}}(n,R)$. Its objects are the monomorphisms of (left) R-modules $\alpha\colon R^n\longrightarrow R[U]$. The group G acts on objects by $g\alpha = g\circ\alpha\circ g^{-1}$. The group $GL(n,R)$ acts on objects by $\alpha\tau=\alpha\circ\tau\colon R^n\longrightarrow R[U]$. Since $\widetilde{\mathscr{GL}}(n,R)$ is chaotic, this determines the actions on morphisms.

Proposition 8.5. [13, 6.18] *The actions of G and $GL(n,R)$ on $\widetilde{\mathscr{GL}}(n,R)$ determine an action of $GL(n,R)\rtimes G$, and the classifying space $|\widetilde{\mathscr{GL}}(n,R)|$ is a universal principal $(G,GL(n,R)_G)$-bundle.*

Definition 8.6. The general linear G-category $\mathscr{GL}_G(R)$ of finite dimensional free R-modules is the coproduct over $n\geq 0$ of the orbit categories $\widetilde{\mathscr{GL}}(n,R)/GL(n,R)$. By Proposition 8.5, $B\mathscr{GL}_G(R)$ is the coproduct over $n\geq 0$ of classifying spaces $B(G,GL(n,R)_G)$. Explicitly, by [13, 6.20], the objects of $\mathscr{GL}(R)$ are the finite dimensional free R-submodules M of $R[U]$. The morphisms $\alpha\colon M\longrightarrow N$ are the isomorphisms of R-modules. The group G acts by translation on objects, so that $gM=\{gm\,|\,m\in M\}$, and by conjugation on morphisms, so that $(g\alpha)(gm)=\alpha(m)$ for $m\in M$ and $g\in G$.

Proposition 8.7. *The G-categories $\widetilde{\mathscr{GL}}_G(R)$ and $\mathscr{GL}_G(R)$ are $\mathscr{P}_G$-categories and passage to orbits over general linear groups defines a map $\widetilde{\mathscr{GL}}_G(R)\longrightarrow\mathscr{GL}_G(R)$ of $\mathscr{P}_G$-categories.*

Proof. Define a functor

$$\theta_j \colon \mathscr{P}_G(j) \times \mathscr{GL}_G(R)^j \longrightarrow \mathscr{GL}_G(R)$$

as follows. On objects, for $\phi \in \mathscr{P}_G(j)$ and $M_i \in \mathscr{O}b\mathscr{GL}(R)$, $1 \le i \le j$, define

$$\theta_j(\phi; M_1, \cdots, M_j) = R[\phi](M_1 \oplus \cdots \oplus M_j),$$

where $R[\phi] \colon R[{}^jU] \longrightarrow R[U]$ is induced by $\phi \colon {}^jU \longrightarrow U$ and M_i is viewed as a submodule of the i^{th} copy of $R[U]$ in $R[{}^jU] = \oplus_j R[U]$. For a morphism

$$(\iota; \alpha_1, \cdots, \alpha_j) \colon (\phi; M_1, \cdots, M_j) \longrightarrow (\psi; N_1, \cdots, N_j),$$

define $\theta_j(\iota; \alpha_1, \cdots, \alpha_j)$ to be the unique isomorphism of R-modules that makes the following diagram commute.

$$\begin{array}{ccc} M_1 \oplus \cdots \oplus M_j & \xrightarrow{R[\phi]} & R[\phi](M_1 \oplus \cdots \oplus M_j) \\ {\scriptstyle \alpha_1 \oplus \cdots \oplus \alpha_j} \downarrow & & \downarrow {\scriptstyle \theta_j(\iota; \alpha_1, \cdots, \alpha_j)} \\ N_1 \oplus \cdots \oplus N_j & \xrightarrow[R[\psi]]{} & R[\psi](N_1 \oplus \cdots \oplus N_j) \end{array}$$

Then the θ_j specify an action of $\mathscr{P}_G$ on $\mathscr{GL}_G(R)$. Since the $\widetilde{\mathscr{GL}}(R,n)$ are chaotic, to define an action of $\mathscr{P}_G$ on $\widetilde{\mathscr{GL}}(R)$, we need only specify the required G-functors

$$\tilde{\theta}_j \colon \mathscr{P}_G(j) \times \widetilde{\mathscr{GL}}_G(R)^j \longrightarrow \widetilde{\mathscr{GL}}_G(R)$$

on objects. A typical object has the form $(\phi; \alpha_1, \cdots, \alpha_j)$, $\alpha_i \colon R^{n_i} \longrightarrow R[U]$, and, with $n = n_1 + \cdots + n_j$, $\tilde{\theta}_j$ sends it to

$$R[\phi] \circ (\alpha_1 \oplus \cdots \oplus \alpha_j) \colon R^n \longrightarrow R[U].$$

Again, the $\tilde{\theta}_j$ specify an action. The compatibility with passage to orbits is verified by use of canonical orbit representatives for objects that are obtained by choosing reference maps $\eta_M \colon R^n \longrightarrow M$ for each M dimensional free R-module $M \subset R[U]$; compare [13, 6,18, 6.20]. □

On passage to classifying spaces and then to G-spectra via our infinite loop space machine $\mathbb{E}_G$, we obtain a model $\mathbb{E}_G B\mathscr{GL}_G(R)$ for the K-theory spectrum $\mathbb{K}_G(R)$ of R. The following result compares the two evident models in sight.

