Enriched Model Categories and Diagram Categories

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Outline

1. Enriched Category Theory
   - Enriched Categories & Functors
   - Change of Enrichment
   - Enriched Homotopy Theory

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1. Enriched Category Theory
   - Enriched Categories & Functors
   - Change of Enrichment
   - Enriched Homotopy Theory
2. Enriched Categories as Diagram Categories
   - Diagram Categories
   - Creating Diagram Model Structures
   - Recognizing Diagram Model Structures
3. Equivariant Categories
   - Two Contexts
   - Equivariant Homotopy Theory
   - Equivariant Spectra

$\mathscr{V}$-categories

Let $(\mathscr{V}, \otimes, U)$ be a closed symmetric monoidal category.

Definition

A $\mathscr{V}$**-category** $\mathscr{M}$ consists of

- a set $Ob(\mathscr{M})$
- for each $X, Y \in \mathscr{M}$, an object $\mathscr{M}^{\mathscr{V}}(X, Y) \in \mathscr{V}$
- unit morphisms $U \to \mathscr{M}^{\mathscr{V}}(X, X)$ for each X
- (unital, associative) composition morphisms

$$\mathscr{M}^{\mathscr{V}}(Y, Z) \otimes \mathscr{M}^{\mathscr{V}}(X, Y) \to \mathscr{V}(X, Z).$$

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$$\mathscr{M}(X, Y) = \mathrm{Hom}_{\mathscr{V}}(U, \mathscr{M}^{\mathscr{V}}(X, Y)).$$

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- $\mathscr{V} = sSet$: a $sSet$-category is a simplicial category
- $\mathscr{V} = Top$: a Top-category is a topological category
- $\mathscr{V} = Cat$: a Cat-category is a (strict) 2-category

Examples (continued)

A category may have multiple enrichments: Fix a group G, let $X, Y \in GTop$ (G-spaces). Define

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- $GTop(X, Y) \in Top =$ space of G-maps.

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- $Top_G(X, Y) \in GTop$ = space of all maps with conjugation action
- $GTop(X, Y) \in Top$ = space of G-maps.

Then Top_G is a $GTop$-category while $GTop$ is a Top-category. Moreover, $\underline{Top_G} = \underline{GTop}$.

Tensors

Definition

We say that $\mathscr{M}$ is **tensored** and **cotensored** (or bitensored) over $\mathscr{V}$ if there exist objects $V \odot X$ and X^V for $V \in \mathscr{V}$, $X \in \mathscr{M}$, together with binatural isomorphisms

$$\mathscr{M}^{\mathscr{V}}(V \odot X, Y) \cong \mathscr{V}(V, \mathscr{M}^{\mathscr{V}}(X, Y)) \cong \mathscr{M}^{\mathscr{V}}(X, Y^V).$$

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- a function $Ob(\mathscr{M}) \to Ob(\mathscr{N})$
- for each $X, Y \in \mathscr{M}$ a morphism

$$\mathscr{M}^{\mathscr{V}}(X, Y) \to \mathscr{N}^{\mathscr{V}}(F(X), F(Y))$$

in $\mathscr{V}$ that is compatible with units and composition.

Change of Enrichment

Let $\mathscr{V}_1$, $\mathscr{V}_2$ be closed symmetric monoidal categories.

Proposition

Let $\mathbb{T} : \mathscr{V}_1 \rightleftarrows \mathscr{V}_2 : \mathbb{U}$ *be adjoint, with* $\mathbb{T}$ *strong symmetric monoidal. If* $\mathscr{M}^{\mathscr{V}_2}$ *is a bitensored* $\mathscr{V}_2$*-category, one can define a bitensored* $\mathscr{V}_1$*-category by*

$$\mathscr{M}^{\mathscr{V}_1}(X, Y) := \mathbb{U}\mathscr{M}^{\mathscr{V}_2}(X, Y), \quad V_1 \odot X := \mathbb{T}(V_1) \odot X,$$

$$\textit{and} \qquad X^{V_1} := X^{\mathbb{T}(V_1)}.$$

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$$\text{and} \qquad X^{V_1} := X^{\mathbb{T}(V_1)}.$$

Since $\mathbb{T}(U_1) \cong U_2$, we have $\underline{M^{\mathscr{V}_1}} = \underline{M^{\mathscr{V}_2}}$.

