MODELS OF G-SPECTRA AS PRESHEAVES OF SPECTRA

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Contents

1. Background

The equivariant stable homotopy category is of fundamental importance in algebraic topology. It is the natural home in which to study equivariant stable homotopy theory, a subject that has powerful and unexpected nonequivariant applications. For example, it plays a key role in the recent solution of the Kervaire invariant problem (Hill, Hopkins, and Ravenel [13]), it is central to calculation of topological cyclic homology and therefore to calculations in algebraic K-theory (Hesselholt and Madsen [12]), it plays an interesting role by analogy and comparision in motivic homotopy theory (Voevodsky [33, 34]), and it is the motivational starting point of recent work that develops homological algebra parallelling the topological structure visible in the equivariant stable homotopy category (Kaledin [16]).

Setting up the equivariant stable homotopy category with its attendant model structures takes a fair amount of work. The original version was due to Lewis and May [19], and details of more modern versions have been given by Mandell and May [20, 21, 27]. Most of these sources work with compact Lie groups of equivariance.

Motivated by analogous work of Dugger, Schwede, Shipley, and others, in [11] we gave a general theory describing when enriched model categories are equivalent to categories of enriched presheaves. We start with a good model category $\mathscr{V}$ in which to enrich things and a $\mathscr{V}$-model category $\mathscr{M}$. We show that there is often a small $\mathscr{V}$-category $\mathscr{D}$ such that $\mathscr{M}$ is Quillen equivalent to the category $\mathscr{V}^{\mathscr{D}}$ of enriched presheaves $\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{V}$. Technically, the model structure on $\mathscr{V}^{\mathscr{D}}$ is very simple, just being the (projective) model structure induced levelwise from the model structure on $\mathscr{V}$, which is conceptually simple and well understood. In much of the theory in

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[11], as in earlier theory [31], the category $\mathscr{D}$ is a full $\mathscr{V}$-subcategory of $\mathscr{M}$ whose objects are generators of $\mathscr{M}$ in an appropriate sense[1].

From the point of view of applications and calculations, this is just a starting point. One wants a more concrete understanding of the category $\mathscr{D}$. In general, its hom objects $\underline{\mathscr{D}}(d,e)$ in $\mathscr{V}$ and their composition and unit maps

$$\underline{\mathscr{D}}(e,f)\otimes\underline{\mathscr{D}}(d,e)\longrightarrow\underline{\mathscr{D}}(d,f) \quad\text{and}\quad \mathbf{I}\longrightarrow\underline{\mathscr{D}}(d,d), \tag{1.1}$$

where $\mathbf{I}$ is the unit object of $\mathscr{V}$, may be little easier to understand than for general objects of $\mathscr{M}$. To illustrate the point, we begin by stating two specializations of [11, Theorem 6.7], both of which apply to any compact Lie group G.

Let $\mathscr{I}\mathscr{S}$ be the category of orthogonal spectra and let $G\mathscr{I}\mathscr{S}$ be the category of orthogonal G-spectra, where G is a compact Lie group. The maps are G-maps. The category $G\mathscr{I}\mathscr{S}$ is closed symmetric monoidal under its smash product, with internal hom objects the function G-spectra $F_G(X,Y)$. It is also enriched over $\mathscr{I}\mathscr{S}$, with hom objects the fixed point spectra $F_G(X,Y)^G$. Enriched model categories are discussed in [8, 11, 14] and elsewhere, and $G\mathscr{I}\mathscr{S}$ is an $\mathscr{I}\mathscr{S}$-model category. As usual, we write X_+ for the disjoint union of a G-space and a G-fixed basepoint.

Theorem 1.2. *Let G be a compact Lie group and let $G\mathscr{B}$ be the full $\mathscr{I}\mathscr{S}$-subcategory of $G\mathscr{I}\mathscr{S}$ whose objects are fibrant approximations of the orbit suspension G-spectra $\Sigma^\infty_G(G/H_+)$, where H runs through the closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathscr{I}\mathscr{S}^{G\mathscr{B}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{I}\mathscr{S},$$

and it is a Quillen equivalence.

The letter $\mathscr{B}$ stands for "Burnside". This is a generalized version of the Burnside category of G. When G is finite, the homotopy category of $G\mathscr{B}$ has several equivalent algebraic descriptions (see for example [27, §IX.4]).

As something of a joke, but a serious one that will be central to the mathematical philosophy of our work, we state a companion theorem. Let $\mathscr{M}$ be the category of S-modules of [9] and let $G\mathscr{M}$ be the category of S_G-modules, details of which are given in [21]. Here S_G is the appropriate sphere G-spectrum. The category $G\mathscr{M}$ is closed symmetric monoidal under its smash product, with internal hom objects the function S_G-modules $F_G(X,Y)$. It is also enriched over $\mathscr{M}$, with hom objects the fixed point S-modules $F_G(X,Y)^G$.

Theorem 1.3. *Let G be a compact Lie group and let $G\mathscr{B}$ be the full $\mathscr{M}$-subcategory of $G\mathscr{M}$ whose objects are cofibrant approximations of the orbit suspension G-spectra $\Sigma^\infty_G(G/H_+)$, where H runs through the closed subgroups of G. Then there is an enriched Quillen adjunction*

$$\mathscr{I}\mathscr{S}^{G\mathscr{B}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{I}\mathscr{S},$$

and it is a Quillen equivalence.

Theorem 6.7 of [11] applies to prove both Theorem 1.2 and Theorem 1.3. The orbit G-spectra give compact generating sets in both $\mathrm{Ho}(G\mathscr{I}\mathscr{S})$ and $\mathrm{Ho}(G\mathscr{M})$.

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[1]The generic notation for $\mathscr{V}$ as an enriching category will occasionally be used below, but we never again use $\mathscr{M}$ in this generic sense; it will shortly be given a quite different specific meaning.

We require bifibrant representatives. In Theorem 1.2, the orbit G-spectra are cofibrant, and fibrant approximation makes them bifibrant. However, we do not have simple enough fibrant approximations to allow easy concrete understanding of these bifibrant orthogonal G-spectra. In Theorem 1.3, all S_G-modules are fibrant, and cofibrant approximation makes them bifibrant. Here cofibrant approximation is given by a convenient strong symmetric monoidal left adjoint.

The "goodness" hypothesis of [11, 6.7] holds in Theorem 1.2 because $\mathscr{I}\mathscr{S}$ satisfies the monoid axiom of [31], by [22, 12.5]. It holds in Theorem 1.3 by use of the "Cofibration Hypothesis" of [9, p. 146].

