1. G-SPECTRA

Recall from [?] that for any compact Lie group G and any G-universe U there is a closed symmetric monoidal category $G\mathscr{S}_U$ of equivariant orthogonal spectra. In the case of a complete G-universe U, these are called genuine G-spectra, and they define $RO(G)$-graded equivariant cohomology theories on G-spaces. In the case of a trivial G-universe, these are naive G-spectra or simply spectra with a G-action, and they only define $\mathbb{Z}$-graded equivariant cohomology theories on G-spaces.

Choose a complete G-universe U, and let $i\colon U^G \longrightarrow U$ be the inclusion of the G-fixed universe. Write $\mathscr{S}$ for the closed symmetric monoidal category of orthogonal spectra indexed on the universe U^G. This will be our enriching category $\mathscr{V}$, and our focus will be on categories enriched over $\mathscr{S}$. There are adjunctions

$$\mathscr{T} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathscr{S} \underset{(-)^G}{\overset{\varepsilon^*}{\rightleftarrows}} G\mathscr{S}_{U^G} \underset{i^*}{\overset{i_*}{\rightleftarrows}} G\mathscr{S}_U$$

in which the left adjoints $\Sigma^\infty = F_0$, ε^*, and i_* are strong symmetric monoidal and therefore the right adjoints $\Omega^\infty = \mathrm{ev}_0$, Ω^∞, and i^* are lax symmetric monoidal [**?**, V.1.5][1]. Moreover, equipping the three categories of spectra with their stable model structures, these are all monoidal model categories [**?**, III.4.2, III.7.4] with cofibrant unit objects and the above adjunctions are all Quillen pairs. Thus Propositions **??** and **??** apply to show that $G\mathscr{S}_{U^G}$ and $G\mathscr{S}_U$ are $\mathscr{S}$-model categories, hence $\mathscr{T}$-model categories.

We can use **??** to describe $G\mathscr{S}_{U^G}$ as an $\mathscr{S}$-model category. Write $\Sigma^\infty_+ Y = \Sigma^\infty(Y_+)$ for an unbased G-space Y, where Σ^∞ is the suspension naive G-spectrum functor. Recall that $\mathscr{O}_G$ denotes the full topological subcategory of G-spaces on the orbits G/H. We may adjoin disjoint basepoints to the hom spaces and so view $\mathscr{O}_G$ as enriched in $\mathscr{T}$ rather than $\mathscr{U}$. Analogously, let $\mathscr{S}\mathscr{O}_G \subset G\mathscr{S}_{U^G}$ denote the full $\mathscr{S}$-subcategory of naive G-spectra on the orbit G-spectra $\Sigma^\infty_+ G/H$.

Theorem 1.1. *There is a Quillen $\mathscr{S}$-adjunction*

$$\mathscr{S}^{\mathscr{S}\mathscr{O}_G} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S}_{U^G},$$

and it is a Quillen equivalence.

Proof. The adjunction is immediate from **??**. To see that it is a Quillen equivalence, it suffices to check that the fixed point functor $(-)^G\colon G\mathscr{S}_{U^G} \longrightarrow \mathscr{S}$ preserves the appropriate cotensors and colimits. Since these constructions are performed levelwise, the conclusion follows from [**?**, III.1.6]. Note that since fixed point functors on naive G-spectra preserve weak equivalences, it is not necessary to take fibrant replacements of the objects of $\mathscr{S}\mathscr{O}_G$. □

Lemma 1.2. *There are natural isomorphisms*

$$\underline{\mathscr{S}\mathscr{O}}_G(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K) \cong \Sigma^\infty_+ \mathscr{O}_G(G/H, G/K) \cong S^0_G \odot \mathscr{O}_G(G/H, G/K)_+,$$

where S^0_G is the naive 0-sphere G-spectrum. Therefore the $\mathscr{S}$-categories $\mathscr{S}\mathscr{O}_G$ and $S^0_G[\mathscr{O}_G]$ are isomorphic.

[1] In fact, i^* is also strong symmetric monoidal, but that is not true homotopically, where fibrant approximation intervenes in the specification of i^* on homotopy categories.

Proof. Abbreviate $G\mathscr{S}_{U^G}$ to $G\mathscr{S}$ in this proof. This is a category enriched in $\mathscr{S}$, with morphism spectra denoted $\underline{G\mathscr{S}}(X,Y) \in \mathscr{S}$ for naive G-spectra X and Y. Write S^0_H for S^0_G regarded as an H-spectrum for $H \subset G$. For the first isomorphism,

$$\begin{aligned} \underline{\mathscr{S}\mathscr{O}}_G(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K) &\equiv \underline{G\mathscr{S}}(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K) \\ &\cong \underline{H\mathscr{S}}(S^0_H, \Sigma^\infty_+ G/K) \\ &\cong (\Sigma^\infty_+ G/K)^H \\ &\cong \Sigma^\infty_+ (G/K)^H \cong \Sigma^\infty_+ \mathscr{O}_G(G/H, G/K). \end{aligned}$$