Change of Enrichment - Examples

Some examples:

- Let $U[-] : Set \to \mathscr{V}$ be the functor $U[X] = \coprod_X U$. Then have $U[-] : Set \rightleftarrows \mathscr{V} : \mathrm{Hom}_{\mathscr{V}}(U, -)$. Changes $\mathscr{M}^{\mathscr{V}}$ to $\underline{\mathscr{M}}$.

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- $Triv : Top \rightleftarrows GTop : (-)^G$. This change of enrichment produces *GTop* from Top_G.
- $\Sigma^\infty : Top_* \rightleftarrows \mathscr{S} : \Omega^\infty$, for $\mathscr{S}$ some good category of spectra. (function spectra versus function spaces).
- $\mathcal{P} : sSet \rightleftarrows Cat : \mathcal{N}$

Monoidal Model Categories

Definition

A **monoidal model category** $\mathscr{V}$ is a monoidal category with a model structure on $\underline{\mathscr{V}}$ such that

- (Pushout-product) $V \hookrightarrow W$, $Y \overset{(\sim)}{\hookrightarrow} Z \Rightarrow$

$$V \otimes Z \coprod_{V \otimes Y} W \otimes Y \overset{(\sim)}{\hookrightarrow} W \otimes Z$$

- (unit) For $\emptyset \hookrightarrow V$ have $QU \otimes V \xrightarrow{\sim} U \otimes V \cong V$.

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The homotopy category $Ho(\mathscr{V})$ is monoidal.

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- (SM7) Given $A \overset{(\sim)}{\hookrightarrow} B$ and $X \overset{(\sim)}{\twoheadrightarrow} Y$ in $\underline{\mathscr{M}}$, have

$$\mathscr{M}^{\mathscr{V}}(B, X) \overset{(\sim)}{\twoheadrightarrow} \mathscr{M}^{\mathscr{V}}(A, X) \times_{\mathscr{M}^{\mathscr{V}}(A,Y)} \mathscr{M}^{\mathscr{V}}(B, Y) \quad \text{in } \underline{\mathscr{V}}.$$

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- (Unit) Given $\emptyset \hookrightarrow X$ in $\underline{\mathscr{M}}$, have

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Change of Enrichment Revisited

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be monoidal model categories.

Proposition

Let $\mathbb{T} : \mathscr{V}_1 \rightleftarrows \mathscr{V}_2 : \mathbb{U}$ *be a Quillen pair such that* $\mathbb{T}$ *is strong symmetric monoidal and such that* $\mathbb{T}(QU_1) \to \mathbb{T}(U_1)$ *is a weak equivalence. Then* $\mathbb{L}\mathbb{T} : Ho(\mathscr{V}_1) \rightleftarrows Ho(\mathscr{V}_2) : \mathbb{R}\mathbb{U}$ *is strong symmetric monoidal.*

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Proposition

Let $(\mathbb{T}, \mathbb{U})$ *be as above, and let* $\mathscr{M}$ *be a* $\mathscr{V}_2$*-model category. Then the* $\mathscr{V}_1$*-model category* $\mathscr{M}^{\mathscr{V}_1}$ *is a* $\mathscr{V}_1$*-model category.*

Outline

Setup

Given a small $\mathscr{V}$-category $\mathscr{D}$, let $\mathscr{P} = \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V})$ be the $\mathscr{V}$-category of contravariant $\mathscr{V}$-functors $\mathscr{D}^{op} \to \mathscr{V}$.

Construction

Let $\mathbb{I} : \mathscr{D} \to \mathscr{M}$ be a $\mathscr{V}$-functor. Define a $\mathscr{V}$-functor $\mathbb{I}^* : \mathscr{M} \to \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V})$ by

$$\mathbb{I}^*(M)(d) = \mathscr{M}^{\mathscr{V}}(\mathbb{I}(d), M).$$

Define $\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \to \mathscr{M}$ by $\mathbb{I}_!(X) = X \odot_{\mathscr{D}} \mathbb{I}$. Then

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \rightleftarrows \mathscr{M} : \mathbb{I}^*$$

defines an adjoint pair.

