Let $\mathscr{V}$ be a monoidal model category (cofibrantly generated), and let $\mathscr{B}$ be an $\mathscr{V}$-cateogry. Suppose given a functor $\mathscr{C} \xrightarrow{\mathbb{I}} \mathscr{B}_0$ which is a bijection on objects. Equivalently, one has a $\mathscr{V}$-functor $S[\mathscr{C}] \to \mathscr{B}$. Suppose moreover that $\mathscr{B}$ is $\mathscr{V}$-good, so that enriched presheaf category $\mathscr{P}_{\mathscr{V}}(\mathscr{B}, \mathscr{V})$ has the projective model structure.

Let $\mathbb{N} : \mathscr{P}_{\mathscr{V}}(\mathscr{B}, \mathscr{V}) \to \mathscr{P}_{\mathscr{V}}(S[\mathscr{C}], \mathscr{V})$ be pullback along $S[\mathscr{C}] \to \mathscr{B}$. Left Kan extension gives a left adjoint, denoted $\mathbb{M}$. Note also that there is a canonical isomorphism $\mathscr{P}_{\mathscr{V}}(S[\mathscr{C}], \mathscr{V}) \cong \mathscr{P}(\mathscr{C}, \mathscr{V}_0)$, and that this ordinary presheaf category also has a projective model structure.

We are interested in when the adjunction

$$\mathbb{M} : \mathscr{P}(\mathscr{C}, \mathscr{V}_0) \rightleftarrows \mathscr{P}_{\mathscr{V}}(\mathscr{B}, \mathscr{V}) : \mathbb{N}$$

is a Quillen equivalence. The right adjoint $\mathbb{N}$ preserves weak equivalences and fibrations, so the adjunction is already a Quillen pair. In fact, since $\mathbb{I}$ is surjective on objects, $\mathbb{N}$ creates the weak equivalences. Thus in order to show that the adjunction is a Quillen equivalence, it suffices to show that for every cofibrant $X \in \mathscr{P}(\mathscr{C}, \mathscr{V}_0)$, the unit

$$X \xrightarrow{\eta} \mathbb{N}\mathbb{M}X$$

is a weak equivalence. Begin by supposing that $X = F_c S_i$ is a domain of a generating cofibration for $\mathscr{P}(\mathscr{C}, \mathscr{V}_0)$.

Lemma. *The left adjoint $\mathbb{M}$ preserves the generating cofibrations; that is, $\mathbb{M}(F_c S_i) \cong F_{\mathbb{I}c} S_i$.*

Proof. This is an easy Yoneda argument:

$$\begin{aligned}\mathscr{P}_{\mathscr{V}}(\mathscr{B}, \mathscr{V})(\mathbb{M}(F_c S_i), Z) &\cong \mathscr{P}(\mathscr{C}, \mathscr{V}_0)(F_c S_i, Z \circ \mathbb{I}) \\ &\cong \mathscr{V}(S_i, Z \circ \mathbb{I}(c)) \cong \mathscr{P}_{\mathscr{V}}(F_{\mathbb{I}c} S_i, Z)\end{aligned}$$

□

Thus we want to know if the map

$$F_c S_i \to \mathbb{N} F_{\mathbb{I}c} S_i$$

is a weak equivalence. This means that for each $c' \in \mathscr{C}$, the map

$$F_c S_i(c') = \bigvee_{\mathscr{C}(c',c)} S_i \to \mathbb{N} F_{\mathbb{I}c} S_i(c') = \mathscr{B}(\mathbb{I}c', \mathbb{I}c) \wedge S_i$$

is a weak equivalence.