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Since I don't have time to think right now, this will be a direct transcription of my notes from the early 1980's on permutative G -categories. No changes from what I wrote then, minor updating comments in brackets.

1. THE CATEGORICAL E_∞ G -OPERAD \mathcal{D}_G

We begin by constructing a monstrous big G -monoid of canonical G -sets.

[Draft one(?)] Thus let M be the free monoid generated by $\coprod_{H \subset G}^\infty (G/H)$, the disjoint union of a countably infinite number of copies of each G/H , with unit element the first copy of G/G . Let ${}^j M$ denote the disjoint union of j copies of M .

[Draft two(?)] Thus let M be the disjoint union of countably many copies of each finite product $G/H_1 \times \cdots \times G/H_r$, where the H_i run over all finite sets of subgroups of G . The product on M is given by the identity maps

$$\begin{array}{c}
 (G/H_1 \times \cdots \times G/H_r)_m \times G/J_1 \times \cdots \times G/J_s)_n \\
 \downarrow \\
 (G/H_1 \times \cdots \times G/H_r \times G/J_1 \times \cdots \times G/J_s)_{m+n}
 \end{array}$$

If r and m (or s and n) are 1 and H_1 (or J_1) is G/G , delete the corresponding point from the product on the right. Here copies are indexed on the positive integers and the subscripts m and n specify which copies we are considering. The first copy of the point G/G is a two-sided identity element, and this multiplication is evidently associative. Let ${}^j M$ denote the disjoint union of j copies of M .

[There is a Draft 3 below that is perhaps the definitive version of M .]

Define a category $\mathcal{S}_G(j)$ as follows. The objects are ordered j -tuples $\langle m_1, \cdots, m_j \rangle$ of distinct points of the ambient G -space ${}^j M$. [i^{th} point in i^{th} copy ?] We note that M is a G -monoid with action inherited from that on the generating set via the requirement that the multiplication $M \times M \rightarrow M$ be a G -map, where G acts diagonally on $M \times M$. The morphisms of $\mathcal{S}_G(j)$ are bijections of (ordered) tuples, so that there is precisely one morphism from any one object to any other. In particular, every object is initial and terminal, so the category is contractible. Define a left G -action and a right Σ_j -action on $\mathcal{S}_G(j)$ by the specifications

$$g \langle m_1, \cdots, m_j \rangle = \langle gm_1, \cdots, gm_j \rangle \quad \text{and} \quad \langle m_1, \cdots, m_j \rangle \sigma = \langle m_\sigma(1), \cdots, m_\sigma(j) \rangle$$

on objects. Visibly, these actions commute and so (by use of inverses in Σ_j) specify a left action of $G \times \Sigma_j$. Let $G \times \Sigma_j$ act on morphisms the only way possible (since there is a unique morphism between each pair of objects).

For a representation $\rho: H \rightarrow \Sigma_j$, $H \subset G$, the fixed subcategory $\mathcal{S}_G(j)^{(H,\rho)}$ is the full subcategory with objects those $\langle m_1, \dots, m_j \rangle$ with

$$h\langle m_1, \dots, m_j \rangle = \langle m_{\rho(h)(1)}, \dots, m_{\rho(h)(j)} \rangle$$

for all $h \in H$. This category is nonempty and, again, every object is initial. Thus each $\mathcal{S}_G(j)^{(H,\rho)}$ is contractible. The action by Σ_j is free.

Define functors

$$\tilde{\gamma}: \mathcal{S}_G(k) \times \mathcal{S}_G(j_1) \times \mathcal{S}_G(j_k) \rightarrow \mathcal{S}_G(j_1 + \dots + j_k)$$

by

$$\begin{aligned} \tilde{\gamma}(\langle m_1, \dots, m_k \rangle; \langle n_{1,1}, \dots, n_{1,j_1} \rangle, \dots, \langle n_{k,1}, \dots, n_{k,j_k} \rangle) \\ = \langle m_1 n_{1,1}, \dots, m_1 n_{1,j_1}, \dots, m_k n_{k,1}, \dots, m_k n_{k,j_k} \rangle. \end{aligned}$$

This requires interpretation. Each m_r is a point of ${}^k M$ and so lies in a copy of M . We pay no attention to which copy. Each $n_{r,s}$ is a point of ${}^{j_r} M$. We use the evident ‘‘sum’’ identification

$${}^{j_1} M \amalg \dots \amalg {}^{j_k} M = {}^{j_1 + \dots + j_k} M$$

to regard $n_{r,s}$ as a point in some particular copy of M in ${}^{j_1 + \dots + j_k} M$. Then $m_r n_{r,s}$ is interpreted as lying in that same copy of M , and is of course obtained by the monoid structure. The use of blocks ensures that we obtain a $(j_1 + \dots + j_k)$ -tuple of distinct points.