Definition 8.8. Define the naive permutative G-category $GL_G(R)$ to be the G-groupoid whose objects are the $n \ge 0$ and whose set of morphisms $m \longrightarrow n$ is empty if $m \ne n$ and is the G-group $GL(n,R)$ if $m = n$, where G acts entrywise. The product is given by block sum of matrices. Applying the chaotic groupoid functor to the groups $GL(n,R)$ we obtain another naive permutative G-category $\widetilde{GL}_G(R)$ and a map $\widetilde{GL}_G(R) \longrightarrow GL_G(R)$ of naive permutative G-categories. Applying the functor $\mathscr{G}$ from Proposition 2.5, we obtain a map of genuine permutative G-categories $\mathscr{G}\widetilde{GL}_G(R) \longrightarrow \mathscr{G}GL_G(R)$.

It is convenient to write $\mathscr{GL}_G^{\mathscr{O}}(R)$ for the $\mathscr{O}_G$-category $\mathscr{G}GL_G(R)$ and $\mathscr{GL}_G^{\mathscr{P}}(R)$ for the $\mathscr{P}_G$-category $\mathscr{GL}_G(R)$, and similarly for their total space variants $\mathscr{G}\widetilde{GL}_G(R)$ and $\widetilde{\mathscr{GL}}_G(R)$. We have the following comparison theorem.

Theorem 8.9. *The G-spectra $\mathbb{K}_G\mathscr{GL}_G^{\mathscr{O}}(R)$ and $\mathbb{K}_G\mathscr{GL}_G^{\mathscr{P}}(R)$ are weakly equivalent, functorially in G-rings R.*

Proof. We again use the product of operads trick from [21]. Projections and quotient maps give a commutative diagram of $(\mathscr{O}_G \times \mathscr{P}_G)$-categories

$$\begin{array}{ccccc}
\widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{O}}_G(R) & \longleftarrow & \widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{O}}_G(R) \times \widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{P}}_G(R) & \longrightarrow & \widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{P}}_G(R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{G}\mathscr{L}^{\mathscr{O}}_G(R) & \longleftarrow & \mathscr{G}\mathscr{L}^{\mathscr{O}\times\mathscr{P}}_G(R) & \longrightarrow & \mathscr{G}\mathscr{L}^{\mathscr{P}}_G(R).
\end{array}$$

The middle term at the top denotes the diagonal product, namely

$$\coprod_n \widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{O}}_G(n,R) \times \widetilde{\mathscr{G}\mathscr{L}}^{\mathscr{P}}_G(n,R).$$

The middle term on the bottom is the coproduct over n of the orbits of these products under the diagonal action of $GL(n,R)$. The product of total spaces of universal principal $(G, GL(R,n)_G)$-bundles is the total space of another universal principal $(G, GL(R,n)_G)$-bundle. Therefore, after application of the classifying space functor, the horizontal projections display two equivalences between universal principal $(G, GL(R,n)_G)$-bundles. The conclusion follows by hitting the resulting diagram with the functor $\mathbb{K}_G$ defined with respect to $(\mathscr{O}_G \times \mathscr{P}_G)$-categories and using evident equivalences to the functors $\mathbb{K}_G$ defined with respect to $\mathscr{O}_G$ and $\mathscr{P}_G$-categories when the input is given by $\mathscr{O}_G$ or $\mathscr{P}_G$-categories. □

8.3. **Multiplicative actions on $\mathscr{E}_G$ and $\mathscr{G}\mathscr{L}_G(R)$.** We agree to think of $\mathscr{Q}_G$-categories as "multiplicative", whereas we think of $\mathscr{P}_G$-categories as "additive".

Proposition 8.10. *The G-category $\mathscr{E}_G$ is a $\mathscr{Q}_G$-category.*

Proof. Define a G-functor

$$\xi_j \colon \mathscr{Q}_G(j) \times \mathscr{E}_G^j \longrightarrow \mathscr{E}_G$$

as follows. On objects, for $\phi \in \mathscr{Q}_G(j)$ and $A_i \in \mathscr{E}_G$, $1 \le i \le j$, define

$$\xi_j(\phi; A_1, \cdots, A_j) = \phi(A_1 \times \cdots \times A_j).$$

For a morphism

$$(\iota; \alpha_1, \cdots, \alpha_j) \colon (\phi; A_1, \cdots, A_j) \longrightarrow (\psi; B_1, \cdots, B_j)$$

define $\xi_j(\iota; \alpha_1, \cdots, \alpha_j)$ to be the unique bijection that makes the following diagram commute.

$$\begin{array}{ccc}
A_1 \times \cdots \times A_j & \xrightarrow{\ \phi\ } & \phi(A_1 \times \cdots \times A_j) \\
\Big\downarrow{\scriptstyle \alpha_1 \times \cdots \times \alpha_j} & & \Big\downarrow{\scriptstyle \xi_j(\iota; \alpha_1, \cdots, \alpha_j)} \\
B_1 \times \cdots \times B_j & \xrightarrow[\ \psi\ ]{} & \psi(B_1 \times \cdots \times B_j)
\end{array}$$

Then the ξ_j specify an action of $\mathscr{Q}_G$ on $\mathscr{E}_G$. □

Proposition 8.11. *If R is a commutative G-ring, then $\mathscr{G}\mathscr{L}_G(R)$ is a $\mathscr{Q}_G$-category.*

Proof. Define a functor

$$\xi_j \colon \mathscr{Q}_G(j) \times \mathscr{G}\mathscr{L}(R)^j_G \longrightarrow \mathscr{G}\mathscr{L}_G(R)$$

as follows. Identify $R[U^j]$ with $\otimes_j R[U]$, where $\otimes = \otimes_R$. On objects, for $\phi \in \mathscr{P}_G(j)$ and R-modules $M_i \subset R[U]$, $1 \le i \le j$, define

$$\xi_j(\phi; M_1, \cdots, M_j) = R[\phi](M_1 \times \cdots \times M_j).$$

For a morphism

$$(\iota; \alpha_1, \cdots, \alpha_j)\colon (\phi; M_1, \cdots, M_j) \longrightarrow (\psi; N_1, \cdots, N_j)$$

define $\xi_j(\iota; \alpha_1, \cdots, \alpha_j)$ to be the unique isomorphism of R-modules that makes the following diagram commute.