Technically, [11, 6.7] also requires *either* that the unit object of the enriching category $\mathscr{V}$ be cofibrant *or* that every object in $\mathscr{V}$ be fibrant. The first hypothesis holds in $\mathscr{I}\mathscr{S}$ and the second holds in $\mathscr{M}$. It is fundamental to our work that we need both conditions, and it is impossible to have them both in the same model category for the stable homotopy category [18, 29]. Therefore it is no joke that we need both of these results.

Remark 1.4. We have stated Theorem 1.2 in terms of orbits G/H. We could equally well shrink the category $G\mathscr{B}$ by choosing one H in each conjugacy class. However, when G is finite, it is conceptually clearer to work instead with the (skeletally small) full subcategory of $G\mathscr{I}\mathscr{S}$ whose objects are fibrant approximations of the suspension G-spectra $\Sigma^\infty_G(A_+)$, where A runs through the finite G-sets, or, for smallness, one A in each isomorphism class of finite G-sets.

Provided that we understand presheaves to be additive, in the sense that they take finite wedges to finite products (which are weakly equivalent to finite wedges), Theorems 1.2 and 2.2 remain valid with $G\mathscr{B}$ reinterpreted to allow general finite G-sets as objects. We can freely switch back and forth between these two points of view. As a conceptual convenience that simplifies notations, we agree to work with this larger category from now on. We can always reinterpret everything we do in terms of orbit G-spectra.

Theorems 1.2 and 1.3 are related by the following result, which is [21, IV.1.1]; the nonequivariant special case is [21, I.1.1].

Theorem 1.5. *For any compact Lie group G, there is a Quillen equivalence*

$$G\mathscr{I}\mathscr{S}_+ \underset{\mathbb{N}^\#}{\overset{\mathbb{N}}{\rightleftarrows}} G\mathscr{M}.$$

The functor $\mathbb{N}$ is strong symmetric monoidal, hence $\mathbb{N}^\#$ is lax symmetric monoidal.

In Theorem 1.2, $\mathscr{I}\mathscr{S}$ is given its natural stable model structure. It is a particularly nice enriching category since the sphere orthogonal spectrum S is cofibrant and $\mathscr{I}\mathscr{S}$ satisfies the monoid axiom. In Theorem 1.5, $G\mathscr{I}\mathscr{S}$ is given its positive stable model structure and denoted $G\mathscr{I}\mathscr{S}_+$ to indicate the distinction; in that model structure the sphere spectrum S, like the sphere S-module S, is not cofibrant. The identity functor is a Quillen equivalence $G\mathscr{I}\mathscr{S}_+ \longrightarrow G\mathscr{I}\mathscr{S}$. Therefore Theorems 1.2, 1.5, and 1.3 have the following immediate consequence.

Corollary 1.6. *The categories $\mathscr{I}\mathscr{G}^{G\mathscr{B}}$ and $\mathscr{M}^{G\mathscr{B}}$ are Quillen equivalent.*

We reiterate the generality: the results above hold for all compact Lie groups G. In that generality, we do not know how to simplify the description of either of the domain categories $G\mathscr{B}$ to transform it into a weakly equivalent $\mathscr{I}\mathscr{S}$-category or a weakly equivalent $\mathscr{M}$-category that is intuitive and perhaps even familiar, something accessible to those who are not already experts in equivariant stable homotopy theory. The purpose of this paper is to obtain such a description when G is finite and, from here on out, G will be a fixed finite group. It will require quite a bit of work to prove that our more intuitive choice of $\mathscr{D}$ is weakly equivalent to $G\mathscr{B}$. That is the substance of the paper.

2. Introduction

While understanding the stable homotopy category of G-spectra for a finite group G might be regarded as quite sophisticated work, we lighten the context with another serious joke. The crux of our work is to understand the central role played by the following utterly trivial map. For G-spaces X and Y, we agree once and for all that the canonically homeomorphic based G-spaces $X_+ \wedge Y_+$ and $(X \times Y)_+$ can and will be used interchangably. We write basepoints as $*$.

Definition 2.1. Let A be a finite G-set. Define a based G-map

$$\varepsilon\colon (A \times A)_+ \longrightarrow A_+$$

by

$$\varepsilon(a,b) = \begin{cases} a & \text{if } a = b \\ * & \text{if } a \neq b \end{cases}$$

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We describe the relevant $\mathscr{D}$ in §5 and §4. It is obtained by applying a suitably well-behaved infinite loop space machine $\mathbb{K}$ to simple and concrete permutative categories that are defined in terms of finite G-sets. We denote the resulting $\mathscr{I}\mathscr{S}$-category $\mathbb{K}(G\mathscr{E})$. Technically, it is advantageous that the infinite loop space machine $\mathbb{K}$ that we use in §5 takes values in the category $\mathscr{I}\mathscr{S}$ of orthogonal spectra, since with that choice of target there is a theory of pairings [26] that shows how to construct the category $\mathbb{K}(G\mathscr{E})$ as a category enriched in $\mathscr{I}\mathscr{S}$. However, model theoretic considerations will later lead us to a reinterpretation.

Retaining the notations of Theorem 1.2, we will prove the following result.

Theorem 2.2. *Let G be finite. Then the small $\mathscr{I}\mathscr{S}$-categories $G\mathscr{B}$ and $\mathbb{K}(G\mathscr{E})$ are weakly equivalent. Therefore there is a zigzag of enriched Quillen equivalences between $\mathscr{I}\mathscr{S}^{\mathbb{K}(G\mathscr{E})}$ and $\mathscr{I}\mathscr{S}^{G\mathscr{B}}$.*

In effect, Theorems 1.2 and 2.2 combine to say that when G is finite we can study the equivariant stable homotopy category $G\mathscr{I}\mathscr{S}$ in terms of the nonequivariant stable homotopy category and the elementary categorical object $G\mathscr{E}$. Intuitively, $G\mathscr{E}$ is a “category of permutative categories enriched in permutative categories”, where a permutative category is a strictly associative and unital small symmetric monoidal category. The notion in quotes does not make mathematical sense since there is no known monoidal structure on the category of permutative categories. Category theory due to the first author [10], together with the cited theory of pairings of the second author, gets around that.