The unity and associativity formulae for an operad are direct consequences of the fact that M is a monoid, and the equivariance formulae are formal trivialities (?!).

We define $\mathcal{D}_G(j) = B\mathcal{S}_G(j)$ and have an E_∞ G -operad.

Note that we have an orbit category $\mathcal{S}_G(j)/\Sigma_j$ specified simply by use of unordered j -tuples (or j -pointed sets). Let \mathcal{E}_G be the coproduct over j of these categories. We call \mathcal{E}_G the Burnside category of G . It is G -permutative, or will be so as soon as I define what that means, and $B\mathcal{E}_G$ will equal $D_G S^0$ as a \mathcal{D}_G -space. The associated G -spectrum will be $S_G = \Sigma_G^\infty S^0$.

The category $\tilde{\Sigma}_j$ may be thought of as having as objects all ordered j -tuples of ${}^j\{e\}$, where $\{e\}$ is the unit monoid. $M\{e\}$ above is bigger, $\{e\} \subset M\{e\}$ embedding the old \mathcal{D} in $\mathcal{D}_{\{e\}}$ as an equivalent sub E_∞ operad.

2. EQUIVARIANT ALGEBRAIC K -THEORY

Let R be a commutative ring, G a group (possibly both topologized?) and let $R[G]$ denote the group ring. Let \otimes mean \otimes_R and let $R[G/H]$ be the evident left $R[G]$ -module. Let $\mathcal{G}\mathcal{L}_G(R)$ be the disjoint union over $j \geq 0$ of the following categories. Let $R[M]$ be the monoid ring of the monoid M . Thus $R[M]$ is the direct sum of countably many copies of each left $R[G]$ -module

$$R[G/H_1] \otimes \dots \otimes R[G/H_r].$$

Let ${}^j R[M]$ denote the direct sum of j copies of $R[M]$. The objects of the j^{th} category $\mathcal{G}\mathcal{L}_G(R, j)$ are the j -dimensional free sub R -modules of ${}^j R[M]$ with specified *unordered* basis in ${}^j M$. The morphisms are the isomorphisms of R -modules. The

category $\mathcal{GL}_G(R, j)$ is given the evident left G -action, induced from that on jM by conjugation.

It follows that $\mathcal{GL}_G(R, 0)$ is the trivial G -category with one object 0 . There is an evident strictly associative and unital sum operation

$$\oplus : \mathcal{GL}_G(R, j) \times \mathcal{GL}_G(R, k) \longrightarrow \mathcal{GL}_G(j + k).$$

It admits the natural commutativity isomorphism c specified on (V, W) in the obvious way, $c(v + w) = w + v$. All of this structure is G -equivariant.

Define functors

$$\tilde{\theta}_k : \mathcal{S}_G(k) \times \mathcal{GL}_G(R)^k \longrightarrow \mathcal{GL}_G(R)$$

as follows. Write objects of $\mathcal{GL}_G(R)$ as free R -modules on the objects of $\amalg \mathcal{S}_G(j)/\Sigma_j$. Indeed we may think of these categories as having the same objects. Due to the diagonals implicit in $\tilde{\gamma}$, the $\tilde{\gamma}$ induce $\tilde{\theta}_k$ in an evident way on objects, and then $\tilde{\theta}_k$ is $G \times \Sigma_k$ -equivariant. On morphisms

$$(f; g_1, \dots, g_k) : (m; n_1, \dots, n_k) \longrightarrow (m'; n'_1, \dots, n'_k),$$

define $\tilde{\theta}_k$ as the evident composite

$$\begin{array}{c} R[\tilde{\gamma}(m; n_1 \cdots, n_k)] \\ \downarrow \cong \\ R[n_1] \oplus \cdots \oplus R[n_k] \\ \downarrow g_1 \oplus \cdots \oplus g_k \\ R[n'_1] \oplus \cdots \oplus R[n'_k] \\ \downarrow \cong \\ R[\tilde{\gamma}(m'; n'_1 \cdots, n'_k)]. \end{array}$$

The unlabelled isomorphisms are given on bases in the in the obvious way [Huh??] ($m_1 n_1 \amalg \cdots \amalg m_k n_k \leftrightarrow n_1 \amalg \cdots \amalg n_k$, etc). This is $G \times \Sigma_k$ -equivariant, and easy verifications show that we obtain an action of \mathcal{D}_G on $B\mathcal{GL}_G(R)$ via $B\tilde{\theta}$.