$$\begin{array}{ccc} M_1 \otimes \cdots \otimes M_j & \xrightarrow{R[\phi]} & \phi(M_1 \otimes \cdots \otimes M_j)] \\ {\scriptstyle \alpha_1 \otimes \cdots \otimes \alpha_j} \downarrow & & \downarrow {\scriptstyle \xi_j(\iota; \alpha_1, \cdots, \alpha_j)} \\ N_1 \otimes \cdots \otimes N_j & \xrightarrow{R[\psi]} & \psi(N_1 \otimes \cdots \otimes N_j). \end{array}$$

Then the ξ_j specify an action of $\mathscr{Q}_G$ on $\mathscr{G}\mathscr{L}_G(R)$. □

Restricting the action from $\mathscr{Q}_G$ to $\mathscr{R}_G$, the examples above and easy diagram chases prove that the operad pair $(\mathscr{P}_G, \mathscr{R}_G)$ acts on the categories $\mathscr{E}_G$ and $\mathscr{G}\mathscr{L}_G(R)$. This proves the following result.

Theorem 8.12. *The categories $\mathscr{E}_G$ and $\mathscr{G}\mathscr{L}_G(R)$ for a commutative G-ring R are E_∞ ring G-categories in the sense of Definition 7.12.*

Observe that since we lack a clear categorical definition of a genuine symmetric monoidal G-category, we do not know how to give a categorical definition of a genuine symmetric bimonoidal G-category. Even operadically, where we do have a definition of a genuine permutative G-category, we do not have a definition of a genuine bipermutative G-category. The previous examples show that we do have examples of E_∞ ring G-categories. However, we do not know how to construct E_∞ ring G-categories from general naive bipermutative G-categories.

9. The $\mathscr{P}_G$-category $\mathscr{E}_G(X)$ and the BPQ-theorem

We now return to the BPQ-theorem, but thinking in terms of $\mathscr{P}_G$-categories rather than $\mathscr{O}_G$-categories. This gives a more intuitive approach to the G-category of finite sets over a G-space X.

9.1. The G-category $\mathscr{E}_G(X)$ of finite sets over X.

Definition 9.1. Let X be a G-space. We define the G-groupoid $\mathscr{E}_G(X) = \mathscr{E}_G^{\mathscr{P}}(X)$ of finite sets over X. Its objects are the functions $p\colon A \longrightarrow X$, where A is a finite subset of our ambient G-set U. For a second function $q\colon B \longrightarrow X$, a map $f\colon p \longrightarrow q$ is a bijection $\alpha\colon A \longrightarrow B$ such that $q \circ \alpha = p$. Composition is given by composition of functions over X. The group G acts by translation of G-sets and conjugation on all maps in sight. Thus, for an object $p\colon A \longrightarrow X$, $gp\colon gA \longrightarrow X$ is given by $(gp)(ga) = g(p(a))$. For a map $f\colon p \longrightarrow q$, $gf\colon gA \longrightarrow gB$ is given by $(gf)(ga) = g(f(a))$.

To topologize $\mathscr{E}_G(X)$, give U and X disjoint basepoints $*$. View the set $\mathscr{O}b$ of objects of $\mathscr{E}_G(X)$ as the set of based functions $p\colon U_+ \longrightarrow X_+$ such that $p^{-1}(*)$ is the complement of a finite set $A \subset U$. Topologize $\mathscr{O}b$ as a subspace of $X_+^{U_+}$. View the set $\mathscr{M}or$ of morphisms of $\mathscr{E}_G(X)$ as a subset of the set of functions $\alpha\colon U_+ \longrightarrow U_+$

that send the complement of some finite set $A \subset U$ to $*$ and map A bijectively to some finite set $B \subset U$. Topologize $\mathscr{M}or$ as the subspace of points (p, f, q) in $\mathscr{O}b \times U_+^{U_+} \times \mathscr{O}b$, where $U_+^{U_+}$ is discrete. When X is a finite set and thus a discrete space (since points are closed in spaces in $\mathscr{U}$), $\mathscr{E}_G(X)$ is discrete.

Let $\mathscr{E}_G(n, X)$ denote the full subcategory of $\mathscr{E}_G(X)$ of maps $p\colon A \longrightarrow X$ such that A has n elements. Then $\mathscr{E}_G(X)$ is the coproduct of the groupoids $\mathscr{E}_G(n, X)$.

Proposition 9.2. *The operad $\mathscr{P}_G$ acts naturally on the categories $\mathscr{E}_G(X)$.*

Proof. For $j \geq 0$, we must define functors

$$\theta_j\colon \mathscr{P}_G(j) \times \mathscr{E}_G(X)^j \longrightarrow \mathscr{E}_G(X).$$

