

# BUILDING THE BEILINSON MOTIVE AS A CELL MODULE

BERTRAND GUILLOU

## 0. INTRODUCTION

Let  $X = \mathbb{P}_k^1 - \{0, 1, \infty\}$ . It is known ([D],[DG]), at least if  $k$  is a number field, how to consider  $\mathbb{Q}[\pi_1(X, x)]/I^{n+1}$  as a mixed Tate motive, for  $x \in X(k)$  and  $n \geq 0$  and where  $I$  denotes the augmentation ideal. We follow ([KM], §IV) in defining rational mixed Tate motives to be a certain subcategory of the derived category  $\mathcal{D}_{\mathcal{A}}$  of the Bloch cycle dga  $\mathcal{A} = \mathcal{A}(\text{Spec } k)$ . Recall that one model for the derived category of a commutative dga is given by the homotopy category of cell (=semi-free) modules.

Actually, it will be more convenient to consider homotopy classes of paths rather than loops. Thus let  $a, b \in X(k)$ . Let  ${}_bP_a$  denote the set of homotopy classes of paths from  $a$  to  $b$  and note that it is a  $\pi(X, b) - \pi_1(X, a)$ -bimodule. Deligne and Goncharov ([DG], §3) show that (the dual of)  $\mathbb{Q}[{}_bP_a]/I^{n+1}$  is given by

$$\mathcal{A}(X^n) \rightarrow \mathcal{A}(a \times X^{n-1}) \oplus \mathcal{A}(\Delta_X \times X^{n-2}) \oplus \cdots \oplus \mathcal{A}(X^{n-1} \times b) \rightarrow \cdots \rightarrow \mathcal{A}^{\oplus n+1}.$$

The maps in this complex are given by restriction with the appropriate sign. We will be interested in producing an explicit cellular approximation to this complex.

The first three sections serve as background and fix some notations. In §4, we begin to consider the problem in the case  $n = 2$ . Here we see that the Steinberg relation in motivic cohomology allows one to build the cell module. This study is continued in §6, where we consider the case  $n = 3$ . Already here we see that a higher Steinberg relation is required to hold—namely, certain Massey products must contain zero. For this reason, we briefly review Massey products in §5. Finally, we state and prove the general answer in §7; the main result is that the obstruction to building the cell module is given by certain  $n$ -fold Massey products in the motivic cohomology of  $X$ . Strictly speaking, sections 4 and 6 are not necessary in order to understand the results in §7, though the reader may find it helpful to have these lower dimensional examples worked out explicitly. At the moment, the question of the vanishing of relevant Massey products in  $H^{*,*}(\mathbb{P}^1 - \{0, 1, \infty\})$  is not addressed.

We are very grateful to Spencer Bloch for bringing this question to our attention and for several helpful discussions. This work owes much to the letter [B1]; indeed, much of section 4 comes directly from that source. We also thank Peter May for much guidance and support.

## 1. BLOCH'S CYCLE COMPLEX

Recall the algebraic standard cosimplicial scheme  $\Delta^\bullet$ : we define  $\Delta^n \subseteq \mathbb{A}^{n+1}$  to be the closed subvariety defined by

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{A}^{n+1} \mid \sum_i x_i = 1\}.$$

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The usual formulae for the face maps and degeneracy maps from topology make  $\Delta^\bullet$  into a cosimplicial scheme. Also, note that  $\Delta^n \cong \mathbb{A}^n$ .

**Definition 1.1.** For  $Y$  a quasi-projective variety over a field  $k$ , **Bloch's cycle complex** is defined as follows. Let  $z^q(Y, n)$  be the free abelian group on the codimension  $q$  subvarieties of  $Y \times \Delta^n$  which meet all faces

$$Y \times \Delta^m \subseteq Y \times \Delta^n$$

properly, meaning that the intersection is either empty or of codimension  $q$ . For fixed  $q$ , the groups  $z^q(Y, \bullet)$  form a simplicial abelian group and thus a chain complex in the usual way. Writing

$$N^p(Y)(q) = z^q(Y, 2q - p),$$

this makes  $N^*(Y)(q)$  into a cochain complex. The cohomology groups of this cochain complex, denoted  $\text{CH}^q(Y, p)$ , are called the **higher Chow groups** of  $Y$  of codimension  $q$ .

**1.1. Cubical variant.** For the purpose of defining products, it is more convenient to work with a cubical variant. Let  $\square^n = (\mathbb{A}^1)^n \cong \mathbb{A}^n$ . For  $1 \leq j \leq n$  and  $\epsilon \in \{0, 1\}$ , there is an inclusion  $i_{j, \epsilon} : \square^{n-1} \hookrightarrow \square^n$  given by inserting  $\epsilon$  in the  $j$ th coordinate. In addition, for each  $1 \leq j \leq n - 1$ , there is a projection  $\pi_j : \square^n \rightarrow \square^{n-1}$  which omits the  $j$ th coordinate.

**Definition 1.2.** Let  $\tilde{N}^p(Y)^c(q)$  be the free abelian group on the codimension  $q$  subvarieties of  $Y \times \square^{2q-p}$  which meet all faces

$$Y \times \square^m \subseteq Y \times \square^n$$

properly. Let

$$D^p(Y)(q) \subseteq \tilde{N}^p(Y)^c(q)$$

be the subgroup of “degenerate cycles”, i.e., the sum of the images of the projections  $\pi_j$ . Then we define the **cubical Bloch complex** to be

$$N^p(Y)^c(q) = \tilde{N}^p(Y)^c(q) / D^p(Y)(q).$$

The following proposition says that the cubical cycle complexes also compute the higher Chow groups.

**Proposition 1.3** ([L1], Thm. 4.7; [BK], Prop. 5.1). *There is a canonical quasi-isomorphism*

$$N^p(Y)^c(q) \xrightarrow{\sim} N^p(Y)(q).$$

**1.2. Products.** If  $W \subseteq X \times \square^m$  and  $Z \subseteq Y \times \square^n$  meet all faces properly, then the product

$$W \times Z \subseteq X \times \square^m \times Y \times \square^n \cong X \times Y \times \square^{m+n}$$

also meets all faces properly. Thus we get a product

$$N^m(X)^c(p) \otimes N^n(Y)^c(q) \rightarrow N^{m+n}(X \times Y)^c(p+q)$$

inducing a product

$$\text{CH}^p(X, m) \otimes \text{CH}^q(Y, n) \rightarrow \text{CH}^{p+q}(X \times Y, m+n).$$

Although the cycle complexes only have contravariant functoriality with respect to *flat* maps, the higher Chow groups nevertheless have contravariant functoriality with respect to *all* morphisms when the codomain is smooth ([B2], Thm. 4.1).

Thus, if  $X$  is smooth, one can pull back along the diagonal  $X \hookrightarrow X \times X$  to make  $\mathrm{CH}^*(X, *)$  into a bigraded ring. Moreover, this product is graded commutative ([B2], Cor. 5.7) with respect to the homological, or cubical, grading (as opposed to the grading by codimension).

**1.3. Alternating cycle complex.** When  $Y = \mathrm{Spec} k$ , the above cubical complex  $N^p(Y)^c(q)$  yields a homotopy-commutative dga. After tensoring with  $\mathbb{Q}$ , one can produce an honest cdga by restricting to *alternating cycles*, as we explain below.

