Genuine equivariant objects as presheaves with transfers

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June 3, 2009

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Let G be a finite group and let $\mathscr{O}_G$ denote the orbit category of G, with objects the orbits G/H and maps being G-maps. Then in any category $\mathscr{C}$, a (naive) G-object is simply an object $X \in \mathscr{C}$ equipped with a map of monoids $G \to \mathrm{End}_{\mathscr{C}}(X)$. If $\mathscr{C}$ has limits, so that one can make sense of fixed points, one has an adjunction

$$\mathbb{T} : P(\mathscr{O}_G, \mathscr{C}) \rightleftarrows G\mathscr{C} : \mathbb{U}$$

between $\mathscr{C}$-valued presheaves on the orbit category and the category $G\mathscr{C}$ of G-objects in $\mathscr{C}$. The functor $\mathbb{U}$ assigns to any G-object X the diagram of fixed points of X, together with restriction maps $X^H \to X^K$ for each conjugation of H inside K. The left adjoint $\mathbb{T}$ is defined simply by evaluation at the orbit G/e, using that $\mathscr{O}_G(G/e, G/e) \cong G$. If $\mathscr{C}$ has a well-behaved homotopy theory (is a cofibrantly generated model category), then the above induces an equivalence of homotopy theories (Quillen equivalence).

In many situations, most notably that of the stable homotopy category of G-spectra, the above homotopy theory is not suitably rich; one wants to also build in "transfer maps" that have the opposite variance as the restriction maps above. To do this, one may formally add wrong-way morphisms to the category $\mathscr{O}_G$ to produce a new category B_G. The category B_G has the same objects as $\mathscr{O}_G$ but the set $B_G(G/H, G/K)$ of morphisms is defined to the be set of zig-zags, or spans,

$$\begin{tikzcd}
 & G/L \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
G/H & & G/K.
\end{tikzcd}$$

Composition of spans is defined by a pullback, as in the diagram

$$\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] & & \\
 & G/K_0 \arrow[dl] \arrow[dr] & & G/K_1 \arrow[dl] \arrow[dr] & \\
G/H_0 & & G/H_1 & & G/H_2.
\end{tikzcd}$$

To be precise, in order to have a well-defined composition, one must make a choice of pullback for each pair of compatible spans. The object X is not an orbit in general but rather a finite G-set, which decomposes into a disjoint union of orbits. So in order to define compositions we need to allow spans $G/H \leftarrow X \to G/K$ with arbitrary finite G-sets X. Having defined composition in terms of pullbacks, one now finds that composition is not associative. Rather, there is a canonical isomorphism of G-sets $X \circ (Y \circ Z) \cong (X \circ Y) \circ Z$. So rather than a category B_G, the above defines a *bicategory* B_G^{span} with

- objects the orbits G/H
- 1-morphisms the spans of finite G-sets $G/H \leftarrow X \rightarrow G/K$
- 2-morphisms the maps of G-sets making

$$\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[dd] & \\
G/H & & G/K \\
& Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

  commute.

In other words, the category $B_G^{span}(G/H, G/K)$ is the category of finite G-sets over $G/H \times G/K$.

Note that each category $B_G^{span}(G/H, G/K)$ has in addition the structure of a symmetric monoidal category by taking disjoint unions of finite G-sets. Moreover, composition maps of the bicategory are bilinear with respect to the monoidal structures.

Lemma. *There is a biequivalence $B_G^{span} \to \mathbf{B}_G$ to a permutative 2-category (? terminology).*

This permutative 2-category $\mathbf{B}_G$ then feeds into the group-completion (= $\mathbb{K}$-theory) machine of Elmendorf-Mandell to produce an $\mathscr{S}$-category $\mathscr{B}_G^{\mathbb{K}}$.

Theorem 1 ([GM])**.** *We have a Quillen equivalence*

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{S}}(\mathscr{B}_G) \rightleftarrows G\mathscr{S}_U : \mathbb{I}^*,$$

where $\mathscr{B}_G \subseteq G\mathscr{S}_U$ is the full $\mathscr{S}$-category on the orbits $\Sigma^\infty_U G/H_+$.