3. THE BURNSIDE CATEGORY $\mathcal{E}_G = \amalg_{j \geq 0} \mathcal{E}_G(j)$: BIPERMUTATIVE VERSION

[Draft 3: not integrated with drafts above, a bit different notation, maybe better.]

Let S be the disjoint union over $H \subset G$ of the disjoint union of countably many copies of G/H . Let M be the free monoid generated by the set S . The evident G action on S and the trivial G action on the unit 1 induce a G action on M such that the product is a G -map. Let jM be the disjoint union of j copies of M .

The objects of $\mathcal{E}_G(j)$ are the (unordered) j -point subsets of jM with one point in each copy of M . The morphisms are the bijections. G acts on $\mathcal{E}_G(j)$ by translation of objects and conjugation of morphisms. The sum

$$\oplus : \mathcal{E}_G(j) \times \mathcal{E}_G(k) \longrightarrow \mathcal{E}_G(j + k)$$

is obtained by identifying ${}^jM \amalg {}^kM$ with ${}^{j+k}M$ in the obvious way and taking disjoint unions of sets and maps. The category $\mathcal{E}_G(0)$ is the trivial category with one object 0 (thought of as the empty set) and is a two-sided unit for \oplus . Clearly \oplus is associative, and the obvious transposition reordering of copies of M as ${}^{j+k}M$ gives a natural commutativity isomorphism.

Define

$$\otimes: \mathcal{E}_G(j) \times \mathcal{E}_G(k) \longrightarrow \mathcal{E}_G(jk)$$

as follows. Identify $\{1, 2, \dots, jk\}$ with $\{(a, b) | 1 \leq a \leq j, 1 \leq b \leq k\}$, ordered lexicographically. For objects P of $\mathcal{E}_G(j)$ and Q of $\mathcal{E}_G(k)$, the element of $P \otimes Q$ in the $(a, b)^{\text{th}}$ copy of M is the product in M of the element of P in the $(a, b)^{\text{th}}$ copy of M is the product in M of the element of P in the a^{th} copy of M and the element of Q in the b^{th} copy of M . The definition of \otimes on maps is then evident, $(f \otimes g)(pq) = f(p)g(q)$. The set $\{1\}$ regarded as an object of $\mathcal{E}_G(1)$ is a two-sided unit for \otimes . Again, \otimes is associative, and the commutativity isomorphism $P \otimes Q \longrightarrow Q \otimes P$ sends pq in the $(a, b)^{\text{th}}$ copy to qp in the $(b, a)^{\text{th}}$ copy of M ; its naturality is clear.

The right distributive law $(P \oplus Q) \otimes R = (P \otimes R) \oplus (Q \otimes R)$ is satisfied and the coherence diagrams required of a bipermutative category are easily seen to commute. The key to understanding this definition is to keep firmly in mind that pigeonholing of points in copies of M is not to be regarded as prescribing an ordering of the points of objects. Thus $\mathcal{E}_G(j)$ is the Σ_j -orbit category of an analogous category of ordered sets. The size of S ensures that $B\mathcal{E}_G(j)$ is a model for $B(G, \Sigma_j)$.

