CHAOTIC CATEGORIES AND EQUIVARIANT CLASSIFYING SPACES

B.J. GUILLOU, J.P. MAY, AND M. MERLING

Abstract. Starting categorically, we give simple and precise models of equivariant classifying spaces. We need these models for work in progress in equivariant infinite loop space theory and equivariant algebraic K-theory, but the models are of independent interest in equivariant bundle theory.

Contents

Introduction

Let Π and G be topological groups and let G act on Π, so that we have a semi-direct product $\Gamma = \Pi \rtimes G$ and a split extension

$$1 \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{0.1}$$

The underlying space of Γ is $\Pi \times G$, and the product is given by

$$(\sigma, g)(\tau, h) = (\sigma(g \cdot \tau), gh).$$

There is a general theory of $(\Pi;\Gamma)$-bundles [2, 6, 7, 11], or more explicitly (Π, α, G)-bundles where α denotes the action of G on Π. It is especially familiar when G acts trivially on Π. With $\Pi = O(n)$ or $U(n)$, the G-trivial case gives classical equivariant bundle theory and equivariant topological K-theory.

For applications in equivariant infinite loop space theory and equivariant algebraic K-theory, we need to understand classifying G-spaces for $(\Pi;\Gamma)$-bundles as classifying spaces of categories. The results we need are close to those of [6, 7, 10] and those stated by Murayama and Shimakawa [13][1], but we require a more precise and rigorous categorical and topological understanding than the literature affords. This is intended as a service paper that displays the relevant constructions in their fullblown simplicity.

We start with the topologized equivariant version of the elementary theory of chaotic categories in §1. We analyze a general construction that specializes to give our classifying G-spaces in §2. We show how it gives universal equivariant bundles in §3. The main theorems are Theorem 3.11 and Theorem 3.12 there. Our explicit description of the classifying spaces of $(\Pi;\Gamma)$-bundles as classifying spaces of categories allows us to compute their fixed point spaces categorically in §4. This gives precise information already on the category level, before passage to classifying spaces, which is what we need in the applications. We compare the categorical construction with an earlier mapping space construction [10] in §5.

We assume that the reader is familiar with the classifying space functor B from categories to spaces, or more generally from topological categories to spaces. It works equally well to construct G-spaces from topological G-categories. It is the composite of the nerve functor N from topological categories to simplicial spaces (e.g. [9, §7]) and geometric realization $|-|$ from simplicial spaces to spaces (e.g. [8, §11]), both of which are product-preserving functors. The letter B would sometimes be awkward in our context, since the classifying space functor will often be used to construct universal bundles rather than classifying spaces for bundles, hence we agree to write out $|N-|$ rather than B whenever B seems likely to confuse.

This notation also displays a key technical problem that is sometimes overlooked in the literature. The functor $|-|$ is a left adjoint and therefore preserves all colimits, such as passage to orbits in the equivariant setting. The functor N is a right adjoint and it generally does not preserve colimits or passage to orbits, as we illustrate with elementary examples. This problem is the subject of the paper [1] by Babson and Kozlov. For topological categories, there is no discussion in the literature. Exceptionally, N does commute with passage to orbits in the key examples that appear in equivariant bundle theory. Clear understanding of passage to orbits is essential to our calculations of fixed point spaces.

1. Preliminaries on chaotic and translation categories

The definitions we start with are familiar and elementary. However, to keep track of categorical data and group actions later, we shall be pedantically precise.

1.1. **Preliminaries on topological G-categories.** Let $\mathscr{C}at$ be the category of categories and functors. We may also view it as the 2-category of categories, with 0-cells, 1-cells, and 2-cells the categories, functors, and natural transformations. Equivalently, we let $\mathscr{C}at(\mathscr{A},\mathscr{B})$ be the internal hom category whose objects are the functors $\mathscr{A} \longrightarrow \mathscr{B}$ and whose morphisms are the natural transformations between them, thus enriching $\mathscr{C}at$ over itself.

[1] But see Scholium 3.13.

For a group G, a G-category $\mathscr{A}$ is a category with an action of G specified by a homomorphism from G to the automorphism group of $\mathscr{A}$. We shall write $\mathbf{G}$ for G regarded as a category with a single object, and then the action can be specified by a functor $\mathbf{G} \times \mathscr{A} \longrightarrow \mathscr{A}$ that is the identity on objects. We have the 2-category $G\mathscr{C}at$ of G-categories, G-functors, and G-natural transformations, where the latter notions are defined in the evident way: everything must be equivariant.

We may view $G\mathscr{C}at$ as the underlying 2-category of a category enriched over $G\mathscr{C}at$. The 0-cells are still G-categories, but now we have the G-category $\mathscr{C}at_G(\mathscr{A}, \mathscr{B})$ as the internal hom between them. Its underlying category is $\mathscr{C}at(\mathscr{A}, \mathscr{B})$, and G acts by conjugation on functors and natural transformations. Thus, for $F\colon \mathscr{A} \longrightarrow \mathscr{B}$, $g \in G$, and A either an object or a morphism of $\mathscr{A}$, $(gF)(A) = gF(g^{-1}A)$. Similarly, for a natural transformation $\eta\colon E \longrightarrow F$ and an object A of $\mathscr{A}$,

$$(g\eta)_A = g\eta_{g^{-1}A}\colon gE(g^{-1}A) \longrightarrow gF(g^{-1}A).$$

The category $G\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is the same as the G-fixed category $\mathscr{C}at_G(\mathscr{A}, \mathscr{B})^G$, and we sometimes vary the choice of notation.

We can topologize the definitions so far, starting with the 2-category of categories internal to the category $\mathscr{U}$ of (compactly generated) spaces, together with continuous functors and continuous natural transformations. Recall that a category $\mathscr{A}$ internal to a cartesian monoidal category $\mathscr{V}$ has object and morphism objects in $\mathscr{V}$ and structure maps Source, Target, Identity and Composition in $\mathscr{V}$. These maps are denoted S, T, I, and C, and the usual category axioms must hold. When $\mathscr{V} = \mathscr{U}$, we refer to internal categories as topological categories; we refer to them as topological G-categories when $\mathscr{V} = G\mathscr{U}$. These are more general than (small) topologically enriched categories, which have discrete sets of objects. We can now allow G to be a topological group in the equivariant picture. We continue to use the notations already given in the more general topological situation.

1.2. Chaotic topological G-categories.

Definition 1.1. A small category $\mathscr{C}$ is *chaotic* if there is exactly one morphism, sometimes denoted generically by ι, from b to a for each pair of objects a and b. The unique morphism from a to b must then be inverse to the unique morphism from b to a. Thus $\mathscr{C}$ is a groupoid, and its classifying space is contractible since every object is initial and terminal. A topological category $\mathscr{C}$ is chaotic if its underlying category is chaotic. Its classifying space is again contractible (see Remark 2.11). Similarly, a topological G-category is chaotic if its underlying category is chaotic. It is then contractible but not usually G-contractible.

The senior author remembers hearing the name "chaotic" long ago, but we do not know its source. The idea is that everything is the same as everything else, which does seem rather chaotic[2].

Lemma 1.2. *If $\mathscr{A}$ is any category and $\mathscr{B}$ is a chaotic category, then the category $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ is again chaotic.*

Proof. The unique natural map $\zeta\colon E \longrightarrow F$ between functors $E, F\colon \mathscr{A} \longrightarrow \mathscr{B}$ is given on an object A of $\mathscr{A}$ by the unique map $\eta_A\colon E(A) \longrightarrow F(A)$ in $\mathscr{B}$. □

[2] Some category theorists suggest the name "indiscrete category", by formal analogy with indiscrete spaces in topology. The key difference is that indiscrete spaces are of no interest, whereas we hope to convince the reader that chaotic categories are of considerable interest.

Lemma 1.3. *If $\mathscr{A}$ is any topological G-category and $\mathscr{B}$ is a chaotic topological G-category, then the topological G-category $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$ and its G-fixed category $G\mathscr{C}at(\mathscr{A},\mathscr{B})$ are again chaotic.*

Proof. Since $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$ is just the category $\mathscr{C}at(\mathscr{A},\mathscr{B})$ with its conjugation action by G, Lemma 1.2 implies the conclusion for $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$. The conclusion is inherited by $G\mathscr{C}at(\mathscr{A},\mathscr{B})$ since the unique natural transformation between G-functors E and F is necessarily a G-natural transformation. □

Definition 1.4. The chaotic topological category $\tilde{X}$ generated by a space X is the topological category with object space X and morphism space $X\times X$. The source, target, identity, and composition maps are defined by

$$S=\pi_2\colon X\times X\longrightarrow X,\quad T=\pi_1\colon X\times X\longrightarrow X,\quad I=\Delta\colon X\longrightarrow X\times X,\quad \text{and}$$

$$C=\mathrm{id}\times\varepsilon\times\mathrm{id}\colon (X\times X)\times_X(X\times X)\cong X\times X\times X\longrightarrow X\times X,$$

where $\varepsilon\colon X\longrightarrow *$ is the trivial map. On elements, $S(y,x)=x$, $T(y,x)=y$, $I(x)=(x,x)$, and $C(z,y,x)=(z,x)$. Forgetting the topology, the element (y,x) is the unique morphism $\iota\colon x\longrightarrow y$; the reverse ordering of source and target here will be helpful later. When X is a (left or right) G-space, we give $\tilde{X}$ the action specified by the given action on the object space X and the diagonal action on the morphism space $X\times X$; $\tilde{X}$ is then a chaotic topological G-category.

