1. Road map, where we are

Notations are not definitive. I've deleted the superscript fib since the double superscripting looks terrible after texing.

We have the categories $\mathscr{E}_G$ and $G\mathscr{E}$ of finite G-sets. We give $\mathscr{E}_G$ the bicategorical horizontal composition pairing using ε, as is forced by studying tom Dieck splitting.

We have the NU categories $\mathscr{F}_G$ and $G\mathscr{F}$ of orthogonal suspension spectra of finite G-sets. Write $\mathscr{F}_G \vee \mathbf{I}$ and $G\mathscr{F} \vee \mathbf{I}$ for adding units (notation will surely be changed later).

Unwritten NU version of 7.2 in the prequel, together with that result as written, should give:

Theorem 1.1. *The categories $\mathscr{I}\mathscr{S}^{G\mathscr{F}}$ (non-unital presheaves), $\mathscr{I}\mathscr{S}^{G\mathscr{F}\vee\mathbf{I}}$, $\mathscr{I}\mathscr{G}^{\mathbb{K}G\mathscr{E}}$ (non-unital presheaves), and $\mathscr{I}\mathscr{G}^{\mathbb{K}G\mathscr{E}\vee\mathbf{I}}$ are Quillen equivalent.*

We can apply $\mathbb{N}$ to these, or we can apply $\mathbb{M}$; notations as in Mandell-May.

The same arguments applied to presheaves in $\mathscr{M}$ should prove the analogue.

Theorem 1.2. *The categories $\mathscr{M}^{\mathbb{N}G\mathscr{F}}$ (non-unital presheaves), $\mathscr{M}^{\mathbb{N}G\mathscr{F}\vee\mathbf{I}}$, $\mathscr{M}^{\mathbb{N}\mathbb{K}G\mathscr{E}}$ (non-unital presheaves), and $\mathscr{M}^{N\mathbb{K}G\mathscr{E}\vee\mathbf{I}}$ are Quillen equivalent.*

If Dugger-Shipley works technically and the gods are good one can hope for direct comparisons among these. I'm pretty sure we know that $\mathscr{I}\mathscr{S}^{\mathscr{B}}$ is Quillen equivalent to $\mathscr{M}^{\mathscr{B}}$. My guess is that this would want to go in two steps as follows, but I wrote this before sketching out the Dugger-Shipley theory, so it should be adapted to that. This may be overoptimistic.

Theorem 1.3. *The four presheaf categories of Theorem 1.2 are Quillen equivalent respectively to four presheaf categories $\mathscr{I}\mathscr{S}^{\mathbb{N}^{\#}\mathbb{N}G\mathscr{F}}$ (non-unital presheaves), $\mathscr{I}\mathscr{S}^{\mathbb{N}^{\#}\mathbb{N}G\mathscr{F}\vee\mathbf{I}}$, $\mathscr{I}\mathscr{G}^{\mathbb{N}^{\#}\mathbb{N}\mathbb{K}G\mathscr{E}}$ (non-unital presheaves), and $\mathscr{I}\mathscr{G}^{\mathbb{N}^{\#}\mathbb{N}\mathbb{K}G\mathscr{E}\vee\mathbf{I}}$. In turn these are Quillen equivalent equivalent respectively to the four presheaf categories of Theorem 1.1.*

I'm queazy about one thing in the unit/non-unit comparison. In the EKMM version, we have canonical quasi-identities $S \longleftarrow \mathbf{S} \longrightarrow G\mathscr{F}(A,A)$. Now the equivalence between $G\mathscr{M}$ and $\mathscr{M}^{\mathscr{B}}$ for sure talked about genuine unital sheaves. We pull back along $\mathscr{F} \longrightarrow \mathscr{B}$, and it is unclear to me how precisely we account for the unital/non-unital dichotomy in asserting that $\mathscr{M}^{\mathscr{B}}$ "$=$" $\mathscr{M}^{G\mathscr{F}}$, even though it feels almost tautological [Notations about when to write G and when not are a mess, but I'm sure you understand.] There is something about this that doesn't feel right. Maybe a version of the quasi-equivalence trick will work here, perhaps after transferring enrichement from $\mathscr{M}$ to $\mathscr{I}\mathscr{S}$ to get a cofibrant unit object. Since we don't need the "pointed" condition, there might be a variant of this trick that rigorously compares the thing we get by adjoining identities to the version of $\mathscr{B}$ that we are seeing. The $\mathscr{F}$ in 7.5, fortuitously, should be the $\mathscr{F}$ here. Modulo fiddling, the quasi-identies should give something close enough to the ζ_d there, since nothing strict is needed. Then $\mathscr{B}$ on the left, since $\mathscr{F}$ maps into $\mathscr{B}$, and $G\mathscr{F} \vee \mathbf{I}$ on the right, since it maps onto $G\mathscr{F}$, seems to give just the right kind of structure. This is very rough, but at least gives an idea.