4. PERMUTATIVE G -CATEGORIES

4.1. Segalic definition. Let $(\mathcal{A}, \square, *, c)$ be a weakly permutative G -category [undefined, naive permutative category with G action] and let $\tilde{\mathcal{A}}_n$ be its weakly permutative G -category of n -fold sums as specified in Step 1 of Construction 10. [Reference to: J.P. May. The spectra associated to permutative categories. Topology 17(1978), 225–228.] Step 2 of Construction 10 gives $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}_n\}$ a structure of weak $\mathcal{F}G$ -category [naive version of Segal Γ - G -space]. To ensure that this is actually an $\mathcal{F}G$ -category, so that $B\tilde{\mathcal{A}}$ is an $\mathcal{F}G$ -space, we need functors $\nu_\rho: (\mathcal{A}^n)^\phi \longrightarrow (\tilde{\mathcal{A}}_n)^\phi$ which provide inverse equivalences to the restrictions $\delta^\rho: (\tilde{\mathcal{A}}_n)^\phi \longrightarrow (\mathcal{A}^n)^\phi$ for each $\rho: H \longrightarrow \Sigma_n$. If ρ is trivial, we can take $\nu_\rho = \nu^\rho$. [Notation ν from op cit]. Otherwise ν will not be ρ -equivariant and this fails.

The existence of the ν_ρ is of importance. They should not be regarded as part of the structure: we are really concerned only with the properties of δ . This condition, the existence of the ν_ρ , provides our first way of specifying the notion of a permutative G -category.

4.2. Operadic definition. Our second way is simply to require a categorical action of the categorical E_∞ G -operad on the G -category \mathcal{A} . We do not have to presuppose a weakly permutative structure on \mathcal{A} , since this results via the embedding of the translation operad $\tilde{\Sigma}$ in \mathcal{S}_G . Here we are given the commutative diagrams

$$\begin{array}{ccc} \mathcal{S}_G(k) \times \mathcal{S}_G(j_1) \times \cdots \times \mathcal{S}_G(j_k) \times \mathcal{A}^j & \xrightarrow{\tilde{\gamma} \times \text{id}} & \mathcal{S}_G(j) \times \mathcal{A}^j \\ \downarrow \cong & & \downarrow \tilde{\theta}_j \\ \mathcal{S}_G(k) \times \mathcal{S}_G(j_1) \times \mathcal{A}^{j_1} \cdots \times \cdots \times \mathcal{S}_G(j_k) \times \mathcal{A}^{j_k} & \xrightarrow{\text{id} \times \tilde{\theta}_{j_1} \times \cdots \times \tilde{\theta}_{j_k}} & \mathcal{S}_G(k) \times \mathcal{A}^k \\ & & \uparrow \tilde{\theta}_k \end{array}$$

where $j = j_1 + \cdots + j_k$.

Let e_k be the canonical unit object in $\mathcal{S}_G(k)$, the ordered k -tuple in ${}^k M$ given by the units of the k copies of M . Then e_0 gives the unit $* \in \mathcal{A}$, e_1 induces the identity of \mathcal{A} , and e_2 induces a product $\square: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$. The unique isomorphism $e_2 \rightarrow e_2 \tau$ in $\mathcal{S}_G(2)$, where $\tau \in \Sigma_2$ is the transposition, induces $c: \square \rightarrow \square \circ \tau$. The diagram and easy calculations with $\tilde{\gamma}$ demonstrate that $(\mathcal{A}, \square, *, c)$ is a weakly permutative G -category.

4.3. Operadic vs Segalic. I claim that the operadic structure implies the Segalic properties. That is, I can construct the ν_ρ if I am given the $\tilde{\theta}_k$. [Take $k = n$ to reconcile notations.] Given $\rho: H \rightarrow \Sigma_n$, we use ρ to regard \underline{n} as an ordered “ n -pointed” H -set. Write $\underline{n} = \amalg_r n_r H / K_R$ as an H -set. Form

$$G \times_{H_r} \underline{n} \cong \amalg_r n_r G \times_H (H/K_R) \cong \amalg_r n_r G / K_r.$$

We have $\underline{n} = H \times_H \underline{n} \subset G \times_H \underline{n}$ as a sub H -set. Fix an embedding of $G \times_H \underline{n}$ as a sub G -set of M and regard M as the first copy of M in ${}^n M$. Define

$$\nu_\rho: (\mathcal{A}^n)^\rho \rightarrow (\tilde{\mathcal{A}}_n)^\rho$$

as follows. [Notations op cit. and/or suspect.] For $s \subset \underline{n}$ with $|s - \{0\}| = q$ and for $A_i \in (\mathcal{A}^n)^\rho$, so that $hA_i = A_{\rho(h)(i)}$ for all $h \in H$, set

$$A_s = \tilde{\theta}_q(s)(A_{s_1}, \dots, A_{s_q}) \in \mathcal{A},$$

where $0 < s_1 < \cdots < s_q$, $s_a \in s$. This makes sense both on objects and, with s interpreted as id_s , on morphisms. We then have