Note that the symmetric group  $\Sigma_n$  acts on  $\square^n$  and therefore on  $N^p(Y)^c(q)$ . One can then consider the subgroup of cycles which are alternating with respect to this action, that is, the cycles  $Z$  such that

$$\sigma(Z) = \mathrm{sgn}(\sigma) \cdot Z$$

for all  $\sigma \in \Sigma_n$ . In fact, one can show ([B3], Lemma 1.1) that the alternating cycles are closed under the differential, so that the alternating cycles form a subcomplex. However, the product of alternating cycles need no longer be alternating. One can rectify this, by projecting onto the subspace of alternating cycles, but it is necessary to pass to  $\mathbb{Q}$  coefficients. Let  $\mathrm{alt}_n \in \mathbb{Q}[\Sigma_n]$  be the element

$$\mathrm{alt}_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \cdot \sigma.$$

Then, letting

$$A^p(Y)(q) \subseteq N^p(Y)^c(q) \otimes \mathbb{Q}$$

be the subspace of alternating cycles, multiplication with  $\mathrm{alt}_{2q-p}$  gives a projection

$$\mathrm{alt}_{2q-p} : N^p(Y)^c(q) \otimes \mathbb{Q} \rightarrow A^p(Y)(q).$$

We now define a product by the composition

$$A^m(X)(p) \otimes A^n(Y)(q) \rightarrow N^{m+n}(X \times Y)(p+q) \xrightarrow{\mathrm{alt}} A^{m+n}(X \times Y)(p+q).$$

When  $X = Y = \mathrm{Spec} k$ , this makes  $A^*(\mathrm{Spec} k)(*)$  into a commutative dga by construction.

## 2. FRIEDLANDER-SUSLIN-VOEVODSKY VARIANT

As we have already mentioned, the higher Chow groups have good functoriality for maps between smooth schemes, but the cycle complexes we have introduced do not. In this section, we discuss cycle complexes of Friedlander-Suslin-Voevodsky with good functoriality.

**Definition 2.1.** Given smooth schemes  $X$  and  $Y$  Let  $z_{\mathrm{equi}}(Y, r)(X)$  denote the free abelian group on the closed and irreducible subvarieties  $Z \subseteq X \times Y$  which are dominant and equidimensional of relative dimension  $r$  over  $X$ .

As is discussed in ([MVW], 16.1), for fixed  $Y$  and  $r$  this defines a presheaf in  $X$ .

**Definition 2.2.** We will write

$$\mathcal{Z}^p(Y)(q) = C_{2q-p}(z_{\mathrm{equi}}(\mathbb{A}^q, 0))(Y).$$

Again, this is the free abelian group on the (codimension  $q$ ) closed, irreducible subvarieties of  $\Delta^{2q-p} \times Y \times \mathbb{A}^q$  which are quasifinite and dominant over  $\Delta^{2q-p} \times Y$ . For fixed  $q$ , this defines a cochain complex.

Since  $C_{2q-p}(z_{\text{equi}}(\mathbb{A}^q, 0))$  determines a presheaf on the category of smooth  $k$ -schemes  $\text{Sm}_k$  (and in fact an étale sheaf), we get functoriality of these complexes with respect to all maps between smooth schemes.

Note that any codimension  $q$  cycle  $Z \subseteq \Delta^{2q-p} \times Y \times \mathbb{A}^q$  which is quasifinite over  $\Delta^{2q-p} \times Y$  must meet the faces  $\Delta^m \times Y \times \mathbb{A}^q$  properly, so we have an inclusion

$$\mathcal{Z}^p(Y)(q) \hookrightarrow N^p(Y \times \mathbb{A}^q)(q).$$

Moreover, homotopy invariance of the higher Chow groups ([B2], Thm. 2.1) says that the pullback along the projection  $Y \times \mathbb{A}^q \rightarrow Y$  induces a quasi-isomorphism

$$N^p(Y)(q) \xrightarrow{\sim} N^p(Y \times \mathbb{A}^q)(q).$$

Thus, in order to show that our new complexes compute the higher Chow groups, it remains to show that the inclusion  $\mathcal{Z}(Y) \hookrightarrow N(Y \times \mathbb{A}^q)$  is a quasi-isomorphism. For  $Y = \text{Spec } k$ , this is given by the following theorem of Suslin.

**Theorem 2.3** ([S], Thm. 2.1). *The inclusion*

$$\mathcal{Z}^*(\text{Spec } k)(q) \hookrightarrow N^*(\mathbb{A}^q)(q)$$

*is a quasi-isomorphism.*

Note that, generalizing what we discussed above, there is an inclusion of groups

$$C_{2q-p}(z_{\text{equi}}(Y \times \mathbb{A}^q, r))(\text{Spec } k) \hookrightarrow N^{p-2\dim Y+2r}(Y \times \mathbb{A}^q)(q + \dim Y - r)$$

Specializing to  $r = \dim Y$ , this gives

$$C_{2q-p}(z_{\text{equi}}(Y \times \mathbb{A}^q, \dim Y))(\text{Spec } k) \hookrightarrow N^p(Y \times \mathbb{A}^q)(q)$$

Assuming resolution of singularities, Suslin also proved

**Theorem 2.4** ([S], Thm. 3.2). *If  $k$  satisfies resolution of singularities then for any equidimensional quasiprojective scheme  $Y$ , the natural inclusion*

$$C_{2q-p}(z_{\text{equi}}(Y \times \mathbb{A}^q, \dim Y))(\text{Spec } k) \hookrightarrow N^p(Y \times \mathbb{A}^q)(q)$$

*is a quasi-isomorphism.*

The duality theorem of Friedlander and Voevodsky reads

**Theorem 2.5** ([FV], Thm. 7.4). *If  $k$  satisfies resolution of singularities,  $Y$  is a smooth quasiprojective  $k$ -scheme, and  $X$  is a  $k$ -scheme of finite type, then for any  $r$  the inclusion*

$$z_{\text{equi}}(X, r)(Y) \hookrightarrow z_{\text{equi}}(Y \times X, r + \dim Y)(\text{Spec } k)$$

*induces a quasi-isomorphism*

$$C_*(z_{\text{equi}}(X, r))(Y) \xrightarrow{\sim} C_*(z_{\text{equi}}(Y \times X, r + \dim Y))(\text{Spec } k).$$

Combining the previous two theorems gives the desired quasi-isomorphism

$$\mathcal{Z}^p(Y)(q) \xrightarrow{\sim} N^p(Y \times \mathbb{A}^q)(q).$$

**2.1. Products.** Now that we have our nice functorial cycle complexes, we want to modify them to yield cdga's.

As above, the first step is to pass from simplicial complexes to cubical complexes. Again ([L2], Lemma 5.26.1), one can show that the cubical complexes  $\mathcal{Z}^*(Y)^{c(*)}$  are quasi-isomorphic to the simplicial ones.

At this point, we have functorial dga's computing the higher Chow groups (assuming resolution of singularities). It only remains to make the dga's commutative. To clarify, the products we are considering are

$$\mathcal{Z}^m(Y)^c(p) \otimes \mathcal{Z}^n(Y)^c(q) \rightarrow \mathcal{Z}^{m+n}(Y \times Y)^c(p+q) \xrightarrow{\Delta^*} \mathcal{Z}^{m+n}(Y)^c(p+q).$$

Recall that  $\mathcal{Z}^m(Y)^c(p)$  is a subgroup of the group of cycles on  $\square^{2p-m} \times Y \times \mathbb{A}^p$ . There are thus two sources of noncommutativity: the box coordinate  $m$  and the affine coordinate  $p$ . As above, we tensor our dga's with  $\mathbb{Q}$  and consider the subspaces of cycles which are alternating with respect to permutation of the box coordinates. It can be shown ([L2], Lemma 5.26.2) that the result is quasi-isomorphic.