As already said, we are going to use infinite loop space theory. There is a fundamental issue that must be resolved in order to prove Theorem 2.2 using that tool. There is no known infinite loop space machine that knows about function spectra. That is, given input data X and Y (permutative categories, E_∞-spaces, Γ-spaces, etc) for an infinite loop space machine $\mathbb{K}$, we do not know what input data will have as output the function spectrum $F(\mathbb{K}X, \mathbb{K}Y)$. The problem does not make sense as just stated because the output spectra $\mathbb{K}Z$ are always connective, whereas $F(\mathbb{K}X, \mathbb{K}Y)$ is generally not. The most that one could hope for is to detect the connective cover of $F(\mathbb{K}X, \mathbb{K}Y)$. Even if we could solve that problem, the problem of detecting composition maps of function spectra would remain.

We use a combination of methods to get around this to prove Theorem 2.2. Our use of "categories of permutative categories" will mimic categorically the structure of suspension G-spectra and their fixed point spectra. We state the relevant results from infinite loop space theory in §3, §4, and §5. The results here are of independent interest and work more generally than is needed to prove Theorem 2.2. We describe in §3 how infinite loop space theory reproves the tom Dieck splitting theorem for the fixed point spectra of suspension G-spectra of based G-spaces when G is finite. This is based on use of the equivariant Barratt-Quillen theorem, and we describe a multiplicative elaboration of that result in §4. When specialized to finite G-sets, our proof of the tom Dieck splitting theorem leads us in §5 to the construction of the $\mathscr{I}\mathscr{S}$-category $\mathbb{K}(G\mathscr{E})$ referred to in Theorem 2.2.

However, to connect the category $\mathbb{K}(G\mathscr{E})$ with the category $G\mathscr{B}$, we need a more geometric starting point. We will use an elementary understanding of Atiyah duality for finite G-sets to obtain a concrete understanding of $G\mathscr{B}$ that transforms it into an $\mathscr{I}\mathscr{S}$-category that resembles the category $\mathbb{K}(G\mathscr{E})$.

Everything we do depends on starting equivariantly and then passing to fixed points. As a matter of general context, it will help to fix some generic notations. Let $\mathscr{C}$ be any category whose objects have actions by our finite group G. We write $G\mathscr{C}$ for the category of G-objects and G-maps, and write $\mathscr{C}_G$ for the category of G-objects and nonequivariant maps, which we refer to as 'arrows' to distinguish them from G-maps. This is a G-category with G acting by conjugation on morphism sets, or morphism objects in enriched contexts. The two categories are related conceptually by $G\mathscr{C} = (\mathscr{C}_G)^G$. The objects, being G-objects, are already G-fixed; we apply the G-fixed point functor to hom sets. This point of view on equivariance in enriched contexts is discussed in detail and justified conceptually in [21, II§1].

Thus we write $\mathscr{I}\mathscr{S}_G$ for the category of orthogonal G-spectra and non-equivariant maps. The category $\mathscr{I}\mathscr{S}_G$ is symmetric monoidal and thus enriched over itself, with hom objects the G-spectra $F_G(X,Y)$. To understand $G\mathscr{I}\mathscr{S}$ as an $\mathscr{I}\mathscr{S}$-category, we must first understand $\mathscr{I}\mathscr{S}_G$ as an $\mathscr{I}\mathscr{S}_G$-category. That is, to understand spectra $F_G(X,Y)^G$, we must first understand G-spectra $F_G(X,Y)$. We let $\mathscr{B}_G^{\mathrm{fib}}$ denote the full subcategory of $\mathscr{S}_G$ whose objects are those of $G\mathscr{B}^{\mathrm{fib}}$.

We describe how Atiyah duality helps us understand $\mathscr{B}_G$ concretely in §6. Technically, however, we work in the EKMM category $G\mathscr{M}$ of modules over the sphere G-spectrum S_G, rather than in the Quillen equivalent category $G\mathscr{I}\mathscr{S}$. Both are closed symmetric monoidal. As recalled in §**??**, $G\mathscr{M}$ and $G\mathscr{I}\mathscr{G}$ are Quillen equivalent, and there is a precise analogue of Theorem 1.2 that applies to $G\mathscr{M}$. Following our general rule, we write $\mathscr{M}_G$ for the category of S_G-modules and non-equivariant maps. The category $\mathscr{M}_G$ is enriched over itself, with hom objects again denoted by $F_G(X,Y)$, and $G\mathscr{M}$ is enriched over the category $\mathscr{M}$ of S-modules, with hom objects $F_G(X,Y)^G$.

In any good model for the equivariant stable homotopy category, $F_G(X,Y)$ is equivalent to $Y \wedge D(X)$ when either X or Y is a dualizable G-spectrum, where $D(X) = F_G(X, S_G)$. When X is of the form $X = \Sigma_G^\infty(A_+)$ for a finite G-set A, X

is self-dual. If, further, Y is of the form $Y = \Sigma^\infty_G(B_+)$ for a finite G-set B, we have

$$F_G(X,Y) \simeq \Sigma^\infty_G B_+ \wedge \Sigma^\infty_G A_+ \cong \Sigma^\infty_G (B\times A)_+ \tag{2.3}$$

This is an obvious starting point, and it already shows that our function spectra are connective, but it raises the question of how to compute composition. If $Z = \Sigma^\infty_G C_+$ for a third finite G-set C, we want to compute the composition

$$F_G(Y,Z)\wedge F_G(X,Y) \longrightarrow F_G(X,Z)$$

in terms of A, B, and C. It turns out that Atiyah duality tells us how to do this.

It is to show this that it is convenient to start work in $G\mathscr{M}$ rather than $G\mathscr{I}\mathscr{S}$. We explain why. The identification of the category $G\mathscr{B}$ is a point-set level rather than merely an up to homotopy problem. The choice of bifibrant models for the generating objects does not matter in Theorem 1.2, since any two choices give Quillen equivalent categories of presheaves [11, §7]. The choice does matter when trying to give an explicit model for $G\mathscr{B}$. Every S_G-module is fibrant. We have an appropriate suspension S_G-module functor $\mathbf{S}^\infty_G$, and it is very nearly strong symmetric monoidal. We have categories $G\mathscr{B}$ and $\mathscr{B}_G$ of S_G-modules whose objects are the $\mathbf{S}^\infty_G(A_+)$, and these objects are bifibrant, with no use of fibrant or cofibrant approximation. This makes the identification problem considerably simpler in $G\mathscr{M}$ than in $G\mathscr{I}\mathscr{S}$. It also allows us to work in a framework in which the technical details most closely match the intuition.