Projective Model Structure

Definition

We say that $\mathscr{D}$ is $\mathscr{V}$-good if each $\mathscr{D}^{\mathscr{V}}(d, d') \otimes - : \mathscr{V} \to \mathscr{V}$ preserves acyclic cofibrations.

Theorem (Kan, Schwede-Shipley, Shulman)

If $\mathscr{D}$ is $\mathscr{V}$-good, then $\mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V})$ is a (cofib. gen.) $\mathscr{V}$-model category, with weak equivalences and fibrations given objectwise.

Question

Given a $\mathscr{V}$-functor $\mathscr{D} \xrightarrow{\mathbb{I}} \mathscr{M}$, can one define a $\mathscr{V}$-model structure on $\mathscr{M}$ so that

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \rightleftarrows \mathscr{M} : \mathbb{I}^*$$

becomes a Quillen equivalence?

Creating Diagram Model Structures

Definition

A map $f : M \to N$ in $\mathscr{M}$ is said to be a $\mathscr{D}$-equivalence ($\mathscr{D}$-fibration) if $\mathbb{I}^*(f)$ is a weak equivalence (fibration).

Theorem (Dwyer-Kan, GM)

Let $\mathscr{D}$ be $\mathscr{M}$-good. Then $\mathscr{M}$ has a $\mathscr{V}$-model structure, where the weak equivalences are the $\mathscr{D}$-equivalences and the fibrations are the $\mathscr{D}$-fibrations. The adjoint pair

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \rightleftarrows \mathscr{M} : \mathbb{I}^*$$

is a Quillen adjoint pair. It is a Quillen equivalence if and only if for each cofibrant $X \in \mathscr{P}$, the unit $X \xrightarrow{\eta} \mathbb{I}^*\mathbb{I}_!(X)$ is a weak equivalence.

Example: Modeling G-Spaces

Let G = topological group, consider $GTop$.

- Say $f : X \to Y$ is a weak equivalence (fibration) if $f^H : X^H \to Y^H$ is a weak homotopy equivalence (fibration) for each closed $H \leq G$. Note that $X^H = GTop(G/H, X)$.
- Define Top-category $\mathscr{O}_G$ with objects G/H for $H \leq G$ closed and with $\mathscr{O}_G(G/H, G/K) = GTop(G/H, G/K)$. Have $\mathbb{I} : \mathscr{O}_G \to GTop$.

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- Define Top-category $\mathscr{O}_G$ with objects G/H for $H \leq G$ closed and with $\mathscr{O}_G(G/H, G/K) = GTop(G/H, G/K)$. Have $\mathbb{I} : \mathscr{O}_G \to GTop$.

Theorem (Elmendorf, Dwyer-Kan)

The above induces a Quillen equivalence

$$\mathbb{I}_! : \mathscr{P}_{Top}(\mathscr{O}_G, Top) \rightleftarrows GTop : \mathbb{I}^*.$$

Recognizing Diagram Model Structures - The Question

Question

Given a $\mathscr{V}$-model category $\mathscr{M}$, when can one find a small $\mathscr{V}$-category $\mathscr{D}$ and a $\mathscr{V}$-functor $\mathbb{I} : \mathscr{D} \to \mathscr{M}$, such that the adjunction

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Question'

Given a $\mathscr{V}$-model category $\mathscr{M}$, when can one find a small, full $\mathscr{V}$-subcategory $\mathscr{D} \subseteq \mathscr{M}$ such that the adjunction induced by the inclusion $\mathbb{I} : \mathscr{D} \hookrightarrow \mathscr{M}$

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \rightleftarrows \mathscr{M} : \mathbb{I}^*$$

is a Quillen equivalence?