However, this does not yet produce a cdga, since we have the extra affine coordinates. If we now restrict to the cycles which are *invariant* under permutation of the affine coordinates, then we finally obtain a cdga. Once again, this restriction does not change the quasi-isomorphism type ([L2], Lemma 5.27) (this essentially follows from homotopy invariance of the higher Chow groups). We will denote our resulting presheaf of cdga's by  $\mathcal{A}^p(-)(q)$ .

### 3. THE MAP $k^* \rightarrow \mathrm{CH}^1(\mathrm{Spec} k, 1)$

It is known ([NS], [T]) that

$$\mathrm{CH}^n(\mathrm{Spec} k, n) \cong K_n^M(k),$$

where the groups on the right are the Milnor  $K$ -groups. In particular, this says that

$$\mathrm{CH}^1(\mathrm{Spec} k, 1) \cong k^*,$$

the units in the field (this appeared already in [B2]), and that  $\mathrm{CH}^2(\mathrm{Spec} k, 2)$  is the quotient of  $k^* \otimes k^*$  by the subgroup generated by elements of the form  $a \otimes (1 - a)$ , with  $a \in k - \{0, 1\}$  (these relations are known as the Steinberg relations).

We will be interested in producing an explicit map

$$\mathcal{O}^*(\mathrm{Spec} k) = k^* \rightarrow \mathcal{A}^1(\mathrm{Spec} k)(1)$$

as well as finding explicit cycles in  $\mathcal{A}^1(\mathrm{Spec} k)(2)$  realizing the Steinberg relations. This, and much more, was done in [T] for the cubical Bloch complex. In fact, it suffices to work with  $\mathcal{Z}^*(\mathrm{Spec} k)^{c(*)}$  since we have an explicit projection  $\mathcal{Z} \otimes \mathbb{Q} \rightarrow \mathcal{A}$ .

The map  $k^* \rightarrow N^1(\mathrm{Spec} k)^{c(1)}$  is defined by

$$a \mapsto C_a := Z(t(a-1) - a) \subseteq \square^1.$$

In other words,  $C_a$  is merely the point  $\frac{a}{a-1} \in \square^1 = \mathbb{A}^1$ . To get a codimension 1 cycle in  $\square^1 \times \mathbb{A}^1$ , we pullback along the projection  $\square^1 \times \mathbb{A}^1 \rightarrow \square^1$ , yielding the cycle  $C_a \times \mathbb{A}^1 \subseteq \square^1 \times \mathbb{A}^1$ . Note, however, that  $C_a \times \mathbb{A}^1 \notin \mathcal{Z}^1(k)^{c(1)}$  since this cycle is not quasifinite and dominant over  $\square^1$ .

**Remark 3.1.** When  $a = 1$ , we have

$$C_a = Z(t(1-1) - 1) = Z(-1) = \emptyset \subseteq \square^1,$$

so that  $[C_1] = 0 \in \mathrm{CH}^1(\mathrm{Spec} k)$ .

Recall that the maps

$$\mathcal{Z}^1(k)^c(1) \hookrightarrow N^1(\mathbb{A}^1)^c(1) \xleftarrow{\pi^*} N^1(k)^c(1)$$

are quasi-iso's. We are looking for a cycle in  $\mathcal{Z}^1(k)^c(1)$  which maps to the class of  $C_a \times \mathbb{A}^1$ .

**Definition 3.2.** Denoting the coordinate on  $\square^1$  by  $x$  and the coordinate on  $\mathbb{A}^1$  by  $t$ , the cycle

$$\Gamma_a := Z(tx(x-1) - (x(a-1) - a)) \subseteq \square^1 \times \mathbb{A}^1$$

is quasi-finite and dominant over  $\square^1$ . This defines a map  $\Gamma_{(-)} : k^* \rightarrow \mathcal{Z}^1(k)^c(1)$ .

The cycle  $\Gamma_a$  is the graph of the rational function

$$\varphi(x) = \frac{x(a-1) - a}{x(x-1)} = (a-1) \frac{x - \frac{a}{a-1}}{x(x-1)}.$$

Moreover, we can see that it maps to the class of  $C_a \times \mathbb{A}^1$ , as an explicit homotopy  $C_a \times \mathbb{A}^1 \simeq \Gamma_a$  is given by

$$Z(tx(x-1)h - (x(1-a) - a)) \subseteq \square^2 \times \mathbb{A}^1.$$

Here we are considering  $x$  and  $h$  as coordinates on  $\square^2$ .

It remains to find an explicit cochain which bounds the cochain  $\Gamma_a \cdot \Gamma_{1-a}$ . We do not know of an explicit such cycle.

In fact, one has more generally an isomorphism

$$\mathrm{CH}^1(Y, 1) \cong \mathcal{O}^*(Y).$$

**Definition 3.3.** We define a map  $\mathcal{O}^*(Y) \xrightarrow{\Gamma} \mathcal{Z}^1(Y)^c(1)$  by

$$f \mapsto \Gamma_f := Z(tx(x-1) - [x(f(y)-1) - f(y)]).$$

As above one can show, by use of an explicit homotopy, that this is compatible with the map  $\mathcal{O}^*(Y) \rightarrow N^1(Y)^c(1)$ . We will hereafter write  $[f] = \Gamma_f$ , as the notation is less cumbersome.

Though we do not give here an explicit construction of a cochain bounding  $[f] \cdot [1-f]$ , let us fix a particular choice of  $T_t \in \mathcal{A}^1(\mathbb{G}_m - \{1\})(2)$  such that  $d(T_t) = [t] \cdot [1-t]$ , where  $t$  is the parameter on  $\mathbb{G}_m - \{1\}$ . Note since  $[1-t] \cdot [t] = -[t] \cdot [1-t]$ , we can choose  $T_{1-t} := -T_t$ . Note furthermore that this choice of universal Totaro cycle  $T_t$  gives us Totaro cycles in general. For instance, an element  $a \in k - \{0, 1\}$ , corresponds to a map  $\mathrm{Spec} k \xrightarrow{\varphi_a} \mathbb{G}_m - \{1\}$ , and one can then define

$$T_a := \varphi_a^*(T_t).$$

#### 4. TOWARDS THE BEILINSON MOTIVE, $n = 2$

We assume the Beilinson-Soulé vanishing conjecture, so that

$$\mathcal{A}^* = \mathcal{A}^*(\mathrm{Spec} k)(*)$$

is a cohomologically connected (Adams-graded) cdga.

Let  $X = \mathbb{P}^1 - \{0, 1, \infty\}$ , and let  $a \neq b \in X(k)$ . We will be interested in building a cell  $\mathcal{A}$ -module representing the ‘‘Beilinson complex’’

$$\begin{aligned} \mathcal{A}(X \times X) &\xrightarrow{\beta} \mathcal{A}(\{a\} \times X) \oplus \mathcal{A}(\Delta_X) \oplus \mathcal{A}(X \times \{b\}) \\ &\xrightarrow{\beta'} \mathcal{A}((a, a)) \oplus \mathcal{A}((a, b)) \oplus \mathcal{A}((b, b)). \end{aligned}$$

Here the first map is given by the restriction along the inclusions, where restriction onto the diagonal is taken with a minus sign. The second map is given by restriction to  $a$  minus restriction to  $b$ .