We need quite a long zigzag of weak equivalences between $\mathscr{I}\mathscr{S}$-categories $\mathscr{D}$ to connect $\mathbb{K}(G\mathscr{E})$, $G\mathscr{B}$, and $G\mathscr{B}$. We chart the path in §**??**. We also explain there how the theory of [11] applies to express the category of naive G-spectra as an elementary category of presheaves of spectra. The essential ingredients in the zigzag are equivariant infinite loop space theory and an equivariant version of the Barratt-Quillen theorem that is explained in §3 and §4. Parts of the relevant theory have appeared in such papers as Costenoble-Waner [6] and Shimakawa [32]. Much of the story has been understood by the second author since the early 1980's. In fact, some of the easier work here is transcribed from yellowed pieces of paper dating that far back, although much of this material has been rediscovered several times since, for example in [4, §5]. However, this material has still not been written up in a coherent fashion, and we make essential use of some results that are new. The write-up is in progress, but this is not the place for a full exposition. We will explain what is needed and prove the relevant categorical claims, but we will leave the details of the infinite loop machinery to that work in progress.

3. A categorical reinterpretation of the tom Dieck splitting

The G-fixed point spectra of suspension G-spectra have a well-known splitting, due to tom Dieck [7] on the level of homotopy groups and lifted to the spectrum level in [19, §V.11]. Specialization of our proof of this result is the starting point of our construction of $\mathbb{K}(G\mathscr{E})$. The tom Dieck splitting actually works for all compact Lie groups G, but we have nothing helpful to add in that generality. Our group G is always finite.

Theorem 3.1. *For a based G-space Y,*

$$(\Sigma^\infty_G Y)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} Y^H).$$

Here the wedge runs over the conjugacy classes of subgroups H of G, and $WH = W_G H$ is the group NH/H, where NH is the normalizer of H in G.

We shall construct a categorical version of this splitting by constructing a based permutative G-category $\mathscr{E}_G(Y)$ with fixed point permutative categories $\mathscr{E}_G(Y)^H$ such that application of suitable equivariant and nonequivariant infinite loop space machines $\mathbb{K}_G$ and $\mathbb{K}$ on permutative G-categories and permutative categories, respectively, delivers suspension spectra as outputs.

Theorem 3.2. *There is a natural weak equivalence*

$$\Sigma^\infty_G(Y) \longrightarrow \mathbb{K}_G\mathscr{E}_G(Y).$$

Theorem 3.3. *There is a natural weak equivalence*

$$\bigvee_{(H)} \Sigma^\infty EWH_+ \wedge_{WH} Y^H \longrightarrow \mathbb{K}(\mathscr{E}_G(Y)^G).$$

As in Theorem 3.1, (H) runs through the conjugacy classes of subgroups H of G.

The equivariant and nonequivariant infinite loop space machines commute with passage to fixed points.

Theorem 3.4. *For permutative G-categories $\mathscr{A}$ there is a natural weak equivalence*

$$\mathbb{K}(\mathscr{A}^G) \longrightarrow (\mathbb{K}_G\mathscr{A})^G.$$

Clearly Theorems 3.2, 3.3, and 3.4 together reprove Theorem 3.1.

Remark 3.5. Let $\iota\colon H \longrightarrow G$ be the inclusion of a subgroup. Theorem 3.2 means that the induced maps

$$(\iota^*\Sigma^\infty_G(Y))^H \longrightarrow (\iota^*\mathbb{K}_G\mathscr{E}_G(Y))^H \tag{3.6}$$

are weak equivalences of spectra for all H, where ι^* is the forgetful functor from G-spectra to H-spectra. It is standard that $\iota^*\Sigma^\infty_G$ is naturally equivalent to $\Sigma^\infty_H\iota^*$, and it is also true that $\iota^*\mathbb{K}_G$ is naturally equivalent to $\mathbb{K}_H\iota^*$. Therefore (3.6) can be rewritten as

$$(\Sigma^\infty_H\iota^*Y)^H \longrightarrow (\mathbb{K}_H\iota^*\mathscr{E}_G(Y))^H \tag{3.7}$$

Since the theorems above hold for all G, including H, the right side of (3.7) must be equivalent to $(\mathbb{K}_H\mathscr{E}_H(\iota^*Y))^H$. One might imagine that $\iota^*\mathscr{E}_G(Y)$ is equivalent to $\mathscr{E}_H(\iota^*Y)$, but we shall see that that already fails when Y is a point. There is no contradiction. We will say more about this in [3], but we will display a natural map $\iota^*\mathscr{E}_G(Y) \longrightarrow \mathscr{E}_H\iota^*Y$ and explain why it fails to induce an equivalence on passage to H-fixed points in **??**.

Theorem 3.1 is implied by the equivariant Barratt-Quillen theorem and Theorem 3.2 is implied by the non-equivariant Barratt-Quillen theorem. The version of these results that we shall use can be stated operadically as follows. It is built into the second author's infinite loop space machinery. In fact, with the model theoretic modernization of the original version of that theory that is given in [1], one can redefine the restriction of $\mathbb{K}$ to cofibrant objects Y in such a way that weak equivalence can be replaced by isomorphism in the following result.

Theorem 3.8 (Equivariant Barratt-Quillen theorem)**.** *For an E_∞ operad $\mathscr{O}_G$ of G-spaces and based G-spaces Y, there is a natural weak equivalence*

$$\Sigma^\infty_G(Y) \longrightarrow \mathbb{K}_G\mathbf{O}_G(Y),$$

where $\mathbf{O}_G$ is the monad on based G-spaces whose algebras are the $\mathscr{O}_G$-algebras.

The nonequivariant Barratt-Quillen theorem is the case $G = e$. We are especially interested in the case $Y = X_+$ for an unbased G-space X. We then have

$$\mathbf{O}_G(X_+) = \coprod_{j\geq 0} \mathscr{O}_G(j) \times_{\Sigma_j} X^j$$

The term with $j = 0$ is a single point, which is the basepoint. The space $\mathscr{O}_G(j)$ is a model for the universal (G, Σ_j)-bundle $E_G(\Sigma_j)$. This means that $\mathscr{O}_G(j)$ is a Σ_j-free $G\times\Sigma_j$-space such that, for a subgroup Λ of $G\times\Sigma_j$, $\mathscr{O}_G(j)^\Lambda$ is empty unless $\Lambda\cap\Sigma_j = e$, in which case it is contractible. For a general based G-space Y, we quotient out $\mathbf{O}_G(Y_+)$ by basepoint identifications to obtain $\mathbf{O}_G Y$ [23, 2.4].