1.3. **The adjunction between G-spaces and topological G-groupoids.** Let $G\mathscr{G}pd$ denote the category of topological G-groupoids; it is the full subcategory of $G\mathscr{C}at$ whose objects are topological G-groupoids. We have the object functor $\mathscr{O}b\colon G\mathscr{G}pd\longrightarrow G\mathscr{U}$ from topogical G-groupoids to topological G-spaces.

Lemma 1.5. *The chaotic category functor from G-spaces to topological G-groupoids is right adjoint to the object functor, so that*

$$G\mathscr{G}pd(\mathscr{G},\tilde{X})\cong G\mathscr{U}(\mathscr{O}b\mathscr{G},X)$$

for a topological G-groupoid $\mathscr{G}$ with object space $\mathscr{O}b\mathscr{G}$ and a topological G-space X. If $\mathscr{G}$ is chaotic with object G-space X, then the unit of the adjunction is an isomorphism of topological G-groupoids $\eta\colon\mathscr{G}\longrightarrow\tilde{X}$.

Proof. Let $\mathscr{M}or\mathscr{G}$ be the morphism G-space of $\mathscr{G}$. The functor $\mathscr{G}\longrightarrow\tilde{X}$ determined by a continuous G-map $f\colon\mathscr{O}b\mathscr{G}\longrightarrow X$ is f on object G-spaces and the composite

$$\mathscr{M}or\mathscr{G}\xrightarrow{(T,S)}\mathscr{O}b\mathscr{G}\times\mathscr{O}b\mathscr{G}\xrightarrow{f\times f}X\times X$$

on morphism G-spaces. The last statement rephrases the meaning of chaotic. □

1.4. **Translation categories and the chaotic categories $\tilde{G}$.** We use another simple definition to identify chaotic categories with other familiar categories. Let G be a topological group and Y be a (left) G-space. Then Y together with its action by G can be thought of as the (covariant) topological functor $\mathbf{Y}\colon\mathbf{G}\longrightarrow\mathscr{U}$ that sends the single object $*$ to Y and is given on morphism spaces by the map $G\longrightarrow\mathscr{U}(Y,Y)$ adjoint to the action map $G\times Y\longrightarrow Y$.

Definition 1.6. Let Y be a (left) G-space. Define the translation category $T(G,Y)$ as follows. The object space is Y and the morphism space is $G\times Y$. We think of (g,y) as $g\colon y\longrightarrow gy$. The map $I\colon Y\longrightarrow G\times Y$ sends y to (e,y). The maps S and T send (g,y) to y and gy, respectively. The domain of composition, $(G\times Y)\times_Y(G\times Y)$

can be identified with $(G \times G) \times Y$, and composition sends (h, g, y) to (hg, y). The construction is functorial in Y.

Clearly $T(G, Y)$ is not chaotic in general, but we have the following identification. In view of the differing group actions, it must not be viewed as a tautology.

Lemma 1.7. *There is an isomorphism μ from the translation right G-category $T(G, G)$ to the chaotic right G-category $\tilde{G}$.*

Proof. G acts on itself from the left when defining the category $T(G, G)$ and it acts on itself from the right when defining the right G-action. The object spaces of both $T(G, G)$ and $\tilde{G}$ are G, and μ is the identity map on object spaces. The morphism spaces of both are $G \times G$, and μ is the standard homeomorphism from $G \times G$ with its right action on the second factor to $G \times G$ with its diagonal right action. Explicitly, $\mu(h, g) = (hg, g)$ and $\mu^{-1}(h, g) = (hg^{-1}, g)$. Our reversed order of S and T on the morphism space $X \times X$ of $\tilde{X}$ ensures that μ is a functor. □

Remark 1.8. When we return to the split extension (0.1), the group Π there will play a role close to that of the group denoted G in Definition 1.6 and Lemma 1.7. Nonequivariantly, we would then specialize to $Y = \Pi$ with its natural left Π action and see the usual universal principal Π-bundle. The relevant equivariant specialization is a little less obvious; see Lemma 3.4, which is a follow up of Lemma 1.7.

2. The category $\mathscr{C}at(\tilde{X}, \mathbf{\Pi})$

We let X be a space and Π be a topological group in this section. Recall that $\mathbf{\Pi}$ denote Π regarded as a category with a single object. From now on, functors and natural transformations are to be continuous (in the topological sense), even when we neglect to say so. We are especially interested in the functor categories $\mathscr{C}at(\tilde{X}, \tilde{\Pi})$, which are chaotic by Lemma 1.2, and in the functor categories $\mathscr{C}at(\tilde{X}, \mathbf{\Pi})$, which are not. The right action of Π on $\tilde{\Pi}$ induces a right action of Π on $\mathscr{C}at(\tilde{X}, \tilde{\Pi})$. We take the action for granted without indicating it in the notation.

2.1. **An explicit description of $\mathscr{C}at(\tilde{X}, \mathbf{\Pi})$.** We describe $\mathscr{C}at(\tilde{X}, \mathbf{\Pi})$ explicitly. For clarity, we defer adding in the second group G that appears in the bundle theory until after we have this description in place since a group defined solely in terms of the diagonal on X and the product on Π plays a central role in the description. By the adjunction given in Lemma 1.5 (with $G = e$), the object space of the chaotic category $\mathscr{C}at(\tilde{X}, \tilde{\Pi})$ can be identified with the space $\mathscr{U}(X, \Pi)$ of maps $X \longrightarrow \Pi$ with its standard (compactly generated) function space topology.

Definition 2.1. Define the pointwise product $*$ on $\mathscr{U}(X, \Pi)$ by

$$(\alpha * \beta)(x) = \alpha(x)\beta(x)$$

for $\alpha, \beta \colon X \longrightarrow \Pi$. The unit element ε is given by $\varepsilon(x) = e$ and inverses are given by $\alpha^{-1}(x) = \alpha(x)^{-1}$. The topological group $\mathscr{U}(X, \Pi)$ contains Π as a (closed) subgroup, where we regard an element $\sigma \in \Pi$ as the constant map $\sigma \colon X \longrightarrow \Pi$ at σ. The inclusion of Π in $\mathscr{U}(X, \Pi)$ gives $\mathscr{U}(X, \Pi)$ its right Π-action.

Definition 2.2. Choose a basepoint $x_0 \in X$. There is a unique representative map α such that $\alpha(x_0) = e$ in each orbit of $\mathscr{U}(X, \Pi)$ under the right action by Π. Let $\mathscr{O}(X, \Pi) \subset \mathscr{U}(X, \Pi)$ denote the subspace of such representative maps. It is a subgroup of $\mathscr{U}(X, \Pi)$. The Π-action and the product $*$ on $\mathscr{U}(X, \Pi)$ are related by $\alpha\sigma = \alpha * \sigma$ for $\sigma \in \Pi$, and $*$ restricts to a homeomorphism of Π-spaces $\mathscr{O}(X,\Pi) \times \Pi \longrightarrow \mathscr{U}(X,\Pi)$. Write elements of $\mathscr{U}(X,\Pi)$ in the form $\alpha\sigma$, where $\alpha(x_0) = e$. Passage to orbits restricts to a homeomorphism $\mathscr{O}(X,\Pi) \cong \mathscr{U}(X,\Pi)/\Pi$.

The proofs of the follow three lemmas are exercises from the fact that there is a unique morphism (y,x) from x to y in $\tilde{X}$; compare Lemma 1.2.

Lemma 2.3. *A functor $E\colon \tilde{X} \longrightarrow \mathbf{\Pi}$ is given by the trivial map $X \longrightarrow *$ of object spaces and a map $E\colon X \times X \longrightarrow \Pi$ of morphism spaces such that $E(x,x) = e$ and $E(z,y)E(y,x) = E(z,x)$. Define $\alpha \in \mathscr{O}(X,\Pi)$ by $\alpha(x) = E(x,x_0)$. Then α determines E by the formula*

$$E(y,x) = E(y,x_0)E(x_0,x) = \alpha(y)\alpha(x)^{-1}.$$

Writing $E = E_\alpha$, sending E_α to α specifies a homeomorphism from the space of functors $\tilde{X} \longrightarrow \mathbf{\Pi}$ to $\mathscr{O}(X,\Pi)$.