We will write  $\mathcal{A}\{n\}$  for  $\mathcal{A}\langle n\rangle[n]$ , the complex obtained by shifting both the Adams grading and the homological grading by  $n$ . Recall that there is a quasi-isomorphism  $\mathcal{A} \oplus \mathcal{A}\{-1\} \simeq \mathcal{A}(\mathbb{G}_m)$ . If  $\pi : \mathbb{G}_m \rightarrow \text{Spec}(k)$  is the structure morphism and if  $t$  is the standard parameter on  $\mathbb{G}_m$ , this map is given by

$$(x, y) \mapsto \pi^*(x) + \pi^*(y) \cdot [t].$$

Together with the Mayer-Vietoris sequence associated to the covering

$$(\mathbb{A}^1 - \{0\}) \cup (\mathbb{A}^1 - \{1\}) = \mathbb{A}^1$$

(note that  $(\mathbb{A}^1 - \{0\}) \cap (\mathbb{A}^1 - \{1\}) = X$ ), this gives a quasi-isomorphism

$$\mathcal{A} \oplus \mathcal{A}\{-1\} \oplus \mathcal{A}\{-1\} \rightarrow \mathcal{A}(X).$$

Explicitly, this is

$$(x, y, z) \mapsto \pi^*(x) + \pi^*(y) \cdot [t] + \pi^*(z) \cdot [1 - t].$$

Similarly, one gets a quasi-isomorphism

$$\mathcal{A} \oplus \mathcal{A}\{-1\}^4 \oplus \mathcal{A}\{-2\}^4 \rightarrow \mathcal{A}(X \times X)$$

given explicitly by

$$\begin{aligned} (x, y_1, \dots, y_4, z_1, \dots, z_4) \mapsto & \pi^*x + \pi^*y_1 \cdot [U] + \pi^*y_2 \cdot [1 - U] \\ & + \pi^*y_3 \cdot [V] + \pi^*y_4 \cdot [1 - V] \\ & + \pi^*z_1 \cdot [U] \cdot [V] + \pi^*z_2 \cdot [1 - U] \cdot [V] \\ & + \pi^*z_3 \cdot [U] \cdot [1 - V] + \pi^*z_4 \cdot [1 - U] \cdot [1 - V], \end{aligned}$$

where  $U$  is the parameter on the first factor and  $V$  is the parameter on the second factor. Keeping the above quasi-isomorphisms in mind, we will label the generators of the various summands by  $[U]$ ,  $[1 - U] \cdot [V]$ , etc.

Using these quasi-isomorphic free  $\mathcal{A}$ -modules, we attempt to build a complex

$$\mathcal{A} \oplus \mathcal{A}\{-1\}^4 \oplus \mathcal{A}\{-2\}^4 \xrightarrow{\alpha} \mathcal{A}^3 \oplus \mathcal{A}\{-1\}^6 \xrightarrow{\alpha'} \mathcal{A}^3.$$

There is no trouble defining  $\alpha'$ ; this is defined to be restriction to  $a$  minus restriction to  $b$  as above. Similarly, there is no trouble defining  $\alpha$  on the  $\mathcal{A}$  and  $\mathcal{A}\{-1\}$  summands. The trouble comes when trying to define  $\alpha$  on the  $\mathcal{A}\{-2\}$  summands. Writing  $W$  for the parameter on the diagonal  $\Delta_X$ , one wants to set

$$\alpha([U] \cdot [1 - V]) = [a] \cdot [1 - V] - [W] \cdot [1 - W] + [U] \cdot [1 - b],$$

but of course we can't since  $[W]$  is not an element of  $(\mathcal{A}\{-1\})^2(2) = \mathcal{A}^1(1)$ . Instead, we have no choice but to define

$$\alpha([U] \cdot [1 - V]) = [a] \cdot [1 - V] - [1 - b] \cdot [U]$$

(we have switched the order of  $[U]$  and  $[1 - b]$  here, introducing a sign, because  $[U]$  denotes the generator of the left  $\mathcal{A}$ -module  $\mathcal{A}\{-1\}$ ). We define  $\alpha$  similarly on the other  $\mathcal{A}\{-2\}$  summands.

Unfortunately, the forced definition of  $\alpha$  does not yield a complex. As expected, we have  $\alpha' \circ \alpha = 0$  on the  $\mathcal{A}$  and  $\mathcal{A}\{-1\}$  summands. On the  $\mathcal{A}\{-2\}$  summands we have

$$\alpha' \circ \alpha([U] \cdot [V]) = ([a]^2, 0, -[b]^2)$$

and

$$\alpha' \circ \alpha([1 - U] \cdot [1 - V]) = ([1 - a]^2, 0, -[1 - b]^2).$$

These both vanish by graded-commutativity (we are working over  $\mathbb{Q}$ ). On the other hand,

$$\alpha' \circ \alpha([1 - U] \cdot [V]) = ([1 - a] \cdot [a], 0, -[1 - b] \cdot [b])$$

and

$$\alpha' \circ \alpha([U] \cdot [1 - V]) = ([a] \cdot [1 - a], 0, -[b] \cdot [1 - b]).$$

These do not vanish, but they are coboundaries (since  $CH^2(k, 2) \cong K_2^M(k)$ ). Choices of bounding cycles allow one to define a contracting homotopy  $h : \alpha' \circ \alpha \simeq 0$ .

Let us suppose for a moment that we have chosen appropriate bounding cycles and defined a contracting homotopy. This allows us to build a cell module out of our almost-complex. We must now compare this cell module to the original complex. At this point, we have the (not commutative!) solid arrow diagram

$$\begin{array}{ccccc} \mathcal{A}(X \times X) & \xrightarrow{\beta} & \mathcal{A}({}_a X) \oplus \mathcal{A}(\Delta_X) \oplus \mathcal{A}(X_b) & \xrightarrow{\beta'} & \mathcal{A}_{a,a} \oplus \mathcal{A}_{a,b} \oplus \mathcal{A}_{b,b} \\ \uparrow f & \nearrow H & \uparrow f' & & \parallel \\ \mathcal{A} \oplus \mathcal{A}\{-1\}^4 \oplus \mathcal{A}\{-2\}^4 & \xrightarrow{\alpha} & \mathcal{A}^3 \oplus \mathcal{A}\{-1\}^6 & \xrightarrow{\alpha'} & \mathcal{A}^3 \\ & \searrow h & & & \end{array}$$

where, for instance,  $\mathcal{A}_{a,a}$  denotes  $\mathcal{A}(a, a) \cong \mathcal{A}$ . The right-hand square commutes, but the  $\mathcal{A}\{-2\}$  summands cause the left-hand square to only commute up to homotopy. The composites  $\beta \circ f$  and  $f' \circ \alpha$  differ by  $[W] \cdot [1 - W]$  on the summand  $\mathcal{A}\{-2\}[U] \cdot [1 - V]$  and by  $[1 - W] \cdot [W]$  on the summand  $\mathcal{A}\{-2\}[1 - U] \cdot [V]$ . Defining a commuting homotopy  $H : f' \circ \alpha \simeq \beta \circ f$  thus amounts to finding a bounding cycle for  $[W] \cdot [1 - W]$  and for  $[1 - W] \cdot [W]$ .

Suppose that we have chosen such a homotopy  $H$ . To compare our cell module to the original complex, it suffices to find a homotopy  $\Theta : h \simeq \beta' \circ H$ . But note that given any  $H$ , we can simply define  $h$  to be  $\beta' \circ H$ . Thus  $\Theta$  is not necessary, and it remains only to construct  $H$  (and therefore  $h$ ) explicitly.