Recall that a category $\mathscr{A}$ internal to a cartesian monoidal category $\mathscr{C}$ has object and morphism objects in $\mathscr{C}$ and structure maps Source, Target, Identity and Composition in $\mathscr{C}$. These maps are denoted S, T, I, and C, and the the usual category axioms hold. We have a notion of a symmetric monoidal G-category $\mathscr{A}$ internal to $\mathscr{C}$. It has an equivariant product functor $\mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{A}$ of internal categories and a unit object $\mathbf{I}$, and it has an equivariant symmetry isomorphism $\gamma\colon \mathscr{A}\times\mathscr{A}\longrightarrow\mathscr{A}\times\mathscr{A}$, all satisfying the usual coherence properties. It is permutative if it is strictly associative and unital. When $\mathscr{C}$ is the category $\mathscr{U}$, we refer to internal categories as topological categories, and we refer to them as topological G-categories when $\mathscr{C}$ is $G\mathscr{U}$. These are more general than (small) topologically enriched categories, which have discrete sets of objects.

We define a based category to be a category with an isolated base object $*$, meaning that there are no maps other than the identity to or from $*$ in an equivalent skeletal category. For a G-category, we require $*$ to be G-fixed. Based spaces are understood to have nondegenerate basepoints. We are interested in based G-CW complexes Y and, later, in based finite G-sets Y. Note that a based finite G-set is necessarily of the form A_+ for an unbased finite G-set A. Based maps $A_+\longrightarrow Y$ are the same as unbased maps $A\longrightarrow Y$. For later purposes, it is best to think of them in their based form, even though we write them in unbased form in the following definition.

Definition 3.9. Let Y be a based G-space. We define a based topological permutative G-category $\mathscr{E}_G(Y)$. We take finite G-sets to be contained in some ambient G-set U (a "universe") that contains all finite G-sets up to isomorphism. For example, U might be the coproduct of countably many copies of each orbit G/H. We choose a G-fixed basepoint $*\in U$.

Briefly, $\mathscr{E}_G(Y)$ is the category of finite G-sets over Y and bijections over Y. Its objects are all functions $p\colon S\longrightarrow Y$. For a second function $q\colon T\longrightarrow Y$, a map $f\colon p\longrightarrow q$ is a bijection $f\colon S\longrightarrow T$ such that $q\circ f = p$. Composition is given by composition of functions, and $f = \mathrm{id}$ gives identity maps. The group G acts by its given action on Y and conjugation on all maps in sight. For an object $p\colon S\longrightarrow Y$, $gp\colon S\longrightarrow Y$ is given by $(gp)(s) = g(p(g^{-1}y))$. Similarly, for a map f, $(gf)(s) = g(f(g^{-1}s))$. The product on $\mathscr{E}_G(Y)$ sends (p,q) to the disjoint union $p+q\colon S\coprod T\longrightarrow Y$, and similarly on morphisms. The empty map $i\colon\emptyset\longrightarrow Y$ is the unit. The symmetry is the evident isomorphism $p+q \longrightarrow q+p$. The base object sends $*$ to the basepoint of Y.

We will be precise about the topology in §7. Roughly, we may view the set $\mathscr{O}b$ of objects of $\mathscr{E}_G(Y)$ as the set of functions $U \longrightarrow Y$ that send the complement of some finite G-subset S to the basepoint and topologize it as a subspace of Y^U. We may view the set $\mathscr{M}or$ of morphisms of $\mathscr{E}_G(Y)$ as the functions $f\colon p \longrightarrow q$ that send the complement of some finite subset S to $*$ and topologize it as the subspace of points (p,f,q) in $\mathscr{O}b \times U^U \times \mathscr{O}b$, where U^U is discrete. To be precise, we need to take the choice of S into account. When Y is finite, $\mathscr{E}_G(Y)$ is discrete.

Remark 3.10. We have not yet specified exactly what we mean by a permutative G-category. For equivariant infinite loop space theory, it must be more than a topological G-category with a strict symmetric monoidal structure such that G acts compatibly with the product and symmetry isomorphism. We will say a little more about this in **??** and a lot more in [3], but for now it suffices to say that $\mathscr{E}_G(Y)$ is a typical example.

Starting from this definition, we describe how to see Theorems 3.2 and 3.3 as specializations of the equivariant and nonequivariant Barratt-Quillen theorems. Proofs are deferred to §7. Starting nonequivariantly, we head towards Theorem 3.3 with the following easy pair of results, which were known in the early 1980's.

Proposition 3.11. *There is a based category $\mathscr{F}(G,Y)$ with classifying space*

$$B\mathscr{F}(G,Y) \cong EG_+ \wedge_G Y.$$

Let $\mathscr{E}$ be the category of finite sets $\underline{n}$, $n \geq 0$, and their isomorphisms (the symmetric groups Σ_n). Recall from [23, p. 161] or [24, 4.8] that there is an E_∞ operad $\mathscr{O}$ of spaces that is obtained by applying the classifying space functor to the E_∞ operad $\tilde{\mathscr{E}}$ of categories whose j^{th} term is the translation category $\tilde{\Sigma}_j$ of Σ_j.

Proposition 3.12. *For a based category $\mathscr{F}$, there is a based category $\mathscr{E}\int\mathscr{F}$ with classifying space*

$$B(\mathscr{E}\textstyle\int\mathscr{F}) \cong \mathbf{O}B\mathscr{F}.$$

Combining the previous two results gives the following conclusion.

Corollary 3.13. $B(\mathscr{E}\int\mathscr{F}(G,Y)) \cong \mathbf{O}(EG_+ \wedge_G Y)$.

The classifying space $B\mathscr{A}$ of a permutative category $\mathscr{A}$ is an $\mathscr{O}$-space, and the infinite loop space machine on permutative categories is defined by $\mathbb{K}\mathscr{A} = \mathbb{K}(B\mathscr{A})$. Now Corollary 3.13 and the nonequivariant case of Theorem 3.8 give the following conclusion.

Theorem 3.14. *There is a natural weak equivalence*

$$\Sigma^\infty(EG_+ \wedge_G Y) \longrightarrow \mathbb{K}(\mathscr{E}\textstyle\int\mathscr{F}(G,Y)).$$

To complete the proof of Theorem 3.3, we shall prove the following result.