Lemma 2.4. *For $E_\alpha, E_\beta\colon \tilde{X} \longrightarrow \mathbf{\Pi}$, a natural transformation $\eta\colon E_\alpha \longrightarrow E_\beta$ is given by a map $\eta\colon X \longrightarrow \Pi$ such that $\eta(y)E_\alpha(y,x) = E_\beta(y,x)\eta(x)$ for $x,y \in X$. If $\sigma \in \Pi$ is defined by $\sigma = \eta(x_0)$, then the pair $(\beta\sigma,\alpha)$ determines η by the formula*

$$\eta(x) = E_\beta(x,x_0)\eta(x_0)E_\alpha(x,x_0)^{-1} = (\beta\sigma * \alpha^{-1})(x).$$

Writing $\eta = \eta_\sigma$, sending η_σ to $(\beta\sigma,\alpha)$ specifies a homeomorphism from the space of morphisms of $\mathscr{C}at(\tilde{X},\mathbf{\Pi})$ to the space $\mathscr{U}(X,\Pi) \times \mathscr{O}(X,\Pi)$.

Lemma 2.5. *Identify the object and morphism spaces of $\mathscr{C}at(\tilde{X},\mathbf{\Pi})$ with*

$$\mathscr{O}(X,\Pi) \quad \text{and} \quad \mathscr{M}(X,\Pi) \equiv \mathscr{U}(X,\Pi) \times \mathscr{O}(X,\Pi)$$

via the homeomorphisms of the previous two lemmas. Then the identity map I sends α to $(\alpha e,\alpha)$ and the source and target maps S and T send $(\beta\sigma,\alpha)$ to α and to β. The $S = T$ pullback

$$\mathscr{M}(X,\Pi) \times_{\mathscr{O}(X,\Pi)} \mathscr{M}(X,\Pi)$$

can be identified with $\mathscr{U}(X,\Pi) \times \mathscr{U}(X,\Pi) \times \mathscr{O}(X,\Pi)$ via

$$((\gamma\tau,\beta),(\beta\sigma,\alpha)) \leftrightarrow (\gamma\tau,\beta\sigma,\alpha)$$

and the composition map C sends $(\gamma\tau,\beta\sigma,\alpha)$ to $(\gamma\tau\sigma,\alpha)$.

Proof. If we compose $\eta_\tau\colon E_\beta \longrightarrow E_\gamma$ with $\eta_\sigma\colon E_\alpha \longrightarrow E_\beta$, we obtain

$$\eta_\tau * \eta_\sigma = \gamma^{-1}\tau * \beta * \beta^{-1}\sigma * \alpha = \gamma^{-1}\tau\sigma * \alpha,$$

which corresponds to the given description. □

2.2. **Two identifications of $\mathscr{C}at(\tilde{X},\mathbf{\Pi})$.** We show here that Lemma 1.7 leads to one identification of $\mathscr{C}at(\tilde{X},\mathbf{\Pi})$, and the work of the previous section leads to a closely related one. These elementary identifications are at the heart of our work. They are not formal, and we will see in the next section why they may be surprising.

Notation 2.6. The category $\mathbf{\Pi}$ is isomorphic to the orbit category $\tilde{\Pi}/\Pi$. The quotient functor $p\colon \tilde{\Pi} \longrightarrow \mathbf{\Pi}$ is the trivial map $\Pi \longrightarrow *$ on object spaces and is given on morphism spaces by the map $p\colon \Pi \times \Pi \longrightarrow (\Pi \times \Pi)/\Pi \cong \Pi$ specified by $p(\tau,\sigma) = \tau\sigma^{-1}$. Let q denote the functor

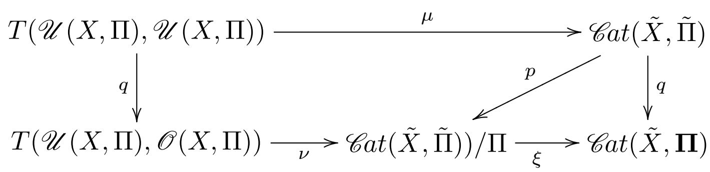
$$\mathscr{C}at(\mathrm{id},p)\colon \mathscr{C}at(\tilde{X},\tilde{\Pi}) \longrightarrow \mathscr{C}at(\tilde{X},\mathbf{\Pi}).$$

We also let q denote the functor

$$T(\mathscr{U}(X,\Pi),\mathscr{U}(X,\Pi)) \longrightarrow T(\mathscr{U}(X,\Pi),\mathscr{O}(X,\Pi))$$

induced by the quotient map $p\colon \mathscr{U}(X,\Pi) \longrightarrow \mathscr{U}(X,\Pi)/\Pi \cong \mathscr{O}(X,\Pi)$.

Theorem 2.7. *There is a commutative diagram of topological categories in which μ, ν, and ξ are isomorphisms.*



Proof. The map p is the quotient map given by passage to orbits over Π. Since q on the right is a Π-map with Π acting trivially on $\mathscr{C}at(\tilde{X},\mathbf{\Pi})$, q factors through a map ξ that makes the triangle commute. Since $\mathscr{C}at(\tilde{X},\tilde{\Pi})$ is the chaotic category whose object space is the topological group $\mathscr{U}(X,\Pi)$, Lemma 1.7 gives the isomorphism μ. Since q on the left is obtained by passage to orbits from the relevant action of Π, it is clear that μ induces an isomorphism ν making the left trapezoid commute.

All that remains is to prove that ξ is an isomorphism, and that follows from the work of the previous section. For a functor $E_\alpha\colon \tilde{X} \longrightarrow \mathbf{\Pi}$, $\alpha\colon X \longrightarrow \Pi$ and $\alpha\times\alpha\colon X\times X \longrightarrow \Pi\times\Pi$ define the object and morphism maps of a functor $F\colon \tilde{X} \longrightarrow \tilde{\Pi}$. The functoriality properties of E_α show that $p\circ F = E_\alpha$, so that q is surjective on objects. If we also have $p\circ F' = E_\alpha$, then a quick check shows that $F(x)^{-1}F'(x) = F(y)^{-1}F'(y)$ for all $x,y\in X$. If the common value is denoted by σ, then $F'(x) = F(x)\sigma$ for all x. In view of the specification of p and q in Notation 2.6, this implies that ξ is a homeomorphism on object spaces.

Now let $E_\alpha, E_\beta\colon \tilde{X} \longrightarrow \mathbf{\Pi}$ be any two functors. For any choices of functors $F,F'\colon \tilde{X} \longrightarrow \tilde{\Pi}$ such that $q\circ F = E_\alpha$ and $q\circ F' = E_\beta$, define $\zeta\colon X \longrightarrow \Pi\times\Pi$ by $\zeta(x) = (F(x),F'(x))$. Then ζ is a map from the object space of $\tilde{X}$ to the morphism space of $\tilde{\Pi}$. A quick check shows that ζ is a natural transformation $F \longrightarrow F'$ such that $\eta = q\circ\zeta$ is a natural transformation $E_\alpha \longrightarrow E_\beta$ with $\eta_{x_0} = F'(x_0)F(x_0)^{-1}$. Via our enumeration of the possible choices, this implies that q restricted to the inverse image of the space of natural transformations $E_\alpha \longrightarrow E_\beta$ can be identified with the quotient map $p\colon \Pi\times\Pi \longrightarrow \Pi$ of Notation 2.6. It follows that ξ is a homeomorphism on morphism spaces. □

2.3. **The nerve functor and classifying spaces.** We recall the definition of the nerve functor N in more detail than might be thought warranted at this late date since, in the presence of the left-right action dichotomy of multiple group actions, the original definitions in category theory can cause real problems arising from categorical dyslexia. There are two standard conventions in the literature, and we must choose. Let $\mathscr{C}$ be a topological category with object space $\mathscr{O}$ and morphism space $\mathscr{M}$. Then $N_0\mathscr{C} = \mathscr{O}$ and, for $q>0$,

$$N_q\mathscr{C} = \mathscr{M}\times_{\mathscr{O}}\cdots\times_{\mathscr{O}}\mathscr{M},$$

with q factors $\mathscr{M}$. The pullbacks are over pairs of maps (S,T). To avoid dyslexia, we remember that $g\circ f$ means first f and then g, and choose to forget the picture

$$\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \longrightarrow \cdots \longrightarrow \bullet \xrightarrow{f_{q-1}} \bullet \xrightarrow{f_q} \bullet$$

of q composable arrows and instead remember that the picture

$$x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \longleftarrow \cdots \longleftarrow x_{q-2} \xleftarrow{f_{q-1}} x_{q-1} \xleftarrow{f_q} x_q \tag{2.8}$$

corresponds to an element $[f_1, \cdots, f_q]$ of $N_q\mathscr{C}$, so that $S(f_i) = T(f_{i+1})$. For $x \in \mathscr{O}$, we write $\mathrm{id} = I(x)$ generically. Then

$$d_0[f] = T(f), \quad d_1[f] = S(f), \quad \text{and} \quad s_0(x) = [\mathrm{id}_x].$$

For $q \geq 2$,

$$d_i[f_1, \cdots, f_q] = \begin{cases} [f_2, \cdots, f_q] & \text{if } i = 0 \\ [f_1, \cdots, f_{i-1}, f_i \circ f_{i+1}, f_{i+2}, \cdots, f_q] & \text{if } 0 < i < q \\ [f_1, \cdots, f_{q-1}] & \text{if } i = q \end{cases}$$

and, for $q \geq 1$,

$$s_i[f_1, \ldots, f_q] = [f_1, \cdots, f_i, \mathrm{id}, f_{i+1}, \cdots, f_q].$$

Of course, these can and should be expressed in terms of the maps S, T, I, and C so as to remember the topology and check continuity.