Conditions on $\mathscr{D}$

Definition

$\mathscr{D}$ is called a

- **reflecting set** if $X \xrightarrow{f} Y$ between fibrant objects is a weak equivalence if $\mathscr{M}^{\mathscr{V}}(\mathbb{I}(d), X) \xrightarrow{\sim} \mathscr{M}^{\mathscr{V}}(\mathbb{I}(d), Y)$ for all $d \in \mathscr{D}$
- **creating set** if $X \xrightarrow{f} Y$ is a weak equivalence if and only if $\mathscr{M}^{\mathscr{V}}(\mathbb{I}(d), X) \xrightarrow{\sim} \mathscr{M}^{\mathscr{V}}(\mathbb{I}(d), Y)$ for all $d \in \mathscr{D}$

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Assumption

- objects of $\mathscr{D}$ are bifibrant (in $\mathscr{M}$)
- $\mathscr{D}$ is $\mathscr{V}$-good

Recognizing Diagram Model Structures

Theorem

The adjunction

$$\mathbb{I}_! : \mathscr{P}_{\mathscr{V}}(\mathscr{D}, \mathscr{V}) \rightleftarrows \mathscr{M} : \mathbb{I}^*$$

is a Quillen pair. It is a Quillen equivalence if and only if $\mathscr{D}$ is a reflecting set and

$$X \xrightarrow{\eta} \mathbb{I}^*\mathbb{I}_!(X) \xrightarrow{\mathbb{I}^*(r)} \mathbb{I}^*(R\mathbb{I}_!(X))$$

is a weak equivalence for cofibrant X.
If $\mathscr{D}$ is a creating set, fibrant replacement is not needed.

Outline

Equivariant Contexts

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- a group object in a cartesian monoidal category $\mathscr{V}$

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Definition

Let

- (discrete) $U[G] = \coprod_G U \in \mathscr{V}$
- (cart. mon.) $U[G] = G \in \mathscr{V}$

Let $*_G$ be one-object $\mathscr{V}$-category with $\mathrm{End}(\star) = U[G]$.
Then $\underline{G\mathscr{M}}$ = category of $\mathscr{V}$-functors $*_G \to \mathscr{M}$.
Thus have morphism $U[G] \to \mathscr{M}^{\mathscr{V}}(X, X)$ in $\mathscr{V}$

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Thus have morphism $U[G] \to \mathscr{M}^{\mathscr{V}}(X, X)$ in $\mathscr{V}$

Note when $\mathscr{V} = Mod_k$, $k[G] = k[G]$.

Orbits, Fixed Points

Let $X : *_G \to \mathscr{M} \in G\mathscr{M}$. Define

$$X/G := \operatorname{colim} X = \operatorname{coeq}\left(X(\star) \odot U[G] \rightrightarrows X(\star) \odot U\right),$$

$$X^G := \lim X = \operatorname{eq}\left(X(\star)^U \rightrightarrows X(\star)^{U[G]}\right).$$

Gives

$$(-)/G : G\mathscr{M} \rightleftarrows \mathscr{M} : Triv, \qquad Triv : \mathscr{M} \rightleftarrows G\mathscr{M} : (-)^G.$$

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$$(-)/G : G\mathscr{M} \rightleftarrows \mathscr{M} : \mathit{Triv}, \qquad \mathit{Triv} : \mathscr{M} \rightleftarrows G\mathscr{M} : (-)^G.$$

Definition

Have $G\mathscr{V}$-category $\mathscr{M}_G$ and its G-fixed $\mathscr{V}$-category $G\mathscr{M}$, changing enrichment along $\mathit{Triv} : \mathscr{V} \rightleftarrows G\mathscr{V} : (-)^G$.

Since $G\mathscr{M} = \mathscr{P}_{\mathscr{V}}(*_G^{op}, \mathscr{M})$, have projective model str. on $G\mathscr{M}$. Correct model str. accounts for subgroups.

- Restriction $G\mathscr{M} \to H\mathscr{M}$ for $H \leq G$ allows to define $(-)/H$ and $(-)^H$.
- Say $f : X \to Y$ in $G\mathscr{M}$ is a weak equivalence (fibration) if $f^H : X^H \to Y^H$ is a weak equivalence (fibration).

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Theorem

If the functors $(-)^H$ are good, then the above determines a $\mathscr{V}$-model structure on $G\mathscr{M}$.

$G\mathscr{M}$ as $\mathscr{O}_G$-diagrams

Would like to model $G\mathscr{M}$ as $\mathscr{O}_G$ for some $\mathscr{O}_G$.
Take $\mathscr{O}_G$ to be full $\mathscr{V}$-subcategory of $G\mathscr{V}$ on $U[G/H]$.
Have adjunction $\mathbb{J}^* : \mathscr{P}_{\mathscr{V}}(\mathscr{O}_G, \mathscr{M}) \rightleftarrows G\mathscr{M} : \mathbb{J}_*$, with $\mathbb{J} : *_G \to \mathscr{O}_G$ picks out $U[G/e]$.