Theorem 3.15. *There is an equivalence of symmetric monoidal categories*

$$\mathscr{E}_G(Y)^G \simeq \textstyle\prod_{(H)} \mathscr{E}\int\mathscr{F}(W_GH, Y^H).$$

For $H \subset G$, there is an equivalence $\mathscr{E}_G(Y)^H \simeq \mathscr{E}_H(Y)^H$, hence an equivalence

$$\mathscr{E}_G(Y)^H \simeq \textstyle\prod_{(K)} \mathscr{E}\int\mathscr{F}(W_HK, Y^K).$$

Applying B and observing that B commutes with products, we see that Theorem 3.15 gives the following result.

Corollary 3.16. *For $H \subset G$,*

$$B\mathscr{E}_G(Y)^H \simeq \prod_{(K)} B(\mathscr{E} \textstyle\int \mathscr{F}(W_H K, Y^K)) \cong \prod_{(K)} \mathbf{O}(EW_H K \times_{W_H K} Y^K).$$

Recall that the natural map from a finite wedge of spectra to a finite product is a weak equivalence. The infinite loop space machine commutes with products.

Proposition 3.17. *For E_∞ spaces X and Y, the canonical map*

$$\mathbb{K}(X \times Y) \longrightarrow \mathbb{K}(X) \times \mathbb{K}(Y)$$

is a weak equivalence.

Putting these results together we see that Theorem 3.3 follows from the nonequivariant version of Theorem 3.8.

Turning to the equivariant context, we shall prove the following analogue of Proposition 3.12. By Theorem 3.8, it will immediately imply Theorem 3.2.

Theorem 3.18. *The classifying G-space $B\mathscr{E}_G(Y)$ is equivalent to $\mathbf{O}_G(Y)$, where $\mathbf{O}_G$ is an E_∞ operad of G-spaces that is obtained by applying the classifying G-space functor B to an E_∞ operad of G-categories.*

4. Pairings and the tom Dieck splitting theorem

The functor Σ^∞_G from based G-spaces to orthogonal G-spectra is strong symmetric monoidal with respect to the smash product. Indeed, the sphere G-spectrum S_G is $\Sigma^\infty_G S^0$ by definition, and we have a coherent natural isomorphism

$$\Sigma^\infty_G X \wedge \Sigma^\infty_G Y \cong \Sigma^\infty_G (X \wedge Y).$$

The functor Σ^∞ is also denoted F_0 and is left adjoint to the zero$^{\text{th}}$ space functor. The isomorphism is a special case of [21, II.4.8], whose proof is the same as in the nonequivariant case [22, 1.8]. It is a long-standing problem in equivariant stable homotopy theory to understand the behavior of the tom Dieck splitting under this isomorphism. That is not an easy problem nor one that we can expect to understand in explicit detail. For example, when $X = Y = S^0$ and we ask for the behavior on components of the canonical map

$$S^G_G \wedge S^G_G \longrightarrow (S_G \wedge S_G)^G \cong S^G_G,$$

we are asking for the multiplication on the Burnside ring $A(G)$, and that entails understanding the decomposition of product G-spaces $G/H \times G/K$ as disjoint unions of orbits. The proof of the tom Dieck splitting in the previous section sheds light on this problem. When specialized to finite G-sets, the same arguments will give an ingredient in the proof of Theorem 2.2.

We shall say more about the categorical background in the next section, but it would only confuse matters to start with that. A notion of pairing of permutative categories was introduced in [26]. As already said, the infinite loop space machine $\mathbb{K}$ takes permutative categories to orthogonal spectra. As is explained in [21, I§8], it also takes pairings $\mathscr{A} \times \mathscr{B} \longrightarrow \mathscr{C}$ of permutative categories to pairings $\mathscr{K}(\mathscr{A}) \wedge \mathscr{K}(\mathscr{B}) \longrightarrow \mathbb{K}(\mathscr{C})$ of orthogonal spectra. The theory of pairings adapts without difficulty to the equivariant setting, as we will explain in [3]. This theory leads to a multiplicative elaboration of the equivariant Barratt-Quillen theorem, Theorem 3.3.

We start from the following addendum to the definition of the category $\mathscr{E}_G(Y)$ in Definition 3.9.

Definition 4.1. Let X and Y be based G-spaces. Define a pairing

$$\wedge\colon \mathscr{E}_G(X)\times\mathscr{E}_G(Y)\longrightarrow \mathscr{E}_G(X\wedge Y)$$

as follows. We may think of objects $p\colon A\longrightarrow X$ and $q\colon B\longrightarrow Y$ as based maps $A_+\longrightarrow X$ and $B_+\longrightarrow Y$. The pairing sends the pair (p,q) to

$$p\wedge q\colon (A\times B)_+\cong A_+\wedge B_+\longrightarrow X\wedge Y.$$

Similarly, the pairing sends $(f\colon p\longrightarrow p', g\colon q\longrightarrow q')$ to $f\wedge g\colon p\wedge q\longrightarrow p'\wedge q'$.

Smash products are bilinear with respect to wedges, in the evident sense that

$$(X\vee X')\wedge Y\cong (X\wedge Y)\vee(X'\wedge Y) \quad\text{and}\quad X\wedge(Y\vee Y')\cong(X\wedge Y)\vee(X\wedge Y').$$

In the language of [26, §2] and [3], the functor $\wedge$ is a pairing of permutative G-categories. Here we do mean nothing more than a pairing in the sense defined in [26], with G acting compatibly.

Theorem 4.2 (Multiplicative Barratt-Quillen theorem)**.** *The pairing*

$$\wedge\colon \mathscr{E}_G(X)\times\mathscr{E}_G(Y)\longrightarrow \mathscr{E}_G(X\wedge Y)$$

of permutative G-categories induces a pairing

$$\mathbb{K}_G\mathscr{E}_G(X)\wedge\mathbb{K}_G(\mathscr{E}_G(Y)\longrightarrow \mathbb{K}_G(\mathscr{E}_G(X\wedge Y)$$

of orthogonal G-spectra such that the following diagram commutes.

$$\begin{array}{ccc}
\Sigma^\infty_G X\wedge\Sigma^\infty_G Y & \xrightarrow{\cong} & \Sigma^\infty_G(X\wedge Y)\\
\downarrow & & \downarrow\\
\mathbb{K}_G\mathscr{E}_G(X)\wedge\mathbb{K}_G\mathscr{E}_G(Y) & \longrightarrow & \mathbb{K}_G\mathscr{E}_G(X\wedge Y)
\end{array}$$

In fact, given the results of the previous section and the theory of pairings, the only thing requiring proof is the commutativity of the diagram, and that can be seen by a direct verification from the explicit definitions of the maps used to prove Theorems 3.8 and 3.18.