To define θ_j on objects, let $\phi\colon {}^jU \longrightarrow U$ be an injective function and $p_i\colon A_i \longrightarrow X$ be a function, $1 \leq i \leq j$, where X_i is a finite subset of U. We define $\theta_j(\phi; p_1, \cdots, p_j)$ to be the composite

$$\phi(A_1 \amalg \cdots \amalg A_j) \xrightarrow{\phi^{-1}} A_1 \amalg \cdots \amalg A_j \xrightarrow{\amalg p_i} {}^jX \xrightarrow{\nabla} X,$$

where ∇ is the fold map, the identity on each of the j copies of X. To define θ on morphisms, let $\psi\colon : {}^jU \longrightarrow U$ be another injective function, and let $\iota\colon \phi \longrightarrow \psi$ be the unique map in $\mathscr{P}_G(j)$. For functions $q_i\colon B_i \longrightarrow X$ and bijections $\alpha_i\colon A_i \longrightarrow B_i$ such that $q_i\alpha_i = p_i$, define $\theta_j(\iota; \alpha_1, \cdots, \alpha_j)$ to be the unique dotted arrow bijection that makes the following diagram commute.

$$\begin{array}{ccccc}
\phi(A_1 \amalg \cdots \amalg A_j) & \xrightarrow{\phi^{-1}} & A_1 \amalg \cdots \amalg A_j & \searrow{\scriptstyle \amalg p_i} & \\
\Big\downarrow{\scriptstyle \theta(\iota;\alpha_1,\cdots,\alpha_j)} & & \Big\downarrow{\scriptstyle \amalg\alpha_i} & & {}^jX \xrightarrow{\nabla} X. \\
\psi(B_1 \amalg \cdots \amalg B_j) & \xrightarrow[\psi^{-1}]{} & B_1 \amalg \cdots \amalg B_j & \nearrow{\scriptstyle \amalg q_i} &
\end{array}$$

Then the maps θ_j specify an action of $\mathscr{P}_G$ on the category $\mathscr{E}_G(X)$. □

We have a multiplicative elaboration, which is similar to [23, VI.1.9]. Regarding a G-space X as a constant G-category with object and morphism space both X, it makes sense to speak of an action of the operad $\mathscr{Q}_G$ on the G-category X. For example, $\mathscr{Q}_G$ acts on X if X is a commutative topological G-monoid.

Proposition 9.3. *If the constant G-category X is a $\mathscr{Q}_G$-category, then $\mathscr{E}_G(X)$ is an E_∞ ring G-category.*

Proof. By analogy with the previous proof, for $j \geq 0$, we have functors

$$\xi\colon \mathscr{Q}_G(j) \times \mathscr{E}_G(X)^j \longrightarrow \mathscr{E}_G(X^j).$$

With the notations of the previous proof, on objects $(\phi; p_1, \cdots, p_j)$, $\xi(\phi; p_1, \cdots, p_j)$ is defined to be the composite

$$\phi(A_1 \times \cdots \times A_j) \xrightarrow{\phi^{-1}} A_1 \times \cdots \times A_j \xrightarrow{\times p_i} X^j.$$

On morphisms $(\iota;\alpha_1,\cdots,\alpha_j)$, $\xi(\iota;\alpha_1,\cdots,\alpha_j)$ is defined to be the unique dotted arrow making the following diagram commute.

$$\begin{array}{ccccc}
\phi(A_1\times\cdots\times A_j) & \xrightarrow{\ \phi^{-1}\ } & A_1\times\cdots\times A_j & \overset{\times p_i}{\searrow} & \\
{\scriptstyle \vartheta(\omega;\alpha_1,\cdots,\alpha_j)}\Big\downarrow & & \Big\downarrow{\scriptstyle \times\alpha_i} & & X^j. \\
\psi(B_1\times\cdots\times B_j) & \xrightarrow{\ \psi^{-1}\ } & B_1\times\cdots\times B_j & \underset{\times q_i}{\nearrow} &
\end{array}$$

Letting ξ denote the action of $\mathscr{Q}_G$ on X, the action ξ of $\mathscr{Q}_G$ on $\mathscr{E}_G(X)$ is defined by the composite maps

$$\mathscr{Q}_G(j)\times\mathscr{E}_G(X)^j \xrightarrow{\Delta\times\mathrm{id}} \mathscr{Q}_G(j)\times\mathscr{Q}_G(j)\times\mathscr{E}_G(X)^j \xrightarrow{\mathrm{id}\times\vartheta} \mathscr{Q}_G(j)\times X^j \xrightarrow{\ \xi\ } X.$$

Further details are the same as in the proof of [23, VI.1.9] or [28, 4.9]. □

9.2. **Free $\mathscr{P}_G$-categories and the $\mathscr{P}_G$-categories $\mathscr{E}_G(X)$.** The categories $\mathscr{E}_G(X)$ are conceptually simple, and they allow us to give a genuinely equivariant variant of Theorem 3.9. To see that, we give a reinterpretation of $\mathscr{E}_G(X)$. Regarding X as a topological G-category as before, we have the topological G-category $\tilde{\mathscr{E}}_G(j)\times_{\Sigma_j}X^j$.

Lemma 9.4. *The topological G-categories $\mathscr{E}_G(j,X)$ and $\tilde{\mathscr{E}}_G(j)\times_{\Sigma_j}X^j$ are naturally isomorphic.*

Proof. For an ordered set $A=(a_1,\cdots,a_j)$ of points of U, let a point $(A;x_1,\cdot,x_j)$ of $\mathscr{O}b\tilde{\mathscr{E}}_G(j)\times_{\Sigma_j}X^j$ correspond to the function $p\colon A\longrightarrow X$ given by $p(a_i)=x_i$. Similarly, let a point $(\alpha\colon A\longrightarrow B;x_1,\cdots,x_j)$ of $\mathscr{M}or\tilde{\mathscr{E}}_G(j)\times_{\Sigma_j}X^j$ correspond to the bijection $\alpha\colon p\longrightarrow q$ over X, where $q\alpha(a_i)=p(a_i)=x_i$. Since we have passed to orbits over Σ_j, our specifications are independent of the ordering of A. These correspondences identify the two categories. □