The second isomorphism is general. For any based G-space X,

$$\Sigma^\infty X \cong S^0 \odot X.$$

In fact, the adjunction

$$\mathscr{S}(S^0 \odot X, Y) \cong \mathscr{T}(X, \underline{G\mathscr{S}}(S^0, Y)_0) \cong \mathscr{T}(X, Y_0),$$

where $Y \in G\mathscr{S}$, implies that $S^0 \odot X \cong F_0 X = \Sigma^\infty X$. The functor $S^0 \odot -$ commutes with wedges and, for a set K, K_+ is the wedge over $k \in K$ of copies of S^0. Comparing with **??** but taking basepoints into account, this implies the last statement. □

Using subscripts to denote the category in which presheaves are enriched, we have the isomorphism

$$\mathscr{S}_{\mathscr{S}}^{S^0[\mathscr{O}_G]} \cong \mathscr{S}_{\mathscr{T}}^{\mathscr{O}_G}.$$

We therefore obtain an identification

$$\mathscr{S}_{\mathscr{S}}^{\mathscr{S}\mathscr{O}_G} \cong \mathscr{S}_{\mathscr{T}}^{\mathscr{O}_G}. \tag{1.3}$$

Thus naive G-spectra can be modelled as the category of topological presheaves of spectra on the topological orbit category, enriched over the category $\mathscr{T}$ of based spaces The objects are Note that the conclusion is trivial when G is trivial. This category is clear and explicit.

We now briefly discuss $G\mathscr{S}_U$ as an $\mathscr{S}$-model category. Let $\mathscr{B}_G \subset G\mathscr{S}_U$ denote the full sub $\mathscr{S}$-category on the genuine suspension G-spectra $\Sigma^\infty_+ G/H$. Since genuine G-spectra are not simply G-objects in some $\mathscr{S}$-category, **??** does not readily apply. On the other hand, since the category $\mathscr{S}$ is stable with $\{\mathbf{I} = S^0\}$ as compact generating set, **??** does apply. Here we must replace $\mathscr{B}_G$ by $\mathscr{B}_G^{fib}$, the full $\mathscr{S}$-subcategory of $G\mathscr{S}_U$ on fibrant replacements for the orbits.

Theorem 1.4. *There is a Quillen $\mathscr{S}$-adjunction*

$$\mathscr{S}^{\mathscr{B}_G^{fib}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{S}_U,$$

and it is a Quillen equivalence.

Proof. The $\mathscr{S}$-category $\mathscr{B}_G^{fib}$ is automatically $\mathscr{S}$-good since $\mathscr{S}$ satisfies the monoid axiom [**?**, 12.5]. Since the orbits give a compact generating set in $\mathrm{Ho}(G\mathscr{S}_U)$, **??** gives the conclusion. □

At least when G is finite, we would like to obtain a further reduction, analogous to (**??**), but that is work in progress. An intuitive starting point is that we would like to have a commutative square of Quillen pairs of $\mathscr{S}$-categories

$$\begin{array}{ccc} G\mathscr{S}_{U^G} & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & G\mathscr{S}_U \\ {\scriptstyle\mathbb{T}}\uparrow\downarrow{\scriptstyle\mathbb{U}} & & {\scriptstyle\mathbb{T}}\uparrow\downarrow{\scriptstyle\mathbb{U}} \\ \mathscr{S}^{\mathscr{S}\mathscr{O}_G} & \underset{j^*}{\overset{j_*}{\rightleftarrows}} & \mathscr{S}^{\mathscr{B}_G^{fib}} \end{array},$$

where the pair (j_*, j^*) is induced by an $\mathscr{S}$-functor $j\colon \mathscr{S}\mathscr{O}_G \longrightarrow \mathscr{B}_G^{fib}$. However, we do not know how to obtain such an $\mathscr{S}$-functor. Restriction of i_* gives an $\mathscr{S}$-functor $\mathscr{S}\mathscr{O}_G \longrightarrow \mathscr{B}_G$, but fibrant replacement functors are rarely enriched and we do not know how to enrich the fibrant replacement functor $\mathscr{B}_G \longrightarrow \mathscr{B}_G^{fib}$ to an $\mathscr{S}$-functor. We shall not give details here, but that problem can be circumvented by replacing the categories $G\mathscr{S}_{U^G}$ and $G\mathscr{S}_U$ in the diagram by by the corresponding categories $G\mathscr{M}_{U^G}$ and $G\mathscr{M}_U$ of EKMM S_G-modules [**?**, **?**], in which all objects are fibrant, but still enriching over the category $\mathscr{S}$ of nonequivariant orthogonal spectra. However, the deeper problem is that of finding a more computable variant of the category $\mathscr{B}_G^{fib}$, and we are now close to solving that.