Now restrict attention to finite (unbased) G-sets A, B, C, and write

5. A category of permutative categories

The tom Dieck splitting theorem Theorem 3.1 concerns objects of the stable homotopy category Ho$G\mathscr{S}$ of G-spectra. Here $\mathscr{S}$ could be any of several candidates for the category of G-spectra, say Lewis-May G-spectra for definiteness. There is a companion result, in fact a corollary, that concerns morphisms [19, V.9.4]. Again, the result applies to compact Lie groups, but we restrict to our finite group G.

Theorem 5.1. *Let X be an unbased G-space. For $H\subset G$, the Abelian group of maps $[\Sigma^\infty_G(G/H_+),\Sigma^\infty_G(X_+)]_G$ is isomorphic to the free abelian group generated by the set of equivalence classes $[\phi,\psi]$ of "spans"*

$$G/H\xleftarrow{\ \phi\ } G/K\xrightarrow{\ \psi\ } X.$$

Here ϕ and ψ are maps of G-spaces (equivalently, G-sets), and the pair (ϕ,ψ) is equivalent to a pair (ϕ',ψ') if there is an isomorphism of G-sets $\xi\colon G/K\longrightarrow G/K'$ such that $\phi'\circ\xi=\phi$ and $\psi'\circ\xi=\psi$.

Starting from this result, there results a description of the full subcategory of Ho$G\mathscr{S}$ whose objects are the $\Sigma_G^\infty(G/H_+)$. However, it is inconvenient to describe composition without generalizing from orbits to finite G-sets. We set up notation to be used from here on out. Consider finite (unbased) G-sets A, B, and C. To simplify notation, we agree to write

$$\mathbb{A} = \Sigma_G^\infty A_+, \quad \mathbb{B} = \Sigma_G^\infty B_+, \quad \text{and} \quad \mathbb{C} = \Sigma_G^\infty C_+.$$

Let Ho$G\mathscr{B}$ denote the full subcategory of Ho$G\mathscr{S}$ with objects the $\mathbb{A}$ for finite G-sets A. Let $[G\mathscr{E}]$ denote the category whose objects are the finite G-sets and whose morphisms $A \longrightarrow B$ are the equivalence classes $[\phi, \psi]$ of spans

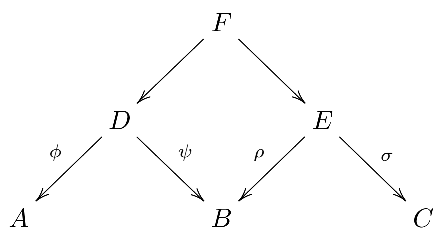
$$A \xleftarrow{\ \phi\ } D \xrightarrow{\ \psi\ } B,$$

where D is a finite G-set. For a span

$$B \xleftarrow{\ \rho\ } E \xrightarrow{\ \sigma\ } C,$$

the composite $[\psi, \phi][\sigma, \rho]$ is the equivalence class of the span displayed in the following diagram, whose diamond is a pullback.

(5.2)



Because we have passed to equivalence classes, composition is well-defined and associative with identity maps the spans $[\mathrm{id}, \mathrm{id}]$. We add spans $[\phi, \psi]$ and $[\phi', \psi']$ from A to B by taking disjoint unions $V \coprod V'$ over A and B. This gives an abelian monoid of morphisms $A \longrightarrow B$, and we apply the Grothendieck construction to obtain an abelian group of morphisms $A \longrightarrow B$. Elementary verifications show that this construction gives an additive category $\mathscr{A}b[G\mathscr{E}]$. To repeat, we start with spans of finite G-sets and build in an additive structure in the most naive possible fashion. The following result is [19, V.9.6].

DANGER: COMPOSITION? CHECK!

Theorem 5.3. *The categories Ho$G\mathscr{B}$ and $\mathscr{A}b[G\mathscr{E}]$ are isomorphic.*

Atiyah duality for smooth G-manifolds specializes to show that the category Ho$G\mathscr{B}$ is self-dual, in the sense that it is isomorphic to its opposite category. It is visible by reversing the direction of spans that the category $\mathscr{A}b[G\mathscr{E}]$ is also self-dual. By [19, V.9.7] the isomorphism of Theorem 5.3 preserves duality.

Duality in a general symmetric monoidal category $\mathscr{V}$ with unit object $\mathbf{I}$ is described categorically in [19, III§1] and [28], for example. Two objects X and Y are dual if there are maps $\eta\colon \mathbf{I} \longrightarrow X \otimes Y$ and $\varepsilon\colon Y \otimes X \longrightarrow \mathbf{I}$ satisfying appropriate identities under composition. If $\mathscr{V}$ is closed, the adjoint of ε is then an isomorphism from Y to the categorical dual $DX = \underline{\mathscr{V}}(X, \mathbf{I})$. The category $G\mathscr{E}$ of finite G-sets is cartesian monoidal, with unit a one-point G-set $*$, and the self-duality of the finite G-set A is given by the equivalence classes $\eta = [\xi, \Delta]$ and $\varepsilon = [\Delta, \xi]$ of the spans

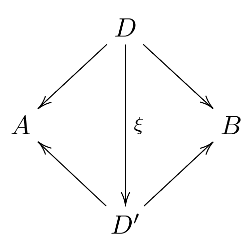
$$(5.4) \qquad * \xleftarrow{\ \xi\ } A \xrightarrow{\ \Delta\ } A \times A \quad \text{and} \quad A \times A \xleftarrow{\ \Delta\ } A \xrightarrow{\ \xi\ } *,$$

where ξ is the trivial function.

We shall lift this structure to point-set categories that are enriched in a suitable symmetric monoidal category of G-spectra. There are several problems to be overcome. One is that the achingly elementary self-duality given by the wrong way maps in the spans of $\mathscr{A}b[G\mathscr{E}]$ is seen in terms of transfer maps in $\mathrm{Ho}G\mathscr{B}$, which are hard to define and work with on the point-set level. We need point-set level descriptions of the self-duality of $\mathbb{A}$ and of composition, before passage to homotopy classes. That is the subject of the next section, and it is equivariant stable homotopy theory.

Another problem is that we must control our hom objects homotopically, paying attention to cofibrancy and fibrancy in our model categories. We ignore this problem in this section, but we show how nonequivariant infinite loop space theory gives rise to a good point-set level description of a category that is a first approximation of the model $\mathbb{K}(G\mathscr{E})$ for the enriched category with hom objects $F_G(\mathbb{B},\mathbb{C})^G$ used in Theorems 1.2 and 2.2.