$$\begin{aligned} hA_s &= h\tilde{\theta}_q(s)(A_{s_1}, \dots, A_{s_q}) \\ &= \tilde{\theta}_q(hs)(hA_{s_1}, \dots, hA_{s_q}) \\ &= \tilde{\theta}_q(\rho(h)(s))(A_{\rho(h)(s_1)}, \dots, A_{\rho(h)(s_q)}) = A_{\rho(h)(s)}, \end{aligned}$$

where the last equality is obtained by using a permutation to rearrange $\rho(h)(s)$ in increasing order and exploiting the Σ_q -equivariance of the the functor $\tilde{\theta}_q$. To obtain the coherent equivalences $i_{(s,t)}: A_{s \cup t} \rightarrow A_s \square A_t$ for $s \cup t \subset \underline{n}$, consider the following instance of the basic diagram, $|t - \{0\}| = r$. [— t — = r as written, sic?]

$$\begin{array}{ccc} \mathcal{S}_G(2) \times \mathcal{S}_G(q) \times \mathcal{S}_G(r) \times \mathcal{A}^{q+r} & \xrightarrow{\tilde{\gamma} \times \text{id}} & \mathcal{S}_G(q+r) \times \mathcal{A}^{q+r} \\ \cong \downarrow & & \downarrow \tilde{\theta}_{q+r} \\ \mathcal{S}_G(2) \times \mathcal{S}_G(q) \times \mathcal{A}^q \times \mathcal{S}_G(r) \times \mathcal{A}^r & \xrightarrow{\text{id} \times \tilde{\theta}_q \times \tilde{\theta}_r} & \mathcal{S}_G(2) \times \mathcal{A}^2 \\ & & \uparrow \tilde{\theta}_2 \end{array}$$

Start with $e_2 \in \mathcal{S}_G(2)$, $s \in \mathcal{S}_G(q)$, $t \in \mathcal{S}_G(r)$. Bottom yields $A_s \square A_t$. $\tilde{\gamma}(e_2; s, t)$ is $s \amalg t$, a $(q+r)$ -pointed subset of ${}^{q+r} M$, with the points of s in the first copy of M and the points of t in the $(r+1)^{\text{st}}$ copy. If (u_1, \dots, u_{q+r}) is $s \cup t$ arranged in increasing order, then

$$A_{s \cup t} = \tilde{\theta}_{q+r}(s \cup t)(A_{u_1}, \dots, A_{u_{q+r}}).$$

Let $\sigma \in \Sigma_{q+r}$ be the permutation which when applied to $(s_1, \dots, s_q, t_1, \dots, t_r)$ yields (u_1, \dots, u_{q+r}) . There is a unique morphism in $\mathcal{S}_G(q+r)$ from $s \cup t$ to

$(s \cup t)\sigma$. Here $s \cup t \subset M \subset^{q+r} M$, M being the first copy. This isomorphism, the diagram, and the equivariance of $\tilde{\theta}_{q+r}$ yield $i_{(s,t)}$ as $\tilde{\theta}_{q_r}(j_{(s,t)}), j_{(s,t)}: s \cup t \cong (s \cup t)\sigma$.

The coherence of the $i_{(s,t)}$ and their ρ -invariance are easy verifications. Since $\tilde{\theta}_1(s)$ is equivalent to the identity for any 1-point set s , $\nu_\rho \circ \delta^\rho$ is certainly equivalent to the identity. The requisite equivalence $\xi: \delta^\rho \circ \nu_\rho \longrightarrow \text{id}$ is “tautological”. Given $\langle A_s, i_{(s,t)} \rangle \in (\mathcal{A}_n)^\rho$, the sum structures specify equivalences

$$A_s \xrightarrow{i} A_{s_1} \square \cdots \square A_{s_q} \xrightarrow{j^{-1}} \tilde{\theta}_q(s)(A_{s_1}, \dots, A_{s_q}),$$

and these yield the required natural equivalence ξ by easy verifications.