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Have adjunction $\mathbb{J}^* : \mathscr{P}_{\mathscr{V}}(\mathscr{O}_G, \mathscr{M}) \rightleftarrows G\mathscr{M} : \mathbb{J}_*$, with $\mathbb{J} : *_G \to \mathscr{O}_G$ picks out $U[G/e]$.

Theorem

The adjunction $(\mathbb{J}^*, \mathbb{J}_*)$ is a Quillen adjunction. It is a Quillen equivalence if the functors $(-)^H$ are cellular.

Take $\mathscr{M} = \mathscr{V} = \mathscr{S}$, some good category of spectra (eg. orthogonal). For G compact Lie, define $G\mathscr{S}$ as above. These are **naive G-spectra**.

Theorem

$G\mathscr{S}$ has an $\mathscr{S}$-model category structure, with weak equivalences and fibrations created by $(-)^H$, $H \leq G$ closed. There is a Quillen equiv. of $\mathscr{S}$-model categories

$$\mathbb{J}^* : \mathscr{P}_{\mathscr{S}}(\mathscr{O}_G, \mathscr{S}) \rightleftarrows G\mathscr{S} : \mathbb{J}_*.$$

Objects of $\mathscr{O}_G$ are $\Sigma^\infty(G/H)_+$.

Equivariant Spectra

Take $\mathscr{M} = \mathscr{V} = \mathscr{S}$, some good category of spectra (eg. orthogonal). For G compact Lie, define $G\mathscr{S}$ as above. These are **naive G-spectra**.

Theorem

$G\mathscr{S}$ has an $\mathscr{S}$-model category structure, with weak equivalences and fibrations created by $(-)^H$, $H \leq G$ closed. There is a Quillen equiv. of $\mathscr{S}$-model categories

$$\mathbb{J}^* : \mathscr{P}_{\mathscr{S}}(\mathscr{O}_G, \mathscr{S}) \rightleftarrows G\mathscr{S} : \mathbb{J}_*.$$

Objects of $\mathscr{O}_G$ are $\Sigma^\infty(G/H)_+$.

But, $G\mathscr{S}$ is wrong category of equivariant spectra.

Genuine Equivariant Spectra

Fix $\mathscr{S} = O\mathscr{S}$ (orthogonal spectra).
Naive category $G\mathscr{S} = G\mathscr{S}_{U^G}$, (only trivial representations).
True category is $G\mathscr{S}_U$, U some complete G-universe (contains all representations).

Definition

For $X \in G\mathscr{S}_U$, $H \leq G$ and $q \in \mathbb{Z}$, define

$$\pi_q^H(X) := \operatorname*{colim}_V \pi_q((\Omega^V X(V))^H).$$

Genuine Equivariant Spectra

Theorem (Mandell-May)

$G\mathscr{S}_U$ is monoidal model cat (we=π_*-iso). There is "change of universe" Quillen pair

$$i_* : G\mathscr{S}_{U^G} \rightleftarrows G\mathscr{S}_U : i^*$$

with i_* strong symmetric monoidal.

Since $Triv : \mathscr{S} \rightleftarrows G\mathscr{S}_{U^G} : (-)^G$ is also Quillen pair with $Triv$ strong symmetric monoidal, can change enrichment to get $\mathscr{S}$-category $G\mathscr{S}_U$.

$G\mathscr{S}_U$ as $G\mathscr{B}$-diagrams

Let $G\mathscr{B} \subseteq G\mathscr{S}_U$ be full $\mathscr{S}$-subcat on $\Sigma^\infty(G/H)_+$.

Theorem

Have Quillen equiv of $\mathscr{S}$-categories

$$\mathbb{I}_! : \mathscr{P}(G\mathscr{B}, \mathscr{S}) \rightleftarrows G\mathscr{S}_U : \mathbb{I}^*.$$

Next?

Now take G finite.

Hope

There is (discrete) category GB with Quillen equivalence of $\mathscr{S}$-categories

$$\mathscr{P}(GB, \mathscr{S}) \rightleftarrows G\mathscr{S}_U.$$

Thanks!