Recall that we write $\mathbb{P}_G$ for the monad on based G-categories associated to the operad $\mathscr{P}_G$, $|\mathscr{P}_G|$ for the operad of G-spaces obtained by applying the classifying space functor B to $\mathscr{P}_G$, and $\mathbf{P}_G$ for the monad on based G-spaces associated to $|\mathscr{P}_G|$. Recall too that X_+ denotes the union of the G-category X with a disjoint trivial basepoint category $*$ and that

$$\mathbb{P}_G(X_+)=\coprod_{j\geq 0}\mathscr{P}_G(j)\times_{\Sigma_j}X^j. \tag{9.5}$$

Theorem 9.6. *There is a natural map*

$$\omega\colon \mathbb{P}_G(X_+)\longrightarrow\mathscr{E}_G(X)$$

of $\mathscr{P}_G$-categories, and it induces a weak equivalence

$$B\omega\colon \mathbf{P}_G(X_+)\longrightarrow B\mathscr{E}_G(X)$$

of $|\mathscr{P}_G|$-spaces on passage to classifying spaces.

Proof. We have a G-fixed basepoint $1 \in U$. Define an inclusion $i\colon X_+ \longrightarrow \mathscr{E}_G(X)$ of based G-categories by identifying $*$ with $\mathscr{E}_G(0,X)$ and mapping X to $\mathscr{E}_G(1,X)$ by sending x to the map $1 \longrightarrow x$ from the 1-point subset 1 of U to X. Since $\mathbb{P}_G(X_+)$ is the free (based) $\mathscr{P}_G$-category generated by X_+, i induces the required natural map ω. Explicitly, it is the composite

$$\mathbb{P}_G(X_+) \xrightarrow{\mathbb{P}_G i} \mathbb{P}_G(\mathscr{E}_G(X)) \xrightarrow{\theta} \mathscr{E}_G(X).$$

More explicitly still, it is the coproduct of the maps

$$\omega_j = i_j \times_{\Sigma_j} \mathrm{id}\colon \mathscr{P}_G(j) \times_{\Sigma_j} X^j \longrightarrow \tilde{\mathscr{E}}_G(j) \times_{\Sigma_j} X^j,$$

where $i_j\colon \mathscr{P}_G(j) \longrightarrow \tilde{\mathscr{E}}_G(j)$ is the $(\Sigma_j \times G)$-functor that sends an object $\phi\colon {}^jU \longrightarrow U$ to the set $\phi(a \amalg \cdots \amalg a) \subset U$ and sends the morphism $\iota\colon \phi \longrightarrow \psi$ to the bijection

$$\phi(a \amalg \cdots \amalg a) \xrightarrow{\phi^{-1}} a \amalg \cdots \amalg a \xrightarrow{\psi} \psi(a \amalg \cdots \amalg a).$$

Passing to classifying spaces, $|i_j|$ is a map between universal principal (G,Σ_j)-bundles, both of which are $(\Sigma_j \times G)$-CW complexes. Therefore $|i_j|$ is a $(\Sigma_j \times G)$-equivariant homotopy equivalence. The conclusion follows. □

Remark 9.7. As a digressive observation, we give an analogue of the nonequivariant equivalence $\mathbf{P}S^0 \longrightarrow B\mathscr{E}$ of $|\mathscr{P}|$-spaces in a classical topological context. We have the little n-cubes operads $\mathscr{C}_n$ and their associated monads $\mathbf{C}_n$. Let $J = (0,1)$ be the interior of I. We have the configuration spaces $F(J^n, j)$ of n-tuples of distinct points in J^n. Sending little n-cubes $c\colon J^n \longrightarrow J^n$ to their center points $c(1/2, \cdots, 1/2)$ gives a homotopy equivalence $f\colon \mathscr{C}_n(j) \longrightarrow F(J^n, j)$. For based spaces Y, we construct spaces $\mathbf{F}_nY$ by replacing $\mathscr{C}_n(j)$ by $F(J^n,j)$ in the construction of $\mathbf{C}_nY$. The maps f induce a homotopy equivalence

$$f\colon \mathbf{C}_nY \longrightarrow \mathbf{F}_nY.$$

That much has been known since [21]. A folklore observation is that although the $F(J^n, j)$ do not form an operad, $\mathscr{C}_n$ acts on $\mathbf{F}_nY$ in such a way that f is a map of $\mathscr{C}_n$-spaces. Indeed, we can evaluate little n-cubes $: J^n \longrightarrow J^n$ on points of J^n to obtain maps $\mathscr{C}_n(j) \times F(J^n,j) \longrightarrow F(J^n,j)$, and any reader of [21] will see how to proceed from there.

9.3. **The Barratt-Priddy-Quillen theorem revisited.** We begin by comparing Theorem 9.6, which is about G-categories, with Theorems 3.5, 3.9, and 3.10, which are about G-fixed categories. Clearly $\mathscr{E}_G(X)^G$ is a $\mathscr{P}$-category, where $\mathscr{P} = (\mathscr{P}_G)^G$. By Theorem 9.6, it is weakly equivalent (in the homotopical sense) to the $\mathscr{P}$-category $(\mathbb{P}_GX_+)^G$. We also have the $\mathscr{O}$-category $\mathscr{F}_G(X)^G$, which by Theorem 3.9 and Remark 3.11 is equivalent to the $\mathscr{O}$-category $(\mathbb{O}_GX_+)^G$. Elaborating Remark 8.3, $\mathscr{E}_G(X)^G$ and $\mathscr{F}_G(X)^G$ are two small models for the category of all finite G-sets and G-isomorphisms over X and are therefore equivalent. To take the operad actions into account, recall the discussion in §2.3.