We know that a choice of  $H$  exists, since the Steinberg relation holds, though we do not know of an explicit choice. Nevertheless, any such choice now produces for us a cell module, together with a quasi-isomorphism to Beilinson's complex. Thus any two choices will produce homotopy equivalent cell modules.

We have seen in this section that to build the Beilinson motive in the case  $n = 2$ , it suffices that the relations  $[t] \cdot [1 - t] = 0$  hold in  $\text{CH}^2(\mathbb{G}_m - \{1\}, 2)$  (this is the universal case). We will generalize this to the  $n = 3$  case in §6, and we give the general result in §7.

## 5. MASSEY PRODUCTS

In this section, we briefly discuss Massey products (see, e.g., [M]), as these will be needed in the following sections.

Throughout this section,  $C^*$  will be a dga. Given a cochain  $c \in C^n$ , we let  $\bar{c}$  denote  $(-1)^{n-1}c$ . This will be convenient later.

Suppose given  $\alpha_1, \alpha_2, \alpha_3 \in H^*(C)$  such that  $\alpha_1 \alpha_2 = 0$  and  $\alpha_2 \alpha_3 = 0$ . Choosing representing cocycles  $a_1, a_2, a_3 \in C^*$ , so that  $[a_i] = \alpha_i$ , this means that there are

cochains  $a_{12}, a_{23} \in C^*$  such that  $d(a_{12}) = a_1 a_2$  and  $d(a_{23}) = a_2 a_3$ . Actually, it is more convenient to choose  $a'_{12}$  and  $a'_{23}$  so that  $d(a'_{12}) = \overline{a_1} a_2$  and  $d(a'_{23}) = \overline{a_2} a_3$ . Then note that that cochain

$$\overline{a'_{12}} a_3 + \overline{a_1} a'_{23}$$

is in fact a cocycle. We denote the cohomology class it represents by  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  and refer to it as the **triple Massey product**.

Unfortunately, the above class is not determined by  $\alpha_1, \alpha_2$ , and  $\alpha_3$ . The cohomology class is sensitive to the choices of  $a'_{12}$  and  $a'_{23}$ ; adding a cocycle to either of these cochains yields a new cohomology class which has equal right to be called the triple Massey product. Thus the triple Massey product  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is not an element of  $H^*(C)$  but rather a coset of  $\alpha_1 H^*(C) + H^*(C) \alpha_3$  in  $H^*(C)$ . The subgroup  $\alpha_1 H^*(C) + H^*(C) \alpha_3 \subseteq H^*(C)$  is called the **indeterminacy** of this Massey product.

We will need more generally the notion of an  $n$ -fold Massey product, which is defined inductively. Suppose given  $\alpha_1, \dots, \alpha_n \in H^*(C)$  with choices of representing cocycles  $a_i \in C^*$ ,  $[a_i] = \alpha_i$ . Suppose moreover that we have defined the notion of 3-fold, 4-fold,  $\dots$ , and  $n-1$ -fold Massey products. Finally, suppose that all of the Massey products  $\langle \alpha_i, \dots, \alpha_j \rangle$  are defined and contain zero for  $j-i < n-1$ . That these Massey products contain zero means that we have bounding cochains  $a_{i, \dots, j}$  for  $\langle \alpha_i, \dots, \alpha_j \rangle$ . Then the cochain

$$\sum_{i=1}^{n-1} \overline{a_{1, \dots, i}} a_{i+1, n}$$

is a cocycle, and we denote the resulting cohomology class by  $\langle \alpha_1, \dots, \alpha_n \rangle$ . Again, there is indeterminacy involved.

The Massey products that will arise later will be iterated Massey products of the elements  $[t]$  and  $[1-t]$  in  $H(\mathcal{A}(\mathbb{G}_m - \{1\})) \cong \text{CH}^*(\mathbb{G}_m - \{1\}, *)$ . In fact, we will see that the necessary (and sufficient) condition for building a cellular approximation to the Beilinson complex will be that these iterated Massey products are defined and contain zero. As we have already used the notation  $T_t$  for a cochain bounding the Massey product  $\langle [t], [1-t] \rangle$ , we will use the notation  $T_{x_1, \dots, x_n}$  for a cochain bounding the Massey product  $\langle x_1, \dots, x_n \rangle$ , and we will refer to these as generalized Totaro cycles (do not confuse cycle with cocycle here!).

## 6. THE $n = 3$ CASE

We now look at the next simplest case, when  $n = 3$ . We are interested in obtaining a cellular approximation to the complex

$$\begin{aligned} \mathcal{A}(X^3) &\xrightarrow{\beta_1} \mathcal{A}(\{a\} \times X^2) \oplus \mathcal{A}(\Delta_X \times X) \oplus \mathcal{A}(X \times \Delta_X) \oplus \mathcal{A}(X^2 \times \{b\}) \\ &\xrightarrow{\beta_2} \mathcal{A}(\{a, a\} \times X) \oplus \mathcal{A}(\{a\} \times \Delta_X) \oplus \mathcal{A}(\Delta_{X^3}) \\ &\quad \oplus \mathcal{A}(\Delta_X \times \{b\}) \oplus \mathcal{A}(X \times \{b, b\}) \oplus \mathcal{A}(\{a\} \times X \times \{b\}) \\ &\xrightarrow{\beta_3} \mathcal{A}(a, a, a) \oplus \mathcal{A}(a, a, b) \oplus \mathcal{A}(a, b, b) \oplus \mathcal{A}(b, b, b) \end{aligned}$$

A Mayer-Vietoris argument again gives us a cellular approximation

$$\mathcal{A} \oplus \mathcal{A}\{-1\}^6 \oplus \mathcal{A}\{-2\}^{12} \oplus \mathcal{A}\{-3\}^8 \xrightarrow{\sim} \mathcal{A}(X^3).$$

A cellular approximation to the above complex will then consist of maps as in the diagram

$$(1) \quad \begin{array}{ccc} \mathcal{A} \oplus \mathcal{A}\{-1\}^6 \oplus \mathcal{A}\{-2\}^{12} \oplus \mathcal{A}\{-3\}^8 & \xrightarrow[\sim]{f_3} & \mathcal{A}(X^3) \\ \downarrow \alpha_3 & & \downarrow \beta_3 \\ \mathcal{A}^4 \oplus \mathcal{A}\{-1\}^{16} \oplus \mathcal{A}\{-2\}^{16} & \xrightarrow[\sim]{f_2} & \mathcal{A}(X^2)^4 \\ \downarrow \alpha_2 & & \downarrow \beta_2 \\ \mathcal{A}^6 \oplus \mathcal{A}\{-1\}^{12} & \xrightarrow[\sim]{f_1} & \mathcal{A}(X)^6 \\ \downarrow \alpha_1 & & \downarrow \beta_1 \\ \mathcal{A}^4 & \xlongequal{\quad\quad\quad} & \mathcal{A}^4, \end{array}$$

together with appropriate homotopies. Repeated application of Lemma A.1, together with the fact that the bottom rectangle strictly commutes, gives that the required homotopies are

$$\begin{aligned} h_3^2 &: \alpha_2 \alpha_3 \simeq 0, \\ h_2^2 &: 0 \simeq \alpha_1 \alpha_2, \\ h_3^3 &: \alpha_1 h_3^2 + h_2^2 \alpha_3 \simeq 0, \\ H_3^1 &: f_2 \alpha_3 \simeq \beta_3 f_3, \\ H_2^1 &: \beta_2 f_2 \simeq f_1 \alpha_2, \\ H_3^2 &: f_1 h_3^2 + H_2^1 \alpha_3 - \beta_2 H_3^1 \simeq 0, \\ H_2^2 &: \beta_1 H_2^1 - h_2^2 \simeq 0, \end{aligned}$$

and

$$H_3^3 : h_3^3 + H_2^2 \alpha_3 - \beta_1 H_3^2 \simeq 0.$$

As we remark in §8, it is often convenient to write the differentials on cofibers in matrix form. Here the matrices for the two cell modules would be