Recall that a (right) action of a group G on a simplicial space Y_* is specified by levelwise group actions such that the d_i and s_i are G-maps; formally, Y_* is a simplicial object in the category of (right) G-spaces. Orbit and fixed point simplicial spaces are constructed levelwise, $(Y_*/G)_q = Y_q/G$ and $(Y_*)^G_q = Y^G_q$. For a G-category $\mathscr{C}$, $N(\mathscr{C}^G) \cong (N\mathscr{C})^G$ since N is a right adjoint, but it is rarely the case that $N(\mathscr{C}/G) \cong (N\mathscr{C})/G$, as the following example should make clear.

Example 2.9. Let G be a group and let G act on itself and therefore on $\mathbf{G}$ by conjugation. Let A be the abelianization of G. Then $\mathbf{G}/G \cong \mathbf{A}$, and $N\mathbf{A}$ is generally much smaller than $(N\mathbf{G})/G$. Here $[g_1, \ldots, g_q]$ and $[h_1, \cdots, h_q]$ are in the same orbit under the conjugation action if and only if there is a single g such that $gg_ig^{-1} = gh_ig^{-1}$ for all i. For example if G is a finite simple group of order n, then A is trivial but $(N_q\mathbf{G})/G$ has at least n^{q-1} elements.

In this example, $N_*\mathbf{G}$ is the simplicial space, usually denoted B_*G, whose geometric realization is the classifying space BG. Parametrizing with a left G-space Y gives a familiar simplicial space $B_*(*, G, Y)$ (e.g. [9, §7]). Write $q\colon E_*G \longrightarrow B_*G$ for the map

$$B_*(*, G, G) \longrightarrow B_*(*, G, *) \cong B_*(*, G, G)/G$$

induced by $G \longrightarrow *$. The isomorphism on the right is obvious, but it is in fact an example of an isomorphism of the form $N(\mathscr{C}/G) \cong (N\mathscr{C})/G$, as the following observations make clear. Recall the translation category from Definition 1.6.

Lemma 2.10. *The simplicial space $NT(G, Y)$ is isomorphic to $B_*(*, G, Y)$.*

Proof. A typical q-tuple (2.8) in $N_qT(G, Y)$ has i^{th} term

$$f_i = (g_i, g_{i+1} \cdots g_q y)\colon g_{i+1} \cdots g_q y \longrightarrow g_i g_{i+1} \cdots g_q y$$

for elements $g_i \in G$ and $y \in Y$. It corresponds to $[g_1, \cdots, g_q]y$ in $B_q(*, G, Y)$. □

Remark 2.11. For any space X, $N\tilde{X}$ is the simplicial space denoted D_*X in [8, p. 97]. Our choice of S and T on $\tilde{X}$ is consistent with (2.8) and the usual notation $(x_0, \cdots, x_q)$ for q-simplices. The claim in Definition 1.1 that $|N\tilde{X}|$ is contractible is immediate from [8, 10.4], which says that D_*X is simplicially contractible. The isomorphism $N\mu\colon NT(G, G) \longrightarrow N\tilde{G}$ implied by Lemma 1.7 coincides with the isomorphism $\alpha_*\colon E_*G \longrightarrow D_*G$ of [8, 10.4].

Applying geometric realization, write $B(*,G,Y) = |B_*(*,G,Y)|$, and similarly for EG and BG. Then $B(*,G,Y) \cong B(*,G,G)\times_G Y = EG\times_G Y$. By Lemma 2.10,

$$BT(G,Y) = EG \times_G Y.$$

A relevant example is $Y = G/H$ for a (closed) subgroup H of G. The space

$$BT(G,G/H) = EG \times_G (G/H) \cong (EG)/H$$

is a classifying space BH since EG is a free contractible H-space.

In particular, take $G = \mathscr{U}(X,\Pi)$ and $H = \Pi$ for a space X and group Π, remembering that $\mathscr{C}at(\tilde{X},\tilde{\Pi})$ is the chaotic category with object space the group $\mathscr{U}(X,\Pi)$. Applying the classifying space functor to the diagram of Theorem 2.7 and using Lemma 2.10, we obtain the following commutative diagram, in which the horizontal maps are homeomorphisms and, up to canonical homeomorphisms, the vertical maps are obtained by passage to orbits over Π.

$$\begin{array}{ccccc}
E(\mathscr{U}(X,\Pi)) & \xrightarrow{\cong} & B\mathscr{C}at(\tilde{X},\tilde{\Pi}) & \xrightarrow{=} & B\mathscr{C}at(\tilde{X},\tilde{\Pi}) \\
\downarrow & & \downarrow & & \downarrow \\
(E\mathscr{U}(X,\Pi))/\Pi & \xrightarrow[\cong]{} & B(\mathscr{C}at(\tilde{X},\tilde{\Pi})/\Pi) & \xrightarrow[\cong]{} & B\mathscr{C}at(\tilde{X},\mathbf{\Pi})
\end{array}$$

Ignoring minor topological niceness conditions[3], for any space X the diagram gives isomorphic categorical models for the universal principal Π-bundle $E\Pi \longrightarrow B\Pi$.

3. Categorical universal principal equivariant bundles

3.1. **Preliminaries on actions by the semi-direct product Γ.** Now return to the split extension (0.1) of the introduction. For a Γ-category or Γ-space, passage to orbits with respect to Π gives a G-category or a G-space. It is standard in equivariant bundle theory to let G act from the left and Π act from the right. Thus suppose that X is a left G and right Π object in any category. Using elementwise notation, turn the right action of Π into a left action by setting $\sigma x = x\sigma^{-1}$.

By an action of Γ on X, we understand a left action that coincides with the given actions when restricted to the subgroups $G = e \times G$ and $\Pi = \Pi \times e$ of Γ. Since $(\sigma, g) = (\sigma, e)(e, g)$, the action must be defined by

$$(\sigma,g)x = (\sigma,e)(e,g)x = (\sigma,e)gx = \sigma gx = (gx)\sigma^{-1}. \tag{3.1}$$

We will denote the action of G on Π by $\cdot$, but we just use juxtaposition for the prescribed actions of G and Π. Since action by g on Π is a group homomorphism, $g\cdot(\sigma\tau) = (g\cdot\sigma)(g\cdot\tau)$ and $g\cdot\sigma^{-1} = (g\cdot\sigma)^{-1}$. The interaction of Π and G in Γ is given by the twisted commutation relation

$$(e,g)(\sigma,e) = (g\cdot\sigma, g) = (g\cdot\sigma, e)(e,g),$$

or the same relation with σ replaced by σ^{-1}. Therefore (3.1) gives an action of Γ if and only if the given actions of Π and G satisfy the twisted commutation relation

$$g(x\sigma) = (gx)(g\cdot\sigma). \tag{3.2}$$

[3]The identity element of the group $\mathscr{U}(X,\Pi)$ should be a nondegenerate basepoint and the space $\mathscr{U}(X,\Pi)$ should be paracompact; see [9, 9.10].

The placement of parentheses is crucial: we are taking group actions in different orders. When the action of G on Π is trivial, $g\cdot\sigma = \sigma$, this is the familiar statement that commuting left and right actions define an action by the product $\Pi \times G$.

Lemma 3.3. *For a G-category $\mathscr{A}$, the left G and right Π-actions on $\mathscr{C}at_G(\mathscr{A}, \tilde{\Pi})$ extend naturally to a Γ-action.*

Proof. We must verify that $g(F\sigma) = (gF)(g\cdot\sigma)$ for $g \in G$, $\sigma \in \Pi$ and a functor $F\colon \mathscr{A} \longrightarrow \Pi$. The unique natural transformation $E \longrightarrow F$ between a pair of functors E and F will then necessarily be given by Γ-maps. The verification is formal from the fact that G acts by conjugation, so that the action of G on Π is part of the prescription of the action of G on F. Recall that the left action of G on $\mathscr{C}at_G(\mathscr{A}, \tilde{\Pi})$ is given by conjugation, $(gF)(a) = g\cdot F(g^{-1}a)$ for $g \in G$ and an object or morphism $a \in \mathscr{A}$. The right action of Π is given by $(F\sigma)(a) = F(a)\sigma$. Then

$$
\begin{aligned}
(g(F\sigma))(a) &= g\cdot(F\sigma)(g^{-1}a)\\
&= g\cdot(F(g^{-1}a)\sigma)\\
&= (g\cdot F(g^{-1}a))(g\cdot\sigma)\\
&= ((gF)(g^{-1}a))(g\cdot\sigma)\\
&= ((gF)(g\cdot\sigma))(a).
\end{aligned}
$$

□

In particular, let $\mathscr{A} = \tilde{X}$ for a left G-space X. Then the given action of G on the object space X and the diagonal action of G on the morphism space $X \times X$ give a left G-action on the category $\tilde{X}$. Lemma 3.3 shows that the left G and right Π-action on $\mathscr{C}at_G(\tilde{X}, \tilde{\Pi})$ give it an action by Γ. Explicitly, the conjugation left action by G and the evident right action by Π on the object space $\mathscr{U}(X, \Pi)$ induce diagonal actions on the morphism space $\mathscr{U}(X, \Pi) \times \mathscr{U}(X, \Pi)$, and these specify left G and right Π-actions on $\mathscr{C}at_G(\tilde{X}, \Pi)$ that satisfy the commutation relation required for a Γ-action.