With less speculation and reliance on [1], for sure we have

$$\mathscr{I}\mathscr{S}^{\mathscr{B}} \simeq G\mathscr{I}\mathscr{S} \simeq G\mathscr{M} \simeq \mathscr{M}^{\mathscr{B}}$$

Barratt-Quillen surely should give the left equivalence (using NU presheaves) in

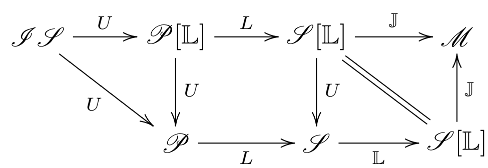
$$\mathscr{I}\mathscr{S}^{\mathbb{K}G\mathscr{E}} \simeq \mathscr{I}\mathscr{S}^{G\mathscr{F}} \simeq \mathscr{I}\mathscr{S}^{G\mathscr{F}\vee\mathbf{I}}.$$

It seems unimaginable that we cannot get equivalences between the last two and their $\mathscr{M}$-counterparts, and depending on queaziness we should have the second equivalence in

$$\mathscr{M}^{G\mathscr{F}} \simeq \mathscr{M}^{G\mathscr{F}\vee\mathbf{I}} \simeq \mathscr{M}^{\mathscr{B}}.$$

I've completely lost the thread about keeping track of when we do or do not need the $\mathscr{D}(i,j)$ to be cofibrant. The only place I guess I'm sure it is important is in interpreting the Atiyah duality comparison. Note for that, by the way, that we have three rather than two things in sight, which is why I kept the middle row in the 3×3 diagram on page 18: we have $F(\mathbb{A},\mathbb{B})$, $B \wedge D\mathbb{A}$, and $\mathbb{B} \wedge \mathbb{A}$. We still have the strict unit in the middle level and it feels "almost" cofibrant.

We have no reason to believe that $\mathbb{K}G\mathscr{E}$ is cofibrant in any sense stronger than levelwise CW homotopy types (which we do have). However, we can rectify this. This is maybe digressive, but here is how. We have the functor $\mathbb{M}\colon \mathscr{I}\mathscr{S} \longrightarrow \mathscr{M}$, which is weakly equivalent to $\mathbb{N}$ [2]. According to the five author paper, up to homotopy equivalence the May machine is a Quillen left adjoint from E_∞-spaces to LMS $\mathscr{S}$. We can hit it with $\mathbb{F}\colon \mathscr{S} \longrightarrow \mathscr{M}$. The functor $\mathbb{M}$ is the top composite in the diagram



We have a natural weak equivalence $\varepsilon\colon \mathbb{L}U \longrightarrow \mathrm{id}$, the unit of an adjunction, and the rest of the diagram commutes. The May machine $\mathbb{K}'$ to $\mathscr{S}$ is related to the May machine $\mathbb{K}$ to $\mathscr{I}\mathscr{S}$ by $\mathbb{K}' = LU\mathbb{K}$. Therefore we have a natural weak equivalence $\varepsilon\colon \mathbb{F}\mathbb{K}' \longrightarrow \mathbb{M}\mathbb{K}$. We can think of $\mathbb{F}\mathbb{K}'$ as the May machine to $\mathscr{M}$, and viewed that way it is a Quillen left adjoint naturally equivalent to $\mathbb{M}\mathbb{K}$. The functor $\mathbb{M}$ is lax symmetric monoidal [2, 7.11]. There is a question about transporting that along ε that I haven't thought about but should be ok. Modulo that, this should give

$$\mathscr{M}^{\mathbb{F}\mathbb{K}'G\mathscr{E}} \simeq \mathscr{M}^{\mathbb{M}\mathbb{K}G\mathscr{E}}$$

showing that, when pushed into $\mathscr{M}$ by the two obvious routes, we get eequivalent presheaf categories with either interpretation. Using the weak equivalence $\mathbb{N} \longrightarrow \mathbb{M}$ and a Dugger-Shipley type argument should give

$$\mathscr{M}^{\mathbb{M}\mathbb{K}G\mathscr{E}} \simeq \mathscr{M}^{\mathbb{N}\mathbb{K}G\mathscr{E}} \simeq \mathscr{I}\mathscr{S}^{\mathbb{K}G\mathscr{E}}.$$

The four displayed chains of equivalences give all the comparisons we should need. More details to go, but the end is in sight.

References

[1] D. Dugger and B. Shipley. Enriched model categories and an application to homotopy endomorphism spectra. Theory and Application of Categories 18(2007), 400–439.

[2] J. A. Mandell and J.P. May. Equivariant orthogonal spectra and S-modules. Memoirs Amer. Math. Soc. Vol 159. 2002.