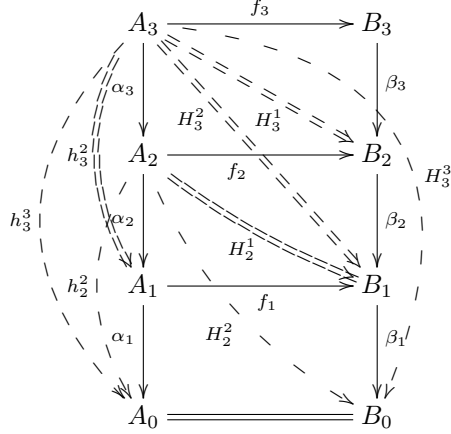
$$A = \begin{pmatrix} d & 0 & 0 & 0 \\ \overline{\alpha_3} & d & 0 & 0 \\ \overline{h_3^2} & \overline{\alpha_2} & d & 0 \\ \overline{h_3^3} & \overline{h_2^2} & \overline{\alpha_1} & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} d & 0 & 0 & 0 \\ \overline{\beta_3} & d & 0 & 0 \\ 0 & \overline{\beta_2} & d & 0 \\ 0 & 0 & \overline{\beta_1} & d \end{pmatrix}$$

The desired quasi-isomorphism of cofibers would be a matrix

$$H = \begin{pmatrix} f_3 & 0 & 0 & 0 \\ H_3^1 & f_2 & 0 & 0 \\ H_3^2 & H_2^1 & f_1 & 0 \\ H_3^3 & H_2^2 & H_1^1 & \text{id} \end{pmatrix}$$

such that  $HA = BH$ .

Replacing the modules on the left column of (1) by  $A_3, \dots, A_0$  and those on the right column by  $B_3, \dots, B_0$ , the desired homotopies are represented in the diagram



As in the  $n = 2$  case, the fact that the bottom rectangle commutes allows us to simplify the above somewhat. Given a choice of  $H_2^1$ , one can define  $h_2^2 := \beta_1 H_2^1$  and  $H_2^2 = 0$ . Similarly, given a choice of  $H_3^2$ , one can define  $h_3^3 := \beta_1 H_3^2$  and  $H_3^3 = 0$ . Thus it suffices to define explicitly homotopies

$$\begin{aligned} h_3^2 : \alpha_2 \alpha_3 &\simeq 0, & H_3^1 : f_2 \alpha_3 &\simeq \beta_3 f_3, \\ H_2^1 : \beta_2 f_2 &\simeq f_1 \alpha_2, & \text{and} & & H_3^2 : f_1 h_3^2 + H_2^1 \alpha_3 - \beta_2 H_3^1 &\simeq 0. \end{aligned}$$

We now describe explicitly what is needed in order to define these homotopies. Let us begin with  $h_3^2$ . As in §4, the composite  $\alpha_2 \alpha_3$  vanishes on the  $\mathcal{A}$  and  $\mathcal{A}\{-1\}$  summands, and so a homotopy is required only on the  $\mathcal{A}\{-2\}$  and  $\mathcal{A}\{-3\}$  summands. We will write  $U$ ,  $V$ , and  $W$  for the parameters on the three copies of  $X$ . As in the  $n = 2$  case,  $\alpha_2 \alpha_3$  vanishes on the summands of the form  $[U] \cdot [V]$  and  $[1 - V] \cdot [1 - W]$ . However,

$$\begin{aligned} \alpha_2 \alpha_3([U] \cdot [1 - V]) &= \alpha_2([a] \cdot [1 - V])|_{aX^2} + ([U] \cdot [1 - V])|_{X\Delta_X} - ([U] \cdot [1 - V])|_{X^2b} \\ &= ([a] \cdot [1 - a])|_{aaX} - ([a] \cdot [1 - V])|_{a\Delta_X} + ([a] \cdot [1 - V])|_{aXb} \\ &\quad + ([a] \cdot [1 - V])|_{a\Delta_X} - ([1 - b] \cdot [U])|_{Xbb} \\ &\quad - ([a] \cdot [1 - V])|_{aXb} + ([1 - b] \cdot [U])|_{Xbb} \\ &= ([a] \cdot [1 - a])|_{aaX} \end{aligned}$$

Thus we define

$$h_3^2([U] \cdot [1 - V]) := (T_a)|_{aaX}.$$

Also, we have

$$h_3^2([1 - U] \cdot [V]) = T_{1-a} := -T_a$$

(note that, since  $a$  is fixed throughout this discussion, the above choice will not cause compatibility problems), and  $h_3^2$  is defined similarly on the other summands.

To see what is needed on the  $\mathcal{A}\{-3\}$  summands, we first note that, for instance

$$\alpha_2 \alpha_3([U] \cdot [V] \cdot [1 - W]) = -[b] \cdot [1 - b] \cdot [U].$$

Thus we may define

$$h_3^2([U] \cdot [V] \cdot [1 - W]) = T_b \cdot [U]$$

(note that there is no sign since  $[U] \cdot [V] \cdot [1 - W]$  is of odd degree). Similarly, one finds that

$$\alpha_2 \alpha_3([U] \cdot [1 - V] \cdot [W]) = [a] \cdot [1 - a] \cdot [W] + [b] \cdot [1 - b] \cdot [U].$$

Thus we may define

$$h_3^2([U] \cdot [1 - V] \cdot [W]) = -T_a \cdot [W] - T_b \cdot [U].$$

The nonzero values of  $h_3^2$  on the  $\mathcal{A}\{-2\}$  and  $\mathcal{A}\{-3\}$  summands are given by

	$h_3^2$
$[U] \cdot [1 - V]$	$T_a,$
$[1 - U] \cdot [V]$	$-T_a,$
$[V] \cdot [1 - W]$	$-T_b,$
$[1 - V] \cdot [W]$	$T_b,$
$[U] \cdot [V] \cdot [1 - W]$	$T_b \cdot [U],$
$[U] \cdot [1 - V] \cdot [W]$	$-T_a \cdot [W] - T_b \cdot [U],$
$[U] \cdot [1 - V] \cdot [1 - W]$	$-T_a \cdot [1 - W],$
$[1 - U] \cdot [V] \cdot [W]$	$T_a \cdot [W],$
$[1 - U] \cdot [V] \cdot [1 - W]$	$T_a \cdot [1 - W] + T_b \cdot [1 - U],$
$[1 - U] \cdot [1 - V] \cdot [W]$	$-T_b \cdot [1 - U].$