Specializing further to $X = G$, we have the following equivariant elaboration of Lemma 1.7. We change the group G there to the group $\mathscr{U}(G, \Pi)$ here and remember that the product on $\mathscr{U}(G, \Pi)$ is just the pointwise product induced by the product on Π, with no dependence on the product of G. We identify the chaotic right $\mathscr{U}(G, \Pi)$-category with object space $\mathscr{U}(G, \Pi)$ with the category $\mathscr{C}at(\tilde{G}, \tilde{\Pi})$. Remember that Π is a subgroup of $\mathscr{U}(G, \Pi)$.

Lemma 3.4. *The isomorphism of right $\mathscr{U}(G, \Pi)$-categories*

$$\mu\colon T(\mathscr{U}(G, \Pi), \mathscr{U}(G, \Pi)) \longrightarrow \mathscr{C}at_G(\tilde{G}, \tilde{\Pi})$$

is an isomorphism of Γ-categories, where the G-action on both source and target categories is given by the conjugation action on the object space $\mathscr{U}(G, \Pi)$ and the resulting diagonal action on the morphism space $\mathscr{U}(G, \Pi) \times \mathscr{U}(G, \Pi)$.

Proof. Since μ is an isomorphism and a Π-map, we can and must give the source category the unique G-action such that μ is a G-map. Since μ is the identity map on object spaces, the action must be the conjugation action on the object space. On an element (β, α) of the morphism space, we must define

$$g(\beta, \alpha) = \mu^{-1}(g\mu(\beta, \alpha)) = \mu^{-1}(g(\beta\alpha), g\alpha) = \mu^{-1}((g\beta)(g\alpha), g\alpha) = (g\beta, g\alpha). \qquad \square$$

Lemma 3.5. *With $X = G$, the diagram of Theorem 2.7 is a commutative diagram of Γ-categories and maps of Γ-categories, where Γ acts through the quotient homomorphism $\Gamma \longrightarrow G$ on the three categories on the bottom row.*

Proof. Since the trapezoid is obtained by passing to orbits under the action of Π, it is a diagram of Γ-categories by Lemma 3.4. The functor $p\colon \tilde{\Pi} \longrightarrow \mathbf{\Pi}$ of Notation 2.6 is a G-map since

$$g\cdot(\tau,\sigma) = g\cdot(\tau\sigma^{-1}) = (g\cdot\tau)(g\cdot\sigma)^{-1} = p(g\cdot\tau, g\cdot\sigma).$$

It follows that the right vertical arrow $q = \mathscr{C}at_G(\tilde{G}, p)$ is a map of Γ-categories. Letting $[F]$ denote the orbit of a functor $F\colon \tilde{G} \longrightarrow \tilde{\Pi}$ under the right action of Π, the functor ξ is specified by $\xi[F] = p\circ F$, and it follows that ξ is Γ-equivariant. □

3.2. **Universal principal $(\Pi;\Gamma)$-bundles.** Observe that for any G-category $\mathscr{A}$, the corepresented functor $\mathscr{C}at_G(\mathscr{A}, -)$ from G-categories to G-categories is a right adjoint and therefore preserves all limits. We take $\mathscr{A}$ to be the G-category $\tilde{G}$ from now on, and we use the functor $\mathscr{C}at_G(\tilde{G}, -)$ to obtain a convenient categorical description of universal principal $(\Pi;\Gamma)$-bundles. Variants of the construction are given in [10, 13].

Definition 3.6. Let G and Π be topological groups, where G acts on Π. Define $E(\Pi;\Gamma)$ to be the Γ-space

$$B\mathscr{C}at_G(\tilde{G}, \tilde{\Pi}) = |N\mathscr{C}at_G(\tilde{G}, \tilde{\Pi})| = |N\mathscr{C}at_G(\tilde{G}, \tilde{\Pi})|.$$

Define $B(\Pi;\Gamma)$ to be the orbit G-space $E(\Pi;\Gamma)/\Pi$ and let $p\colon E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$ be the quotient map.

We need a lemma in order to prove that p is a universal $(\Pi;\Gamma)$-bundle in favorable cases. We defer the proof to the next section. We believe that the result is true more generally, but there are point-set topological issues obstructing a proof. We shall not obscure the simplicity of our work by seeking maximum generality.

Lemma 3.7. *Let Λ be a closed subgroup of Γ. If $\Lambda\cap\Pi \neq e$, then the fixed point category $\mathscr{C}at_G(\tilde{G}, \tilde{\Pi})^{\Lambda}$ is empty. At least if either Π is compact Hausdorff and G is discrete or Γ is discrete, if $\Lambda\cap\Pi = e$, then $\mathscr{C}at_G(\tilde{G}, \tilde{\Pi})^{\Lambda}$ is non-empty and chaotic.*

The following result is [7, Thm. 9], but the details of proof are in [6, §2].

Definition 3.8. Let $p\colon E \longrightarrow B$ be a principal Π-bundle where B is a G-space. Then p is a principal $(\Pi;\Gamma)$-bundle if the (free) action of Π on E extends to an action of Γ and p is a Γ-map, where Γ acts on B through the quotient map $\Gamma \longrightarrow G$.

Theorem 3.9. *A numerable principal $(\Pi;\Gamma)$-bundle $p\colon E \longrightarrow B$ is universal if and only if E^{Λ} is contractible for all (closed) subgroups Λ of Γ such that $\Lambda\cap\Gamma = \{e\}$.*

We explain the hypotheses. Recall from point-set topology that a space X is completely regular if for every closed subspace C and every point x not in C, there is a continuous function $f\colon X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(C) = 1$. This is a weak condition that is satisfied by reasonable spaces, such as CW complexes.

Remark 3.10. By [7, Propositions 4 and 5], a principal $(\Pi;\Gamma)$-bundle with completely regular total space is locally trivial, and a locally trivial principal $(\Pi;\Gamma)$-bundle over a paracompact base space (such as a CW complex) is numerable. Therefore, modulo weak point-set topological conditions, the fixed point condition in Theorem 3.9 is the essential criterion for a universal bundle.

Therefore Lemma 3.7 has the following consequence.

Theorem 3.11. *At least if* Π *is a compact Lie group and* G *is discrete or* Γ *is discrete, the map*

$$p\colon E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$$

obtained by passage to orbits over Π *is a universal principal* $(\Pi;\Gamma)$*-bundle.*

The classsifying space $B(\Pi;\Gamma) = |N\mathscr{C}at_G(\tilde{G},\tilde{\Pi})|/\Pi$ is obtained by first applying the classifying space functor and then passing to orbits. On the other hand, the space $B\mathscr{C}at_G(\tilde{G},\mathbf{\Pi}) = |N\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})|$ is obtained by first passing to orbits on the categorical level and then applying the classifying space functor. The category $\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})$ is thoroughly understood, as explained in §2. The virtue of our model for $B(\Pi;\Gamma)$ is that these two G-spaces can be identified, by Theorem 2.7.

Theorem 3.12. *The canonical map*

$$B(\Pi;\Gamma) = |N\mathscr{C}at_G(\tilde{G},\tilde{\Pi})|/\Pi \longrightarrow |N\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})| = B\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})$$

is a homeomorphism of G*-spaces. Therefore, at least if* Π *is a compact Lie group and* G *is discrete or* Γ *is discrete, the map*

$$Bq\colon B\mathscr{C}at_G(\tilde{G},\tilde{\Pi}) \longrightarrow B\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})$$

is a universal principal $(\Pi;\Gamma)$*-bundle.*

Scholium 3.13. For finite groups G, this result is claimed in [13, p. 1294]. For more general groups G, [13, 3.1] states an analogous result, but with $\tilde{\Pi} \longrightarrow \mathbf{\Pi}$ replaced by a functor defined in terms of the nonequivariant universal bundle $E\Pi \longrightarrow B\Pi$, resulting in a much larger construction. The replacement is needed for the proof of their analogue [13, 3.3] of our Lemma 3.7. A commutation relation of the form $N(\mathscr{C}/\Pi) = (N\mathscr{C})/\Pi$ for their larger construction is stated (five lines above [13, 3.1]), but there is no hint of a proof or of the need for one. It is not altogether clear to us that the commutation relation stated there is true, and we view the commutation relation Theorem 2.7 as the main point of the proof of Theorem 3.12. Nevertheless, [13] had the insightful right idea that led to our work.