To relate $G\mathscr{S}_U$ to $\mathscr{P}_{\mathscr{S}}(\mathscr{B}_G^{\mathbb{K}})$, it suffices to produce a map $\mathscr{B}_G^{\mathbb{K}} \to \mathscr{B}_G$ of $\mathscr{S}$-categories. The equivariant Barratt-Priddy-Quillen theorem says that the mapping spectra in these two $\mathscr{S}$-categories are equivalent, but a correct account of equivariant infinite loop space theory should provide a map of $\mathscr{S}$-categories.

I believe we do have $\Sigma^\infty \mathscr{O}_G \to \mathscr{B}_G^{\mathbb{K}}$, and so an adjoint pair $\mathscr{P}_{\mathscr{S}}(\Sigma^\infty \mathscr{O}_G) \rightleftarrows \mathscr{P}_{\mathscr{S}}(\mathscr{B}_G^{\mathbb{K}})$. We thus have the diagram

$$\begin{tikzcd}
G\mathscr{S}_{U^G} \arrow[r, shift left] \arrow[d, shift left] & G\mathscr{S}_U \arrow[l, shift left] \arrow[d, shift left] \\
\mathscr{P}_{\mathscr{S}}(\Sigma^\infty \mathscr{O}_G, \mathscr{S}) \arrow[r, shift left] \arrow[u, shift left] \arrow[d, equal] & \mathscr{P}_{\mathscr{S}}(\mathscr{B}_G, \mathscr{S}) \arrow[l, shift left] \arrow[u, shift left] \arrow[d, shift left, dashed] \\
\mathscr{P}(\mathscr{O}_G, \mathscr{S}) \arrow[r, shift left] & \mathscr{P}_{\mathscr{S}}(\mathscr{B}_G^{\mathbb{K}}, \mathscr{S}) \arrow[l, shift left] \arrow[u, shift left, dashed]
\end{tikzcd}$$

in which the vertical adjunctions yield Quillen equivalences. This picture says that the category of genuine G-spectra is obtained from the category of naive G-spectra by building in transfer maps.

Question 2 (H. Miller)**.** From this perspective, can one see the $RO(G)$-grading on (co)homology? In other words, how can one see that the representation spheres S^V are invertible?

Question 3. Isn't this result a version of the Lewis-May-McClure result that equivariant cohomology theories (= naive G-spectra) extend to $RO(G)$-graded equivariant cohomology theories if and only if their coefficient systems extend to Mackey functors?

2 Equivariant Barratt-Priddy-Quillen

We want to understand why $\mathbb{K}(GSet) \simeq (\mathbb{S}_G)^G$. According to the tom Dieck splitting, we have, for any G-space X,

$$(\Sigma^\infty_G X_+)^G \simeq \bigvee_{[H]} \Sigma^\infty (EW_G(H)_+ \wedge_{W_G(H)} X^H), \tag{1}$$

where the wedge is over conjugacy classes of subgroups. So it would suffice to identify the latter with the $\mathbb{K}$-theory spectrum.

Since any G-set decomposes into a sum of orbits, there is an identification

$$GSet_{\mathrm{iso}} \simeq \prod_{[H]} Set \wr G/H,$$

where $Set \wr G/H \subseteq GSet_{\mathrm{iso}}$ denotes the full subcategory of G-sets which are sums of copies of G/H. More generally, for any G-space X, there is a decomposition

$$(GSet/X)_{\mathrm{iso}} \simeq \prod_{[H]} (Set \wr G/H)/X, \tag{2}$$

Question 4. If $\mathscr{C}$ and $\mathscr{D}$ are permutative (or symmetric monoidal), is there a natural identification

$$\mathbb{K}(\mathscr{C}) \vee \mathbb{K}(\mathscr{D}) \simeq \mathbb{K}(\mathscr{C} \times \mathscr{D})?$$

Supposing this to be the case, it sufffices to show that $\mathbb{K}(Set \wr G/H) \simeq \Sigma^\infty BW_G(H)_+$. Note that $W_G(W) = Aut_G(G/H)$.