We do not go into as much detail for  $H_3^1$  and  $H_2^1$  but merely claim that suitable definitions are

$$H_3^1([U] \cdot [1 - V] \cdot [1 - W]) = -(T_U \cdot [1 - W])_{|\Delta_X \times X},$$

$$H_3^1([U] \cdot [1 - V] \cdot [W]) = -(T_U \cdot [W])_{|\Delta_X \times X} + ([U] \cdot T_V)_{|X \times \Delta_X},$$

$$H_3^1([U] \cdot [V] \cdot [1 - W]) = -([U] \cdot T_V)_{|X \times \Delta_X},$$

and

$$H_2^1([U] \cdot [1 - V]) = -T_U.$$

Finally, let us see what is needed to define  $H_3^2$ . We compute

$$\begin{aligned} (f_1 h_3^2 + H_2^1 \alpha_3 - \beta_2 H_3^1)([U] \cdot [V] \cdot [1 - W]) &= \\ &= T_b \cdot [U] - [a] \cdot T_U + [a] \cdot T_U - [U] \cdot T_U + [U] \cdot T_b \\ &= -[U] \cdot T_U \end{aligned}$$

Thus  $H_3^2([U] \cdot [V] \cdot [1 - W])$  is a class  $T_{U,U,1-U}$  such that  $d(T_{U,U,1-U}) = [U] \cdot T_U$ . Note that since  $[U]^2 = 0$ , the class  $[U] \cdot T_U$  is a representative for the Massey product  $\langle [U], [U], [1 - U] \rangle$ . Similarly,

$$(f_1 h_3^2 + H_2^1 \alpha_3 - \beta_2 H_3^1)([U] \cdot [1 - V] \cdot [W]) = -T_U \cdot [U] + [U] \cdot T_U = 2 \cdot [U] \cdot T_U.$$

The class  $H_3^2([U] \cdot [1 - V] \cdot [W])$  is then given by a class  $-2T_{U,U,1-U}$ . Note that, since we have  $T_{1-U} = -T_U$ , we have that  $-2[U] \cdot T_U$  represents the Massey product  $\langle [U], [1 - U], [U] \rangle$ .

We have sketched a proof of the following

**Proposition 6.1.** *In order to build the Beilinson motive in the case  $n = 3$ , it suffices that the Massey product  $\langle [t], [t], [1 - t] \rangle$  in*

$$H^2(\mathcal{A}(\mathbb{G}_m - \{1\}))(3) \cong \text{CH}^3(\mathbb{G}_m - \{1\}, 4)$$

*is defined and contains zero.*

## 7. THE GENERAL CASE

From the discussion in the previous sections, it is not difficult to guess at the picture for general  $n$ . One starts with the diagram

$$\begin{array}{ccccccc} B_n & \xrightarrow{\beta_n} & B_{n-1} & \xrightarrow{\beta_{n-1}} & \cdots & \longrightarrow & B_1 & \xrightarrow{\beta_1} & B_0 \\ f_n \uparrow & & f_{n-1} \uparrow & & & & f_1 \uparrow & & \parallel \\ A_n & \xrightarrow{\alpha_n} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & \cdots & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & A_0 \end{array}$$

where the top row is the Beilinson complex, each  $A_i$  is a free  $\mathcal{A}$ -module, and each  $f_i$  is a quasi-isomorphism. In addition, one needs homotopies  $h_i^j$  for each  $3 \leq i \leq n$  and  $2 \leq j \leq n-1$  and  $H_k^l$  for each  $2 \leq k \leq n$  and  $1 \leq l \leq n-1$ . As in the previous cases, the right-hand square commutes on the nose, and no homotopies are required in order to deal with it.

In order to build the  $h_i^2$ 's and the  $H_j^1$ 's, one only needs the Totaro cycles. In order to build the  $h_i^3$ 's and the  $H_j^2$ 's, one needs in addition certain triple Massey products to be defined and to contain zero. In order to define the  $h_i^4$ 's and the  $H_j^3$ 's, one needs certain 4-fold Massey products to be defined and to contain zero. This pattern continues, so that in order to build the  $h_1^{n-2}$  and the  $H_j^{n-3}$ 's, one needs certain  $n-2$ -fold Massey products to be defined and to contain zero, and in order to define  $H_1^{n-2}$  one needs certain  $n-1$ -fold Massey products to be defined and to contain zero. Precise statements are given below.

In order to give precise statements, it will be convenient to introduce some notation. Given a function  $f$  on  $\mathbb{G}_m - \{1\}$ , we will write  $\Phi_f$  to mean either  $[f]$  or  $[1-f]$ . Thus the product  $\Phi_U^1 \Phi_V^2$  can mean any of the following four cohomology classes:  $[U][V]$ ,  $[1-U][V]$ ,  $[U][1-V]$ , or  $[1-U][1-V]$ . Note that a set of generators of  $A_n$  is given by the elements of the form  $\Phi_{U_1}^1 \cdots \Phi_{U_n}^n$ .

**Proposition 7.1.** *For  $2 \leq k \leq n$ , the homotopy  $h_n^k : A_n \rightarrow A_{n-k}[k-1]$  satisfies*

$$d_{n-k}(h_n^k \Phi_{U_1}^1 \cdots \Phi_{U_n}^n) = (-1)^\lambda (\langle \Phi_a^1, \dots, \Phi_a^k \rangle \cdot \Phi_{U_{k+1}}^{k+1} \cdots \Phi_{U_n}^n - \langle \Phi_b^{n-k+1}, \dots, \Phi_b^n \rangle \cdot \Phi_{U_1}^1 \cdots \Phi_{U_{n-k}}^{n-k}),$$

where  $\lambda = n(k-1) + \frac{k(k-1)}{2}$ . Thus  $h_n^k$  can be defined by

$$h_n^k(\Phi_{U_1}^1 \cdots \Phi_{U_n}^n) = (-1)^\lambda (T_{\Phi_a^1, \dots, \Phi_a^k} \cdot \Phi_{U_k}^{k+1} \cdots \Phi_{U_n}^n - T_{\Phi_b^{n-k+1}, \dots, \Phi_b^n} \cdot \Phi_{U_1}^1 \cdots \Phi_{U_{n-k}}^{n-k}).$$

*Proof.* The proof is by double induction on  $k$  and  $n$ . The base case is  $k = n = 2$ , and we have

$$d_0 h_2^2(\Phi_{U_1}^1 \Phi_{U_2}^2) = -\alpha_1 \alpha_2 (\Phi_{U_1}^1 \Phi_{U_2}^2) = -(\Phi_a^1 \Phi_a^2 - \Phi_b^1 \Phi_b^2)$$

as desired. In fact, we similarly find that

$$d_{n-2} h_n^2(\Phi_{U_1}^1 \cdots \Phi_{U_n}^n) = (-1)^{n-1} (\Phi_a^1 \Phi_a^2 \cdot \Phi_{U_3}^3 \cdots \Phi_{U_n}^n - \Phi_b^{n-1} \Phi_b^2 \cdot \Phi_{U_1}^1 \cdots \Phi_{U_{n-2}}^{n-2}).$$

For the inductive step, assume the formula holds for all  $h_j^i$  with  $j < n$  or  $i < k$ . We have

$$\begin{aligned} d_{n-k} h_n^k \Phi_{U_1}^1 \cdots \Phi_{U_n}^n &= -(-1)^n (h_{n-1}^{k-1} \alpha_n \Phi_{U_1}^1 \cdots \Phi_{U_n}^n - h_{n-2}^{k-2} h_n^2 \Phi_{U_1}^1 \cdots \Phi_{U_n}^n + \\ &\quad \cdots + (-1)^k \alpha_{n-k+1} h_n^{k-1} \Phi_{U_1}^1 \cdots \Phi_{U_n}^n). \end{aligned}$$