Lemma 9.8. *The $\mathscr{O}_G$-category $\mathbb{O}_GX_+$ and the $\mathscr{P}_G$-category $\mathbb{P}_GX_+$ are weakly equivalent as $(\mathscr{O}_G \times \mathscr{P}_G)$-categories. Therefore the $\mathscr{O}$-category $(\mathbb{O}_GX_+)^G$ and the $\mathscr{P}$-category $(\mathbb{P}_GX_+)^G$-categories are weakly equivalent*

Proof. The projections

$$\mathbb{O}_GX_+ \longleftarrow (\mathbb{O}_G \times \mathbb{P}_G)(X_+) \longrightarrow \mathbb{P}_GX_+$$

are maps of $(\mathscr{O}_G \times \mathscr{P}_G)$-categories that induce weak equivalences of $|\mathscr{O}_G \times \mathscr{P}_G|$-spaces on passage to classifying spaces. □

Theorem 9.9. *The classifying spaces of the $\mathscr{O}$-category $\mathscr{F}_G(X)^G$ and the $\mathscr{P}$-category $\mathscr{E}_G(X)^G$ are weakly equivalent as $|\mathscr{O} \times \mathscr{P}|$-spaces.*

The conclusion is that, on the G-fixed level, the categories $\mathscr{E}_G(X)^G$ and $\mathscr{F}_G(X)^G$ can be used interchangeably as operadically structured versions of the category of finite G-sets over X. On the equivariant level, $\mathscr{E}_G(X)$ but not $\mathscr{F}_G(X)$ is operadically structured. It is considerably more convenient than the categories $\mathbb{O}_G(X_+)$ or $\mathbb{P}_G(X_+)$. With the notations $\mathbb{K}_G\mathbb{P}_G(X_+) = \mathbb{E}_G B\mathbb{P}_G(X_+) = \mathbb{E}_G\mathbf{P}_G(X_+)$ and $\mathbb{K}_G\mathscr{E}_G(X) = \mathbb{E}_G B\mathscr{E}_G(X)$, we have the following immediate consequence of Theorems 5.2 and 9.6. It is our preferred version of the equivariant BPQ theorem, since it uses the most intuitive categorical input.

Theorem 9.10 (Equivariant Barratt-Priddy-Quillen theorem)**.** *For G-spaces X, there is a composite natural weak equivalence*

$$\alpha\colon \Sigma^\infty_G X_+ \longrightarrow \mathbb{K}_G\mathbb{P}_G X_+ \longrightarrow \mathbb{K}_G\mathscr{E}_G(X).$$

We use this version of the BPQ theorem to reconsider base change functors. We first spell out the naturality statement of Theorem 9.10. For a G-map $f\colon X \longrightarrow Y$, we have the map of $\mathscr{P}_G$-categories

$$f_!\colon \mathscr{E}_G(X) \longrightarrow \mathscr{E}_G(Y)$$

given by post-composition of functions $p\colon A \longrightarrow X$ with the map f. The naturality statement of Theorem 9.10 means that the following diagram commutes.

(9.11)
$$\begin{array}{ccc} \Sigma^\infty_G X_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(X) \\ {\scriptstyle \Sigma^\infty_G f_+}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G f_!} \\ \Sigma^\infty_G Y_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(Y) \end{array}$$

Observe that we do not have a right adjoint f^* to $f_!$ in general. The way to construct such a functor is to pull back a function $q\colon B \longrightarrow Y$ along f to obtain a function $p\colon A \longrightarrow X$. This does not work since A need not be finite when B is, although it is so when X is finite. When both X and Y are finite, we have the kind of contravariant naturality that we saw in Proposition 6.15 (albeit with a clash of notation). The proof is the same as there.

Proposition 9.12. *Let $i\colon X \longrightarrow Y$ be an inclusion of finite G-sets and define a G-map $t\colon Y_+ \longrightarrow X_+$ by $ti(x) = x$ for $x \in X$ and $t(y) = *$ for $y \notin \mathrm{im}(i)$. Then the following diagram of G-spectra commutes.*

$$\begin{array}{ccc} \Sigma^\infty_G Y_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(Y) \\ {\scriptstyle \Sigma^\infty_G t}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G i^*} \\ \Sigma^\infty_G X_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(X) \end{array}$$

From here, it should be possible to reprove the results of [11, §2] using a version $\mathscr{E}_G^{\mathscr{P}}$ of the G-bicategory $\mathscr{E}_G^{\mathscr{O}}$ used in §6.3, but processing the relevant pairings using operadic parametrization via $\mathscr{Q}_G$ and $\mathscr{R}_G$. However, the details are not yet in place.

10. Appendix: the equivariant Steiner operads

In [28, §3], the second author explained the deficiencies of the little cubes and the little discs operads nonequivariantly and showed how the Steiner operads [38] enjoyed all of the good properties of both. The same explanations apply equivariantly. The definitions given in [28, 38] apply verbatim equivariantly, as we indicate below. We must define pairings $\mathscr{K}_V \times \mathscr{K}_W \longrightarrow \mathscr{K}_{V\oplus W}$ and we must explain why the infinite Steiner operad $\mathscr{C}_G$ is an E_∞ operad of G-spaces.