4. Determination of fixed points

4.1. **The fixed point spaces of** $E(\Pi;\Gamma)$**.** We must prove Lemma 3.7. Since Π acts freely on $\mathscr{C}at_G(\tilde{G},\tilde{\Pi})$, it is clear that $\mathscr{C}at_G(\tilde{G},\tilde{\Pi})^\Lambda$ is empty if $\Lambda\cap\Pi \neq e$. Thus assume that $\Lambda\cap\Pi = e$. By Lemma 1.3, the fixed point category $\mathscr{C}at_G(\tilde{G},\tilde{\Pi})^\Lambda$ is chaotic. It remains to prove that it is non-empty, and Lemma 1.5 implies that this is so if and only if the space $\mathscr{U}(G,\Pi)^\Lambda$ is non-empty. Thus it suffices to show that $\mathscr{U}(G,\Pi)$ has a Λ-fixed point, which means that there is a Λ-map $f\colon G\longrightarrow\Pi$. We prove this using the following standard generalization of a homomorphism and a variant needed later.

Definition 4.1. A map $\alpha\colon G\longrightarrow\Pi$ is a crossed homomorphism if

$$\alpha(gh) = \alpha(g)(g\cdot\alpha(h)) \tag{4.2}$$

for all $g,h\in G$. In particular,

$$\alpha(e) = e,\ \ \alpha(g)^{-1} = g\cdot\alpha(g^{-1})\ \text{ and }\ \alpha(g^{-1})^{-1} = g^{-1}\cdot\alpha(g). \tag{4.3}$$

A map $\alpha\colon G\longrightarrow\Pi$ is a crossed anti-homomorphism if

$$\alpha(gh) = (g\cdot\alpha(h))\alpha(g). \tag{4.4}$$

Note that α must be continuous in our general topological context.

Lemma 4.5. *At least if* Π *is compact Hausdorff or* Γ *is discrete, the subgroups* Λ *of* Γ *such that* $\Lambda \cap \Pi = e$ *are the subgroups of the form*

$$\Lambda_\alpha = \{(\alpha(h), h) | h \in H\},$$

where H *is a subgroup of* G *and* $\alpha\colon H \longrightarrow \Pi$ *is a crossed homomorphism.*

Proof. Clearly Λ_α is a subgroup of Γ such that $\Lambda_\alpha \cap \Pi = e$. Conversely, let $\Lambda \cap \Pi = e$. Define H to be the image of the composite of the inclusion $\iota\colon \Lambda \subset \Gamma$ and the projection $\pi\colon \Gamma \longrightarrow G$. Since $\Lambda \cap \Pi = e$, the composite $\pi \circ \iota$ is injective and so restricts to an isomorphism $\nu\colon \Lambda \longrightarrow H$. The hypotheses ensure that ν is a homeomorphism, and that is all that they are needed for. For $h \in H$, define $\alpha(h) = \sigma$, where σ is the unique element of Π such that $(\sigma, h) \in \Lambda$. In other words, α is the composite of $\iota \circ \nu^{-1}\colon H \longrightarrow \Gamma$ and the projection $\rho\colon \Gamma \longrightarrow \Pi$. This description implies that α is continuous. For $h, k \in H$,

$$(\alpha(h), h)(\alpha(k), k) = (\alpha(h)(h \cdot \alpha(k)), hk) \in \Lambda,$$

so $\alpha(hk) = \alpha(h)(h \cdot \alpha(k))$. Thus α is a crossed homomorphism and $\Lambda = \Lambda_\alpha$. □

Proof of Lemma 3.7. We must obtain a Λ-map $f\colon G \to \Pi$, where $\Lambda = \Lambda_\alpha$ for a crossed homomorphism α. By the definition of the action by Λ, this means that

$$f(g) = (h \cdot f(h^{-1}g))\alpha(h)^{-1}$$

or equivalently

$$h \cdot f(h^{-1}g) = f(g)\alpha(h)$$

for all $h \in H$ and $g \in G$. We choose right coset representatives $\{g_i\}$ to write G as a disjoint union of cosets Hg_i. We then define $f : G \longrightarrow \Pi$ by

$$f(kg_i) = \alpha(k)^{-1}$$

for $k \in H$. By using (4.2), writing out the inverse of a product as the product of inverses, using that $h^{-1}\cdot$ and $h\cdot$ are group homomorphisms and that $\cdot$ is a group action, and finally using (4.3) and, again, that $\cdot$ is a group action, we see that

$$\begin{aligned} h \cdot f(h^{-1}kg_i) &= h \cdot \alpha(h^{-1}k)^{-1} \\ &= h \cdot (\alpha(h^{-1})(h^{-1} \cdot \alpha(k))^{-1} \\ &= h \cdot ((h^{-1} \cdot \alpha(k))^{-1}(\alpha(h^{-1}))^{-1}) \\ &= (h \cdot (h^{-1} \cdot \alpha(k)^{-1})(h \cdot (\alpha(h^{-1})^{-1}) \\ &= \alpha(k)^{-1}(h \cdot (h^{-1} \cdot \alpha(h))) \\ &= f(kg_i)\alpha(h). \end{aligned}$$

for all $h \in H$. Thus f is a Λ-map. We have assumed that G is discrete in order to ensure that f is continuous. □

Remark 4.6. If we relax the condition that G is discrete, we do not see how to prove that f is continuous, as would be needed for a more general result.

4.2. **The fixed point categories of $\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})$.** For $H \subset G$, we identify the fixed point categories $\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})^H$ in this section, with no restrictions on Π and G. Since the functor B commutes with fixed points, this gives a categorically precise interpretation of the fixed point space $B(\Pi;\Gamma)^H$ on passage to classifying spaces.

We return to §2, taking $X = G$ there. The H-fixed functors and H-natural transformations in $\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})$ under the conjugation action of G are the H-equivariant functors and the H-natural transformations. Thus

$$\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})^H = H\mathscr{C}at(\tilde{G}, \mathbf{\Pi}). \tag{4.7}$$

Since $\tilde{G}$ and $\tilde{H}$ are both H-free contractible categories, they are equivalent as H-categories. Therefore

$$\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})^H \simeq H\mathscr{C}at(\tilde{H}, \mathbf{\Pi}). \tag{4.8}$$

This implies that we may restrict to the case $G = H$ and deduce conclusions in general. The objects and morphisms of $G\mathscr{C}at(\tilde{G}, \mathbf{\Pi})$ are the G-equivariant functors $E : \tilde{G} \to \mathbf{\Pi}$ and the G-equivariant natural transformations η. In Lemma 2.3, we described a functor E in terms of the map $\alpha \colon G \longrightarrow \Pi$ defined by $\alpha(h) = E(h, e)$.

Lemma 4.9. *The G-action on functors $E \colon \tilde{G} \longrightarrow \mathbf{\Pi}$ induces the G-action on maps $\alpha \colon G \longrightarrow \Pi$ specified by*

$$(g\alpha)(h) = (g \cdot (\alpha(g^{-1}h))(g \cdot \alpha(g^{-1})^{-1})).$$

Proof.

$$(gE)(h, e) = g \cdot E(g^{-1}h, g^{-1}) = g \cdot (E(g^{-1}h, e)E(e, g^{-1})). \qquad \square$$

Lemma 4.10. *The space of objects of $G\mathscr{C}at(\tilde{G}, \mathbf{\Pi})$ can be identified with the subspace of $\mathscr{U}(G, \Pi)$ consisting of the crossed anti-homomorphisms $\alpha \colon G \longrightarrow \Pi$.*

Proof. Setting $g\alpha = \alpha$ and applying $g^{-1} \cdot (-)$ to the formula for the action of G on α, we obtain

$$g^{-1} \cdot \alpha(h) = \alpha(g^{-1}h)\alpha(g^{-1})^{-1}.$$

Replacing g^{-1} by g and multiplying on the right by $\alpha(g)$, this gives

$$\alpha(gh) = (g \cdot \alpha(h))\alpha(g),$$

which says that α is a crossed anti-homomorphism. $\square$

Similarly, as in Lemma 2.4, a natural transformation $\eta \colon E_\alpha \longrightarrow E_\beta$ is determined by $\sigma = \eta(e)$. Explicitly,

$$\eta(g) = E_\beta(g, e)\eta(e)E_\alpha(g, e)^{-1} = \beta(g)\sigma\alpha(g)^{-1}$$

for $g \in G$. Now a G-fixed natural transformation η satisfies $\eta(gh) = g \cdot \eta(h)$ for $g, h \in G$ and thus $\eta(g) = \eta(ge) = g \cdot \eta(e) = g \cdot \sigma$. Therefore the naturality square for G-fixed natural transformations translates into

$$g \cdot \sigma = \beta(g)\sigma\alpha(g)^{-1}$$

or equivalently

$$\beta(g)\sigma = (g \cdot \sigma)\alpha(g). \tag{4.11}$$

We use the following definitions and lemma to put things together.