By induction, we know all of the terms in this sum. The general term is given by

$$h_{n-j}^{k-j} h_n^j (\Phi_{U_1}^1 \cdots \Phi_{U_n}^n) = (-1)^{nk+j+\frac{k(k-1)}{2}} \cdot \left( T_{\Phi_a^1, \dots, \Phi_a^j} (T_{\Phi_a^{j+1}, \dots, \Phi_a^k} \Phi_{U_{k+1}}^{k+1} \cdots \Phi_{U_n}^n - T_{\Phi_b^{n-k+j+1}, \dots, \Phi_b^n} \Phi_{U_{j+1}}^{j+1} \cdots \Phi_{U_{n-k+j}}^{n-k+j}) - T_{\Phi_b^{n-j+1}, \dots, \Phi_b^n} (T_{\Phi_a^1, \dots, \Phi_a^{k-j}} \Phi_{U_{k-j+1}}^{k-j+1} \cdots \Phi_{U_{k-j}}^{k-j} - T_{\Phi_b^{n-k+1}, \dots, \Phi_b^{n-j}} \Phi_{U_1}^1 \cdots \Phi_{U_{n-k}}^{n-k}) \right)$$

One checks that the terms involving a product of a Totaro cycle at  $a$  with a Totaro cycle  $b$  cancel the similar mixed terms appearing in the expression for  $h_{n-k+j}^j h_n^{k-j} \Phi_{U_1}^1 \Phi_{U_n}^n$ , using that  $h_{n-j}^{k-j} h_n^j$  appears with sign  $(-1)^{n+j}$ . Summing the terms gives the result.  $\blacksquare$

One can similarly obtain inductively a formula for the  $H_n^k$ 's. In order to make a precise statement, it is convenient to introduce the following definition.

**Definition 7.2.** We will say that the **length** of an  $n$ -fold Massey product is  $n$ . Similarly, if  $T_{x_1, \dots, x_n}$  is a Totaro cycle bounding an  $n$ -fold Massey product, we will say that it is a Totaro cycle of **length**  $n$ . The **reduced length** of a Massey product or Totaro cycle will be defined to be the length minus 1.

**Proposition 7.3.** For  $1 \leq k \leq n-1$ , the homotopy  $H_n^k : A_n \rightarrow B_{n-k}[k]$  can be defined by

$$H_n^k(\Phi_{U_1}^1 \cdots \Phi_{U_n}^n) = (-1)^\mu \left( \sum \Phi_{U_1}^1 \cdots T_{\Phi_{U_i}^i, \dots, \Phi_{U_i}^j} \Phi_{U_{j+1}}^{j+1} \cdots T_{\Phi_{U_k}^k, \dots, \Phi_{U_k}^i} \cdots \Phi_{U_n}^n \right)$$

where  $\mu = nk + \frac{k(k-1)}{2}$  and where the sum is over all such products such that the sum of the reduced lengths of the Totaro cycles is  $k$ .

The above proposition directly implies the following theorem

**Theorem 7.4.** In order to build the Beilinson motive for general  $n$ , it suffices that the iterated Massey products  $\langle \Phi_t^1, \dots, \Phi_t^k \rangle$  in

$$H^2(\mathcal{A}(\mathbb{G}_m - \{1\}))(k) \cong \mathrm{CH}^k(\mathbb{G}_m - \{1\}, 2k-2)$$

are defined and contain zero for all  $k \leq n$ .

## 8. TRIMMING THE FAT

Rather than just obtaining a cellular approximation to the Beilinson complex, one would like to obtain an approximation which is a minimal  $\mathcal{A}$ -module. We do not go into the details, but the explicit cellular approximations we have obtained lend themselves easily to finding minimal cellular approximations. The resulting picture is

$$\begin{array}{ccccccc} B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0 \\ \sim \uparrow & & \sim \uparrow & & & & \sim \uparrow & & \parallel \\ A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & A_0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \mathcal{A}\{-n\}^{2^n} & \longrightarrow & \mathcal{A}\{-(n-1)\}^{2^{n-1}} & \longrightarrow & \cdots & \longrightarrow & \mathcal{A}\{-1\}^2 & \longrightarrow & \mathcal{A} \end{array}$$

The homotopies  $h_i^k$  descend to the quotients and allow one to build a minimal cell module out of  $\mathcal{A}\{-n\}^{2^n} \rightarrow \cdots \rightarrow \mathcal{A}\{-1\} \rightarrow \mathcal{A}$ , and the epimorphisms  $A_i \twoheadrightarrow \mathcal{A}\{-i\}^{2^i}$  assemble to give a quasi-isomorphism of cell modules.

#### APPENDIX: REVIEW OF COFIBERS

In this section we review the relevant homological algebra, including sign conventions. We are using the sign conventions of [KM], and we will reproduce here some of their discussion from section III.1. Throughout this section,  $A$  will denote a (cohomologically graded) commutative dga.

Recall the unit interval  $A$ -module  $I$ . It is free of rank one in degree  $-1$ , with generator denoted  $[I]$ , and free of rank two in degree  $0$ , with generators denoted  $[0]$  and  $[1]$ . The only nontrivial differential is  $d[I] = [0] - [1]$ . The unit interval  $I$  is introduced in order to define homotopies: given dg  $A$ -modules  $M$  and  $N$  and two maps  $f, g : M \rightarrow N$ , a homotopy from  $f$  to  $g$  is a map  $h : M \otimes I \rightarrow N$  such that  $h(m \otimes [0]) = f(m)$  and  $h(m \otimes [1]) = g(m)$ . Note that since we have chosen to write our unit interval on the right of  $M$ , we have the formula

$$d(m \otimes [I]) = dm \otimes [I] + (-1)^{\deg m} m \otimes ([0] - [1]).$$

This gives the formula

$$d(h(m \otimes [I])) = h(dm \otimes [I]) + (-1)^{\deg m} (f(m) - g(m)).$$

We will follow standard usage and often write  $h(m)$  for  $h(m \otimes [I])$ .

We define the *cone* on  $M$  to be the pushout

$$\begin{array}{ccc} M \otimes [1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M \otimes I & \longrightarrow & CM. \end{array}$$

The induced differential is  $d(m \otimes [I]) = dm \otimes [I] + (-1)^{\deg m} m \otimes [0]$ . Note that there is a canonical map  $M \cong M \otimes [0] \hookrightarrow CM$ . More generally, given any map  $f : M \rightarrow N$  of dg  $A$ -modules, we define the *cofiber* of  $f$  by the pushout diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ CM & \longrightarrow & C(f). \end{array}$$

A typical element of  $C(f)$  can be written in the form  $n + m \otimes [I]$ , and the induced differential is

$$d(n + m \otimes [I]) = dn + (-1)^{\deg m} f(m) + dm \otimes [I].$$

It is often useful to write the differential in the form of the matrix

$$\begin{pmatrix} d_M & 0 \\ \bar{f} & d_N \end{pmatrix},$$

where  $\bar{f}(x) = (-1)^{|x|} f(x)$ . Given the above definitions, it is now straightforward to verify

**Lemma A.1.** *Suppose given dg  $A$ -modules and maps between them as shown in the diagram*

$$\begin{array}{ccc}
 W & \xrightarrow{\alpha} & Y \\
 f \downarrow & \searrow h & \downarrow g \\
 X & \xrightarrow{\beta} & Z,
 \end{array}$$

where  $h$  is a homotopy  $h : \beta f \simeq g\alpha$ . Then the assignment

$$x + w \otimes [I] \mapsto \beta(x) + h(w) + \alpha(w) \otimes [I]$$

defines a map of dg  $A$ -modules  $C(f) \rightarrow C(g)$ .

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