10.1. **The definition of $\mathscr{K}_V$.** Let V be a real representation of G with a G-invariant inner product. Let E_V be the space of embeddings $V \longrightarrow V$, with G acting by conjugation, and let $\mathrm{Emb}_V(j) \subset E_V^j$ be the G-subspace of j-tuples of embeddings with disjoint images. Regard such a j-tuple as an embedding ${}^jV \longrightarrow V$, where ${}^jV$ denotes the disjoint union of j copies of V (where ${}^0V$ is empty). The element $\mathrm{id} \in \mathrm{Emb}_V(1)$ is the identity embedding, the group Σ_j acts on $\mathrm{Emb}_V(j)$ by permuting embeddings, and the structure maps

$$(10.1)\qquad \gamma\colon \mathrm{Emb}_V(k) \times \mathrm{Emb}_V(j_1) \times \cdots \times \mathrm{Emb}_V(j_k) \longrightarrow \mathrm{Emb}_V(j_1 + \cdots + j_k)$$

are defined by composition and disjoint union in the evident way [28, §3]. This gives an operad Emb_V of G-spaces.

Define $R_V \subset E_V = \mathrm{Emb}_V(1)$ to be the sub G-space of distance reducing embeddings $f\colon V \longrightarrow V$. This means that $|f(v) - f(w)| \leq |v - w|$ for all $v, w \in V$. Define a Steiner path to be a map $h\colon I \longrightarrow R_V$ such that $h(1) = \mathrm{id}$ and let P_V be the G-space of Steiner paths, with action of G induced by the action on R_V. Define $\pi\colon P_V \longrightarrow R_V$ by evaluation at 0, $\pi(h) = h(0)$. Define $\mathscr{K}_V(j)$ to be the G-space of j-tuples $(h_1, \cdots, h_j)$ of Steiner paths such that the $\pi(h_r)$ have disjoint images. The element $\mathrm{id} \in \mathscr{K}_V(1)$ is the constant path at the identity embedding, the group Σ_j acts on $\mathscr{K}_V(j)$ by permutations, and the structure maps γ are defined pointwise in the same way as those of Emb_V. This gives an operad of G-spaces, and application of π to Steiner paths gives a map of operads $\pi\colon \mathscr{K}_V \longrightarrow \mathrm{Emb}_V$.

The Steiner operads $\mathscr{K}_V$ are all reduced, $\mathscr{K}_V(0) = *$, and $\mathscr{K}_0$ is the trivial operad with $\mathscr{K}_0(1) = \mathrm{id}$ and $\mathscr{K}_0(j) = \emptyset$ for $j > 1$; its associated monad $\mathbf{K}_0$ is the identity functor on based spaces.

By pullback along π, any space with an action by Emb_V inherits an action by $\mathscr{K}_V$. Exactly as in [21, §5], [23, VII§2], or [28, §3], Emb_V acts naturally on $\Omega^V Y$ for a G-space Y. Evaluation of embeddings at $0 \in V$ gives G-maps $\zeta\colon \mathrm{Emb}_V(j) \longrightarrow F(V, j)$, where $F(V, j)$ is the configuration G-space of j-tuples of distinct points in V. Steiner [38] determines the homotopy types of the $\mathscr{K}_V(j)$ by proving that the composite maps $\zeta \circ \pi\colon \mathscr{K}_V(j) \longrightarrow F(V, j)$ are Σ_j-equivariant deformation retractions. The argument is clever and non-trivial, but for us the essential point is that it uses the metric on V and the contractibility of I in such a way that the construction is clearly G-equivariant. Therefore $F(V, j)$ is a $(\Sigma_n \times G)$-deformation retract of $\mathscr{K}_V(j)$.

10.2. **The pairing $(\mathscr{K}_V, \mathscr{K}_W) \longrightarrow \mathscr{K}_{V\oplus W}$.** In [21, 8.3], a pairing

$$\otimes\colon C_m X \wedge C_n Y \longrightarrow C_{m+n}(X \wedge Y)$$

is defined for based spaces X and Y, where C_n denotes the monad on based spaces induced from the little n-cubes operad $\mathscr{C}_n$. Implicitly, it comes from a pairing of operads $\otimes\colon (\mathscr{C}_m, \mathscr{C}_n) \longrightarrow \mathscr{C}_{m+n}$. The Steiner operad analogue appears in [24, p. 337]. We have a pairing of operads of G-spaces

$$\otimes\colon (\mathscr{K}_V, \mathscr{K}_W) \longrightarrow \mathscr{K}_{V\oplus W}$$

for finite dimensional real inner product spaces V and W. The required maps

$$\otimes\colon \mathscr{K}_V(j) \times \mathscr{K}_W(k) \longrightarrow \mathscr{K}_{V\oplus W}(jk)$$

are given by $(c \otimes d) = e$, where, writing $c = (g_1, \cdots, g_j)$ and $d = (h_1, \cdots, h_k)$, e is the jk-tuple of Steiner paths

$$(g_q, h_r)\colon I \longrightarrow R_V \times R_W \subset R_{V\oplus W},$$

$1 \leq q \leq j$ and $1 \leq r \leq k$, ordered lexicographically. The formulas required in Definition 6.1 are easily verified.