Proposition 5. $\mathbb{K}(Set \wr G/H) \simeq \Sigma^\infty BW_G(H)_+$

Proof. Note that the classical Barratt-Priddy-Quillen theorem says that

$$\mathbb{K}(Set/BW_G(H)) \simeq \Sigma^\infty BW_G(H)_+.$$

We will relate $\mathbb{K}(Set/BW_G(H))$ to $\mathbb{K}(Set \wr G/H)$ via a pair of functors

$$Set \wr G/H \xleftarrow{\mathbf{\Lambda}} W_G(H)Set/EW_G(H) \xrightarrow{\mathbf{R}} Set/BW_G(H).$$

The functor $\mathbf{\Lambda}$ is defined by

$$\mathbf{\Lambda}(X \to EW_G(H)) = G/H \times_{W_G(H)} X,$$

and the functor $\mathbf{R}$ is given by

$$\mathbf{R}(X \to EW_G(H)) = X/W_G(H) \to BW_G(H).$$

Both functors preserve sums and so induce maps on $\mathbb{K}$-theory.

Note that

$$\mathscr{N}(Set \wr G/H_{\mathrm{iso}}) \simeq \coprod_{n \geq 0} B(\Sigma_n \wr W_G(H)),$$

$$\mathscr{N}(W_G(H)Set/EW_G(H)_{\mathrm{iso}}) \simeq \coprod_{n \geq 0} E(\Sigma_n \wr W_G(H)) \times_{\Sigma_n \wr W_G(H)} (EW_G(H))^n,$$

and

$$\mathscr{N}(Set/BW_G(H)_{\mathrm{iso}}) \simeq \coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} (BW_G(H))^n.$$

From this we conclude that the induced maps $\mathscr{N}\mathbf{R}$ and $\mathscr{N}\mathbf{\Lambda}$ are equivalences, so that we get equivalences

$$\mathbb{K}(Set \wr G/H) \xleftarrow{\sim} \mathbb{K}(W_G(H)Set/EW_G(H)) \xrightarrow{\sim} \mathbb{K}(Set/BW_G(H)).$$

on group-completions. □

More generally, one has the following result.

Proposition 6. *Let X be a G-space. Then*

$$\mathbb{K}((Set \wr G/H)/X) \simeq \Sigma^\infty (EW_G(H) \times_{W_G(H)} X^H)_+.$$

Proof. The argument is essentially the same as in the previous proposition. One has a pair of coproduct-preserving functors

$$Set \wr G/H/X \xleftarrow{\mathbf{\Lambda}} W_G(H)Set/EW_G(H) \times X^H \xrightarrow{\mathbf{R}} Set/EW_G(H) \times_{W_G(H)} X^H,$$

and as above these induce equivalences between the nerves of the isomorphism groupoids. □

Theorem 7. $\mathbb{K}(GSet) \simeq (\mathbb{S}_G)^G$.

Proof. This follows from combining the decomposition (2), Proposition 5 and the tom Dieck decomposition (1). □

Corollary 8. $\mathbb{K}(GSet/G/H) \simeq (\mathbb{S}_G)^H$.

Proof. The functor $GSet/G/H \to HSet$ which assigns to $X \xrightarrow{\eta} G/H$ the H-set $\eta^{-1}(eH)$ is an equivalence of categories, so the result follows from the identification $(\mathbb{S}_G)^H = (\mathbb{S}_H)^H$. □

More generally,

Corollary 9. *For any G-space X,*

$$\mathbb{K}(GSet/G/H \times X) \simeq (\Sigma^\infty_H X_+)^H.$$

Proof. Again, the functor $GSet/G/H \times X \to HSet/X$ which assigns to $G/H \xleftarrow{\eta} Z \xrightarrow{\varphi} X$ the object $\eta^{-1}(eH) \xrightarrow{\varphi} X$ is an equivalence of categories. Thus

$$\mathbb{K}(GSet/G/H \times X) \simeq \mathbb{K}(HSet/X) \simeq (\Sigma^\infty_H X_+)^H.$$

Again, the second equivalence uses the decomposition (2), Proposition 6, and the tom Dieck decomposition (1). □