Definition 4.12. Let G act on Π. Define the crossed functor category $\mathscr{C}at_\times(\mathbf{G},\mathbf{\Pi})$ to be the category whose objects are the crossed homomorphisms $G \longrightarrow \Pi$ and whose morphisms $\sigma\colon \alpha \longrightarrow \beta$ are the elements $\sigma \in \Pi$ such that $\beta(g)(g\cdot\sigma) = \sigma\alpha(g)$. The composite $\tau \circ \sigma$, $\tau\colon \beta \longrightarrow \gamma$ is given by $\tau\sigma$. Define the centralizer Π^α of a crossed homomorphism $\alpha\colon G \longrightarrow \Pi$ to be the subgroup

$$\Pi^\alpha = \{\sigma \in \Pi | \alpha(g)(g\cdot\sigma) = \sigma\alpha(g) \ \text{ for all } g \in G\}$$

of Π. It is the automorphism group $\operatorname{Aut}\alpha$ of the object α in $\mathscr{C}at_\times(\mathbf{G},\mathbf{\Pi})$.

Definition 4.13. Define the anti-crossed functor category $\mathscr{C}at^-_\times(\mathbf{G},\mathbf{\Pi})$ to have objects the crossed anti-homomorphisms $\alpha\colon G \longrightarrow \Pi$ and morphisms $\sigma\colon \alpha \longrightarrow \beta$ the elements $\sigma \in \Pi$ such that $\beta(g)\sigma = (g\cdot\sigma)\alpha(g)$, with $\tau\circ\sigma = \tau\sigma$. The centralizer Π^α of a crossed anti-homomorphism $\alpha\colon G \longrightarrow \Pi$ is

$$\Pi^\alpha = \{\sigma \in \Pi | \alpha(g)\sigma = (g\cdot\sigma)\alpha(g) \ \text{ for all } g \in G\}.$$

Again, $\Pi^\alpha = \operatorname{Aut}\alpha$ in $\mathscr{C}at^-_\times(\mathbf{G},\mathbf{\Pi})$.

If the action of G on Π is trivial, then the crossed functor category is just the functor category $\mathscr{C}at(\mathbf{G},\mathbf{\Pi})$ since homomorphisms $\alpha\colon G \to \Pi$ correspond to functors $\alpha\colon \mathbf{G} \longrightarrow \mathbf{\Pi}$ and elements $\sigma \in \Pi$ such that $\beta(g)\sigma = \sigma\alpha(g)$ for $g \in G$ correspond to natural transformations $\alpha \longrightarrow \beta$. In that case,

$$\Pi^\alpha = \{\sigma \in \Pi | \sigma^{-1}\alpha(g)\sigma = \alpha(g) \ \text{ for all } g \in G\}$$

is the usual centralizer of α in Π, and then the following identification is obvious.

Lemma 4.14. *The categories $\mathscr{C}at_\times(\mathbf{G},\mathbf{\Pi})$ and $\mathscr{C}at^-_\times(\mathbf{G},\mathbf{\Pi})$ of crossed homomorphisms and crossed anti-homomorphisms are canonically isomorphic.*

Proof. For a crossed homomorphism $\alpha\colon G \longrightarrow \Pi$, define $\bar\alpha\colon G \longrightarrow \Pi$ by

$$\bar\alpha(g) = g\cdot\alpha(g^{-1}).$$

Then

$$\bar\alpha(gh) = (gh)\cdot\alpha(h^{-1}g^{-1}) = g\cdot h\cdot(\alpha(h^{-1})(h^{-1}\cdot\alpha(g^{-1})) = (g\cdot\bar\alpha(h))(\bar\alpha(g)),$$

so that $\bar\alpha$ is a crossed anti-homomorphism. If σ is a morphism $\alpha \longrightarrow \beta$ in $\mathscr{C}at_\times(\mathbf{G},\mathbf{\Pi})$, then $\beta(g)(g\cdot\sigma) = \sigma\alpha(g)$. It follows that

$$\bar\beta(g)\sigma = (g\cdot\beta(g^{-1}))\sigma = g\cdot(\beta(g^{-1})(g^{-1}\cdot\sigma)) = g\cdot(\sigma\alpha(g^{-1})) = (g\cdot\sigma)\bar\alpha(g),$$

so that σ is also a morphism $\bar\alpha \longrightarrow \bar\beta$ in $\mathscr{C}at^-_\times(\mathbf{G},\mathbf{\Pi})$. The construction of the inverse isomorphism is similar. □

Returning to the G-fixed category of interest, we summarize our discussion in terms of these definitions and results.

Theorem 4.15. *The fixed point category $\mathscr{C}at_G(\tilde G,\mathbf{\Pi})^G$ is isomorphic to the anti-crossed functor category $\mathscr{C}at^-_\times(\mathbf{G},\mathbf{\Pi})$. Therefore it is also isomorphic to the crossed functor category $\mathscr{C}at_\times(\mathbf{G},\mathbf{\Pi})$.*

Corollary 4.16. *For $H \subset G$, the fixed point category $\mathscr{C}at_G(\tilde G,\mathbf{\Pi})^H$ is equivalent to the anti-crossed functor category $\mathscr{C}at^-_\times(\mathbf{H},\mathbf{\Pi})$. Therefore it is also equivalent to the crossed functor category $\mathscr{C}at_\times(\mathbf{H},\mathbf{\Pi})$.*

Remark 4.17. The appearance of anti-homomorphisms in this context is not new; see e.g. [14]. As we have seen, it is also innocuous. We have chosen not to introduce opposite groups, but the anti-isomorphism $(-)^{-1}\colon \Pi \longrightarrow \Pi^{op}$ is relevant.

4.3. **The fixed point spaces of** $B(\Pi, \Gamma)$**.** Since $\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})^G$ is a groupoid, it is equivalent to the coproduct of its subcategories $\mathrm{Aut}\,\alpha$, where we choose one α from each isomorphism class of objects. Thus we have the following result.

Theorem 4.18. *For* $H \subset G$,

$$B(\Pi; \Gamma)^H = B\mathscr{C}at_G(\tilde{G}, \mathbf{\Pi})^H \simeq \coprod B\,\mathrm{Aut}\,\alpha,$$

where the coproduct runs over one α *from each isomorphism class of crossed homomorphisms* $\alpha\colon H \longrightarrow \Pi$ *(or, equivalently, crossed anti-homomorphisms* α*).*

We can identify these isomorphism classes and automorphism groups in more familiar group theoretic terms.

Lemma 4.19. *The isomorphism classes of crossed homomorphisms* $\alpha\colon H \longrightarrow \Pi$ *are in bijective correspondence with the* Π*-conjugacy classes of subgroups* Λ *of* Γ *such that* $\Lambda \cap \Pi = e$ *and* $q(\Lambda) = H$.

Proof. By Definition 4.1, the subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ are of the form

$$\Lambda_\alpha = \{(\alpha(h), h) | h \in H\}$$

for a crossed homomorphism $\alpha\colon H \to \Pi$. If $\sigma \in \Pi$, then $\sigma\Lambda_\alpha\sigma^{-1} \cap \Pi = e$ and therefore $\sigma\Lambda_\alpha\sigma^{-1} = \Lambda_\beta$ for some crossed homomorphism β. The equality forces β and α to be defined on the same subgroup H and to satisfy $\beta(g)(g \cdot \sigma) = \sigma\alpha(g)$. That is, σ is a morphism and thus an isomorphism $\alpha \longrightarrow \beta$ in $\mathscr{C}at_\times(\mathbf{G}, \mathbf{\Pi})$. □

Lemma 4.20. *For a crossed homomorphism* $\alpha\colon H \longrightarrow \Pi$*, the crossed centralizer* Π^α *is the intersection* $\Pi \cap N_\Gamma\Lambda_\alpha$.

Proof. Let $(\pi, g) \in \Pi \rtimes G$ and $h \in H$. Calculating in $\Gamma = \Pi \rtimes G$, we have

$$\begin{aligned}(\sigma, g)^{-1}(\alpha(h), h)(\sigma, g) &= (g^{-1} \cdot \sigma^{-1}, g^{-1})(\alpha(h), h)(\sigma, g)\\ &= ((g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h)), g^{-1}h)(\sigma, g)\\ &= ((g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h))((g^{-1}h) \cdot \sigma), g^{-1}hg).\end{aligned}$$

Therefore (σ, g) is in $N_\Gamma\Lambda_\alpha$ if and only if g is in N_GH and

$$\alpha(g^{-1}hg) = (g^{-1} \cdot \sigma^{-1})(g^{-1} \cdot \alpha(h))((g^{-1}h) \cdot \sigma)$$

for all $h \in H$. When $g = e$, so that $\sigma = (\sigma, e)$ is a typical element of $\Pi \cap N_\Gamma\Lambda_\alpha$, this simplifies to

$$\alpha(h) = \sigma^{-1}\alpha(h)(h \cdot \sigma). \quad \square$$

These lemmas allow us to restate Theorem 4.18 as follows.