4.4. Note the logical situation. Nonequivariantly, $\tilde{\Sigma}$ -categories and permutative categories were one and the same concept. We can take that as an equivariant definition, via \mathcal{S}_G , but here the notion becomes vastly more structured than can be captured by the derived $\mathcal{F}G$ -space, for which the ν_ρ 's are not part of the structure. That is, an operad action makes explicit what is implicit in an $\mathcal{F}G$ -category. Concomitantly, $\mathcal{F}G$ -categories are presumably much more common. [Not so!]. The “correct” notion of permutative category should be $(\mathcal{A}, \square, *, c)$ plus “?”, and it still remains to give the simplest possible “?”, with a concomitant symmetric monoidal notion.

4.5. Reminder on the uniqueness theorem. However defined, it is vital that \mathcal{A}_n should be permutative when \mathcal{A} is. Otherwise uniqueness [of op cit above] fails. This actually seems a bit delicate, and is the heart of the theory. No problem with stronger second definition. We define $\tilde{\theta}_k: \mathcal{S}_G(k) \times \mathcal{A}_n^k \longrightarrow \mathcal{A}_n$ termwise. That is, we let [notations op cit]

$$\tilde{\theta}_k(m, \langle A_s^1 \rangle, \dots, \langle A_s^k \rangle)_s = \tilde{\theta}_k(m, A_s^1, \dots, A_s^k).$$

Σ_k -equivariance, unit, and Adem diagram are clearly inherited. Similarly on morphisms. The $i_{(s,t)}^r: A_{s \cup t}^r \longrightarrow A_s^r \square A_t^r$ yield

$$\tilde{\theta}_k(m, i_{(s,t)}^1, \dots, i_{(s,t)}^k): \tilde{\theta}_k(m, A_{(s,t)}^1, \dots, A_{(s,t)}^k) \longrightarrow \tilde{\theta}_k(m, A_s^1 \square A_t^1, \dots, A_s^k \square A_t^k).$$

The target is equal to

$$\tilde{\theta}_{2k}(\tilde{\gamma}(m; e_2, \dots, e_2), A_s^1, A_t^1, \dots, A_s^k, A_t^k).$$

We must compare this with

$$\tilde{\theta}_k(m, A_s^1, \dots, A_s^k) \square \tilde{\theta}_k(m, A_t^1, \dots, A_t^k),$$

which is equal to

$$\tilde{\theta}_{2k}(\tilde{\gamma}(e_2; m, m), A_s^1, \dots, A_s^k, A_t^1, \dots, A_t^k).$$

We do this via the morphism

$$\tilde{\gamma}(m; e_2, \dots, e_2) \longrightarrow \tilde{\gamma}(e_2; m, m)\mu$$

where μ is the evident shuffle isomorphism. This gives the new $i_{(s,t)}$, and the coherence follows by standard diagram chases. With this definition, the permutative G -category uniqueness theorem is now fully established. With the first definition, \mathcal{A}_n seems *not* to inherit permutativity. For that, one would need a commutativity relation between $\nu_\rho: (\mathcal{A}^n)^\rho \longrightarrow (\mathcal{A}_n)^\rho$ and \square sums, and this appears not to drop out automatically.

4.6. Permutative G -category definition and discussion. [This may have motivated and preceded the introduction of M above.]

Let S be a finite G -set and let $\mathcal{P}(j, S)$ be the category of j -pointed ordered subsets of S and bijective functions. Recall that a weakly permutative G -category $(\mathcal{A}, \square, *, c)$ is equivalent to a G -action of the category level translation operad $\tilde{\Sigma}$. Let ε_j be the trivial j -pointed G -set and note that $\tilde{\Sigma}_j$ may be identified with $\mathcal{P}(j, \varepsilon_j)$. \mathcal{A} is a permutative G -category if it comes with $G \times \Sigma_j$ -equivariant functors

$$\theta_{j,s}: \mathcal{P}(j, S) \times \mathcal{A}^j \longrightarrow \mathcal{A}$$

[sometimes abbreviated θ_j] such that

- (i) $\mathcal{P}(j, e_j) \times \mathcal{A}^j \longrightarrow \mathcal{A}$ specifies $(\mathcal{A}, \square, *, c)$
- (ii) If $S \subset T$, $\theta_{j,S}$ is the composite

$$\mathcal{P}(j, S) \times \mathcal{A}^j \longrightarrow \mathcal{P}(j, T) \times \mathcal{A}^j \xrightarrow{\theta_{j,T}} \mathcal{A}.$$

- (iii) The following diagrams commute.