Moreover, this system of pairings is unital, associative, and commutative. To see this, observe first that the Steiner operads $\mathscr{K}_V$ are all reduced, $\mathscr{K}_V(0) = *$, and that $\mathscr{K}_0$ is the trivial operad with $\mathscr{K}_0(1) = \mathrm{id}$ and $\mathscr{K}_0(j) = \emptyset$ for $j > 1$; its associated monad $\mathbf{K}_0$ is the identity functor on based spaces. The pairing $\otimes\colon \mathscr{K}_0(1) \times \mathscr{K}_V(j) \longrightarrow \mathscr{K}_V(j)$ is the identity map. The following associativity diagram commutes for a triple (V, W, Z) of inner product spaces.

$$\begin{array}{ccc}
\mathscr{K}_V(i) \otimes \mathscr{K}_W(j) \otimes \mathscr{K}_Z(k) & \xrightarrow{\otimes \times \mathrm{id}} & \mathscr{K}_{V\oplus W}(ij) \times \mathscr{K}_Z(k) \\
{\scriptstyle \mathrm{id} \times \otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\
\mathscr{K}_V(i) \times \mathscr{K}_{W\oplus Z}(jk) & \xrightarrow[\otimes]{} & \mathscr{K}_{V\oplus W\oplus Z}(ijk)
\end{array}$$

The following commutatitivity diagram commutes for a pair (V, W).

$$\begin{array}{ccc}
\mathscr{K}_V(j) \times \mathscr{K}_W(k) & \xrightarrow{\otimes} & \mathscr{K}_{V\oplus W}(jk) \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \tau(j,k)} \\
\mathscr{K}_W(k) \otimes \mathscr{K}_V(j) & \xrightarrow[\otimes]{} & \mathscr{K}_{W\oplus V}(kj)
\end{array}$$

Here t is the interchange map and $\tau(j, k)$ is determined in an evident way by the interchange map for V and W and the permutation $\tau(j, k)$ of jk-letters.

Passing to monads as in Lemma 6.2, we obtain a unital, associative, and commutative system of pairings

$$\otimes\colon \mathbf{K}_V X \wedge \mathbf{K}_W Y \longrightarrow \mathbf{K}_{V\oplus W}(X \wedge Y).$$

For the unit property, when $V = 0$ the map $\otimes\colon X \wedge \mathbf{K}_W Y \longrightarrow \mathbf{K}(X \wedge Y)$ is induced by the maps $X \times Y^j \longrightarrow (X \times Y)^j$ obtained from the diagonal map on X and shuffling. We have the following key observation. Its analogue for the little cubes operads is [21, 8.3].

Lemma 10.2. *The following diagram commutes.*

$$\begin{array}{ccc}
\mathbf{K}_V X \wedge \mathbf{K}_W Y & \xrightarrow{\otimes} & \mathbf{K}_{V\oplus W}(X \wedge Y) \\
{\scriptstyle \alpha_V \wedge \alpha_W}\downarrow & & \downarrow{\scriptstyle \alpha_{V\oplus W}} \\
\Omega^V \Sigma^V X \wedge \Omega^W \Sigma^W Y & \xrightarrow[\wedge]{} & \Omega^{V\oplus W} \Sigma^{V\oplus W}(X \wedge Y),
\end{array}$$

(where we implicitly identify $X \wedge Y \wedge S^V \wedge S^W$ *with* $X \wedge S^V \wedge Y \wedge S^W$ *via* $\mathrm{id} \wedge t \wedge \mathrm{id}$*).*

10.3. **The operad $\mathscr{K}_U$ and the action of $\mathscr{L}_U$ on $\mathscr{K}_U$.** Inclusions $V \subset W$ functorially induce inclusions of operads $\mathscr{K}_V \longrightarrow \mathscr{K}_W$. Let U be a complete G-universe, the sum of countably many copies of each irreducible representation of G, and let $\mathscr{K}_U$ be the union over $V \subset U$ of the operads $\mathscr{K}_V$.[8] This is the infinite Steiner operad of G-spaces, and it acts on $\Omega^\infty_G E$ for any genuine G-spectrum E. It is an E_∞ operad since Σ_j-acts freely on $F(U,j)$ and $F(U,j)^\Lambda$ is contractible if $\Lambda \subset \Sigma_j \times G$ and $\Lambda \cap \Pi = e$. Indeed, Λ is isomorphic to a subgroup H of G, and if we let H act on U through the isomorphism, then U is a complete H-universe and U^H is isomorphic to $\mathbb{R}^\infty$. Therefore $F(U,j)^\Lambda$ is equivalent to the contractible space $F(\mathbb{R}^\infty, j)$.

The linear isometries operad $\mathscr{L}$ is defined in many places (e.g. [28, §2]), and it too has an evident equivariant version $\mathscr{L}_U$, which was first used in [18, VII§1]. The $\Sigma_j \times G$-space $\mathscr{L}_U(j)$ is the space of linear isometries $U^j \longrightarrow U$, with G acting by conjugation. Then $\mathscr{L}_U$ is also an E_∞ operads of G-spaces. Again Σ_j acts freely on $\mathscr{L}_U(j)$ and $\mathscr{L}_U(j)^\Lambda$ is contractible if $\Lambda \subset \Sigma_j \times G$ and $\Lambda \cap \Pi = e$. If $\Lambda \cong H$ and H acts on U through the isomorphism, then U is a complete H-universe and $\mathscr{L}_U(j)^H$ is isomorphic the space of H-linear isometries $U^j \longrightarrow U$. The usual argument that $\mathscr{L}(j)$ is contractible (e.g. [23, I.1.2]) adapts readily to prove that this space is contractible.

The formal structure of $(\mathscr{K}_U, \mathscr{L}_U)$ works the same way as nonequivariantly. This is an E_∞ operad pair in the sense originally defined in [23, VI.1.2] and reviewed in [28, §1] and, in more detail, [29, 4.2]. The action of $\mathscr{L}_U$ on $\mathscr{K}_U$ is defined nonequivariantly in [28, §3] and works in exactly the same way equivariantly. From here, multiplicative infinite loop space theory works equivariantly to construct E_∞ ring G-spectra from $(\mathscr{C}_U, \mathscr{L}_U)$-spaces (alias E_∞-ring G-spaces) in exactly the same way as nonequivariantly [23, 28]. The passage from category level data to E_∞-ring G-spaces in analogy with [25, 29] generalizes to equivariant multicategories, as will be explained in [2].

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