Theorem 4.21. *Let* $\Gamma = \Pi \rtimes G$*. For a subgroup* H *of* G,

$$B(\Pi, \Gamma)^H \simeq \coprod B(\Pi \cap N_\Gamma\Lambda),$$

where the union runs over the Π*-conjugacy classes of subgroups* Λ *of* Γ *such that* $\Lambda \cap \Pi = e$ *and* $q(\Lambda) = H$.

Of course, we are only entitled to consider $B(\Pi; \Gamma)$ as a classifying space for principal Γ-bundles when Theorem 3.12 applies. The fixed point spaces $B(\Pi; \Gamma)^H$ of classifying spaces are studied in [7] when Γ is given by a not necessarily split extension of compact Lie groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{q} G \longrightarrow 1. \tag{4.22}$$

For such groups Γ, [7, Theorem 10] gives an entirely different bundle theoretic proof that the conclusion of Theorem 4.21 holds, exactly as stated. When [7] was written, no particularly nice model for the homotopy type $B(\Pi;\Gamma)$ was known.

5. The comparison between $B\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})$ and $\mathscr{U}(EG,B\Pi)$

For a general extension (4.22), with no restrictions on the given groups, a model $p\colon E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$ for a universal principal $(\Pi;\Gamma)$-bundle was later given in terms of mapping spaces [10]. Start with the classical models in §2.3 for universal principal Π, G, and Γ-bundles and let $Eq\colon E\Gamma \longrightarrow EG$ be the map induced by the quotient homomorphism $q\colon \Gamma \longrightarrow G$. Let $\mathrm{Sec}(EG,E\Gamma)$ be the Γ-space of sections $f\colon EG \longrightarrow E\Gamma$, so that $Eq\circ f = \mathrm{id}$. The following result is in [10, Theorem 5].

Theorem 5.1. *The quotient map* $p\colon \mathrm{Sec}(EG,E\Gamma) \longrightarrow \mathrm{Sec}(EG,E\Gamma)/\Pi$ *is a universal principal* $(\Pi;\Gamma)$*-bundle.*

Now let the extension be split, so that $\Gamma = \Pi \rtimes G$.[4] The given action of G induces a left action of G on $E\Pi$ that, together with the free right action by Π, makes it a Γ-space. Taking EG to be a left G-space and letting Γ act through q on EG, we have the product Γ-space $E\Pi\times EG$. It is free as a Γ-space because $E\Pi$ is free as a Π-space and EG is free as a G-space. Since it is obviously contractible, we may as well take $E\Gamma = E\Pi\times EG$. Since the second coordinate of a section $f\colon EG \longrightarrow E\Pi\times EG$ must be the identity, we then have

$$\mathrm{Sec}(EG,E\Gamma) = \mathscr{U}(EG,E\Pi).$$

Its Γ-action is defined just as was the Γ-action on $\mathscr{C}at_G(\tilde{G},\mathbf{\Pi})$ in Lemma 3.3. This gives the following specialization of Theorem 5.1, which is the space level forerunner of the categorical Theorem 3.11.

Theorem 5.2. *The quotient map* $p\colon \mathscr{U}(EG,E\Pi) \longrightarrow \mathscr{U}(EG,E\Pi)/\Pi$ *is a universal principal* $(\Pi;\Gamma)$*-bundle.*

We also have the mapping space $\mathscr{U}(EG,B\Pi)$. The canonical map $E\Pi \longrightarrow B\Pi$ induces a map $q\colon \mathscr{U}(EG,E\Pi) \longrightarrow \mathscr{U}(EG,B\Pi)$. Then there is an induced map ξ that makes the following diagram commute.

$$\begin{array}{ccc} & & \mathscr{U}(EG,E\Pi) \\ & \swarrow^{p} & \downarrow{\scriptstyle q} \\ \mathscr{U}(EG,E\Pi)/\Pi & \xrightarrow[\xi]{} & \mathscr{U}(EG,B\Pi). \end{array}$$

The analogy with the triangle in Theorem 2.7 should be evident. As was also observed in [10, Theorem 5], elementary covering space theory gives the following result, which is the space level forerunner of the categorical Theorem 3.12.

Theorem 5.3. *If* Π *is discrete, then* $\xi\colon \mathscr{U}(EG,E\Pi)/\Pi \longrightarrow \mathscr{U}(EG,B\Pi)$ *is a homeomorphism and therefore* $q\colon \mathscr{U}(EG,E\Pi) \longrightarrow \mathscr{U}(EG,B\Pi)$ *is a universal principal* $(\Pi;\Gamma)$*-bundle.*

---

[4]Ignoring group actions, the spaces $E\Gamma$ and $E\Pi\times EG$ are certainly homeomorphic since both are homeomorphic to the classifying space of the chaotic category with object space $\Pi\times G$.

There is an obvious comparison map relating the categorical and space level constructions. For any G-categories $\mathscr{A}$ and $\mathscr{B}$, we have the evaluation G-functor

$$\varepsilon\colon \mathscr{C}at_G(\mathscr{A},\mathscr{B}) \times \mathscr{A} \longrightarrow \mathscr{B}.$$

Applying the classifying space functor and taking adjoints, this gives a G-map

$$B\mathscr{C}at_G(\mathscr{A},\mathscr{B}) \longrightarrow \mathscr{U}(B\mathscr{A}, B\mathscr{B}).$$

(It factors through the evident simplicial mapping space between nerves.) Taking $\mathscr{A} = \tilde{G}$ and recalling that $E\Pi \longrightarrow B\Pi$ is obtained by applying B to the functor $\tilde{\Pi} \longrightarrow \Pi$, we obtain the following commutative diagram. Here we write $EG = B\tilde{G}$ and $E\Pi = B\tilde{\Pi}$, as we may.

$$\begin{array}{ccc}
B\mathscr{C}at_G(\tilde{G},\tilde{\Pi}) & \longrightarrow & \mathscr{U}(EG, E\Pi) \\
\downarrow & & \downarrow \\
B\mathscr{C}at_G(\tilde{G},\tilde{\Pi})/\Pi & \longrightarrow & \mathscr{U}(EG, E\Pi)/\Pi \\
\downarrow & & \downarrow \\
B\mathscr{C}at_G(\tilde{G},\mathbf{\Pi}) & \longrightarrow & \mathscr{U}(EG, B\Pi)
\end{array}$$

Theorems 3.11 and 5.2 say that that the top two vertical arrows are often universal principal $(\Pi;\Gamma)$-bundles, in which case the top two horizontal arrows are equivalences. Theorems 3.12 and 5.3 say that the lower two vertical arrows and therefore also the bottom horizontal arrow are also often equivalences. More elaborate arguments might prove these results in greater generality.

References

[1] E. Babson and D.N. Kozlov. Group actions on posets. Journal of Algebra 285(2005), 439–450.
[2] T. tom Dieck. Faserbündel mit Gruppenoperation. Arch. Math. (Basel) 20 1969 136143.
[3] B. Guillou and J.P. May. Enriched model categories and diagram categories. Preprint, 2011.
[4] B. Guillou and J.P. May. Models of G-spectra as presheaves of spectra. Preprint, 2011.
[5] B. Guillou and J.P. May. Permutative G-categories and equivariant infinite loop space theory. Preprint, 2011.
[6] R.K. Lashof. Equivariant bundles. Ill. J. Math. 26(1982), 257–271.
[7] R.K. Lashof and J.P. May. Generalized equivariant bundles. Bulletin Belgian Math. Soc. 38 (l986), 265–271.
[8] J.P. May. The geometry of iterated loop spaces. Lecture Notes in Mathematics Vol. 271. Springer-Verlag 1972.
[9] J.P. May. Classifying spaces and fibrations. Memoirs Amer. Math. Soc. No. 155, 1975.
[10] J.P. May. Some remarks on equivariant bundles and classifying spaces. Astérisque 191 (1990), 239–253.
[11] J.P. May, et al. Equivariant homotopy and cohomology theory. NSF-CBMS Regional Conference Series in Mathematics No. 91. Amer. Math. Soc. 1996.
[12] M. Merling. The equivariant $+ = Q$ theorem. In preparation.
[13] M. Murayama and Kazuhisa Shimakawa. Universal equivariant bundles. Proc. Amer. Math. Soc. 123 (1995), 1289–1295.
[14] V. M. Usenko. Subgroups of semidirect products. Ukrainian Mathematical Journal, Vol. 43 (1991), Numbers 7-8, 982-988.

Department of Mathematics, The University of Illinois at Urbana-Champaign, IL 61801
E-mail address: bertg@illinois.edu

Department of Mathematics, The University of Chicago, Chicago, IL 60637
E-mail address: may@math.uchicago.edu

Department of Mathematics, The University of Chicago, Chicago, IL 60637
E-mail address: mmerling@math.uchicago.edu