$$\begin{array}{ccc} \mathcal{P}(k, T) \times \mathcal{P}(j_1, S_1) \times \cdots \times \mathcal{P}(j_k, S_k) \times \mathcal{A}^j & \xrightarrow{\tilde{\gamma} \times \text{id}} & \mathcal{P}(j, \coprod_{1 \leq r \leq k} T \times S_r) \times \mathcal{A}^j \\ \cong \downarrow & & \downarrow \tilde{\theta}_j \\ \mathcal{P}(k, T) \times \mathcal{P}(j_1, S_1) \times \mathcal{A}^{j_1} \cdots \times \mathcal{P}(j_k, S_k) \times \mathcal{A}^{j_k} & \xrightarrow{\text{id} \times \tilde{\theta}_{j_1} \times \cdots \times \tilde{\theta}_{j_k}} & \mathcal{P}(k, T) \times \mathcal{A}^k \\ & & \uparrow \tilde{\theta}_k \end{array}$$

where $j = j_1 + \cdots + j_k$. Here

$$\tilde{\gamma}(\langle t_1, \dots, t_k \rangle; \langle s_{1,1}, \dots, s_{1,j_1} \rangle, \dots, \langle s_{k,1}, \dots, s_{k,j_k} \rangle) = \langle t_1 s_{1,j_1}, \dots, t_1 s_{1,j_1}, \dots, t_k s_{k,1}, \dots, t_k s_{k,j_k} \rangle.$$

(i) and (ii) applied to $S \subset S \coprod \varepsilon_j$ make clear that we are looking at “underlying j -fold sums” with suitably twisted group actions.

Intuition: think of $\tilde{\theta}_j(\langle s_1, \dots, s_j \rangle, A_1, \dots, A_j)$ as a copy of $A_1 \square \cdots \square A_j$ indexed on (s_1, \dots, s_j) . G acts by sending $A_1 \square \cdots \square A_j$ indexed on (s_1, \dots, s_j) to $gA_1 \square \cdots \square gA_j$ indexed on (gs_1, \dots, gs_j) . In particular, if $|S| = j$, then

$$(gs_1, \dots, gs_j) = (s_{\rho(g)(1)}, \dots, s_{\rho(g)(j)})$$

for some representation $\rho: G \longrightarrow \Sigma_j$. Here

$$\tilde{\theta}_j(\langle gs_1, \dots, gs_j \rangle, gA_1, \dots, gA_j) = g\tilde{\theta}_j(\langle s_1, \dots, s_j \rangle, A_1, \dots, A_j).$$

$gA_1 \square \cdots \square gA_j$ is indexed on $(s_{\rho(g)(1)}, \dots, s_{\rho(g)(j)})$, but Σ_j -equivariance says this is $gA_{\rho(g)^{-1}(1)} \square \cdots \square gA_{\rho(g)^{-1}(j)}$ indexed on (s_1, \dots, s_j) . In other words, taking S given as ordered and thinking of

$$\tilde{\theta}_j(\langle s_1, \dots, s_j \rangle, A_1, \dots, A_j)$$

as “ $A_1 \square \cdots \square A_j$ ”, we see that we are giving these objects of \mathcal{A} a ρ -twisted G -action

$$g(“A_1 \square \cdots \square A_j”) = “gA_{\rho(g)^{-1}(1)} \square \cdots \square gA_{\rho(g)^{-1}(j)}”$$

To explain [more clearly], perhaps introduce some such notation as $A_1(s_1) \square \cdots \square A_j(s_j)$.

Actually, this is perhaps an ersatz definition on categorical [size] grounds. Use of all finite sets makes (ii) and (iii) suspect: they should hold only up to coherent

natural isomorphism perhaps. That is, it is too much to expect that they will hold on the nose. In particular, verification of (iii) almost demands

$$\prod_{1 \leq r \leq k} T \times S_r = T \times \left(\prod_{1 \leq r \leq k} S_r \right),$$

whereas they are really canonically isomorphic. Picky, but still it is perhaps better to fix one big infinite S and use only it. This can't quite work since γ cannot be defined for (iii), but it does work if we use ${}^j S = S \amalg \cdots \amalg S$ and $\mathcal{P}(j, {}^j S)$ for the j^{th} ambient category. This leads to equivalent (essentially) operad action definition of G -permutative category and thus leads into such a definition nicely. No problem with exposition that way. [Or so I wrote then.]