To begin with, we should try to explain the categorical structure visible in the category $G\mathscr{E}$ of finite G-sets. We emphasize right away that the equivalence relation used to get a category of spans above must be jettisoned; it can only play a role after passage to homotopy categories, as in Theorem 5.3. The category $G\mathscr{E}$ has a very rich structure, only part of which will be relevant to our work in this paper. To start with, $G\mathscr{E}$ is a bicategory. Its 0-cells are the finite G-sets. For any pair (A,B) of 0-cells, it has a category $G\mathscr{E}(A,B)$ of 1-cells and 2-cells between them. Its 1-cells are the spans $A \longleftarrow D \longrightarrow B$. Its 2-cells are the maps of spans, which are isomorphisms $\xi\colon D \longrightarrow D'$ that make the following diagram commute.



There is a horizontal composition which is given by pullbacks as in (5.2). Since we are not passing to equivalence classes, the composition is only associative and unital up to coherent natural isomorphisms that are given by the universal property of pullbacks. These data are subject to axioms that were first formulated by Benabou [2]. A concise modern discussion is given by Leinster [17].

Unusually for bicategories, there is further structure that is hidden in the structure already described. The maps $A \longrightarrow B$ of finite G-sets are the spans of the form $A = A \longrightarrow B$ and horizontal composition restricts on such spans to the ordinary composition of the category $G\mathscr{E}$. The opposite category of $G\mathscr{E}$ is similarly embedded via the horizontal composition of spans $A \longleftarrow B = B$. The category $G\mathscr{E}$ and its opposite are cartesian monoidal, and this structure is visible in the spans $A \longleftarrow A\times B \longrightarrow B$. In more usual examples of bicategories, such structure would have to be encoded in different ways.

In addition to the structure of a (self-dual) bicategory just described, each category $G\mathscr{E}(A,B)$ is symmetric monoidal. Its product is given by disjoint unions $D \amalg D'$ over A and B, with the empty span $A \longleftarrow \emptyset \longrightarrow B$ as unit object. This structure is encoded in the first author's notion of a weak SymMonCat-category [10, Definition 3], which should be viewed as a bicategory enriched in the category of symmetric monoidal categories.

Infinite loop space machinery works best with strictly associative and unital symmetric monoidal categories, which are called permutative categories. There is no known way of giving the product of permutative categories a structure of a permutative rather than symmetric monoidal category. However, it is well-known ([15, 25] for example) that (small) symmetric monoidal categories are equivalent to permutative categories in a structure preserving way, so that we can regard an infinite loop space machine $\mathbb{K}$ defined on permutative categories to be defined on symmetric monoidal categories.

It is proven in [10, Proposition 14] that a weak SymMonCat can be replaced by a suitably equivalent structure, called a PermCat-category [10, Definition 11], which is a 2-category rather than just a bicategory and which has permutative categories rather than just symmetric monoidal categories (like $G\mathscr{E}(A,B)$) between its 0-cells. Most importantly, it has composition pairings of permutative categories to which the infinite loop space machine $\mathbb{K}$ applies.

We do not want to get bogged down in the categorical details here, but they are essential to the mathematics. For us, the effective conclusion is that we lose no information if we view the $G\mathscr{E}(A,B)$ as permutative rather than symmetric monoidal categories. There is a particular way to do this using well-chosen examples of finite G-sets as 0-cells, rather than using the general theory. We shall give details in [3], but here we just take it as given that the infinite loop space machine $\mathbb{K}$ applies to the categories $G\mathscr{E}(A,B)$. More precisely, we agree to mean by $G\mathscr{E}(A,B)$ a canonically equivalent permutative category. To match this, we have the following notions from the previous section.

Definition 5.5. Consider the fixed point category $G\mathscr{E}(A) \equiv \mathscr{E}_G(A)^G$, where $\mathscr{E}_G(A)$ is as specified in Definition 3.9. Explicitly, $G\mathscr{E}(A)$ is just the slice category of finite G-sets over A_+. Its objects are the G-maps $p\colon D \longrightarrow A_+$, where D is a finite G-set. Its morphisms $p \longrightarrow q$, $q\colon E \longrightarrow A_+$, are the G-maps $f\colon D \longrightarrow E$ such that $q\circ f = p$. More precisely, we agree to mean by $G\mathscr{E}(A)$ a canonically equivalent permutative category.

One point of Theorem 3.15 is to identify $G\mathscr{E}(A)$ in such a way that we can apply Theorem 3.8 to see that $\mathbb{K}(G\mathscr{E}(A))$ is equivalent to $\mathbb{A}^G = (\Sigma^\infty_G(A_+))^G$. Categorically, the intuition is that we have something like a double category, but with 0-cells replaced by permutative categories, so it is a tricategorical rather than 2-categorical structure. We think of this structure as a category of permutative categories enriched in permutative categories.

Definition 5.6. The small $\mathscr{I}\mathscr{S}$-category $\mathbb{K}(G\mathscr{E})$ has objects and morphism objects the orthogonal spectra $\mathbb{K}(G\mathscr{E}(A))$ and $\mathbb{K}(G\mathscr{E}(A,B))$. Its composition is obtained by applying $\mathbb{K}$ to the composition pairings of the PermCat-category $G\mathscr{E}$. Its identity maps from the (orthogonal) sphere spectrum S to $\mathbb{K}(G\mathscr{E}(A,A))$ are induced by the map $S^0 \longrightarrow B(G\mathscr{E}(A,A))$ that sends 1 to the vertex given by the identity span.

The object spectra of $\mathbb{K}(G\mathscr{E})$ are equivalent to the fixed point spectra $\mathbb{A}^G$. We aim to show that the morphism spectra are equivalent to the fixed point spectra $F_G(\mathbb{A},\mathbb{B})^G$, compatibly with composition, but to do so we must first understand these function spectra.

6. Equivariant Atiyah duality for finite G-sets

As explained in the introduction, we work in the category $G\mathscr{M}$ of EKMM S_G-modules in this section, where S_G is the equivariant sphere spectrum. The morphisms are the G-maps. For background on this category see [27, XXIV] and, for the model structure, [21, §IV.2]. We only sketch what we need to know. This category starts with the Lewis-May category $G\mathscr{S}$ of G-spectra. There is a naive intuitive construction of the suspension G-spectrum functor $\Sigma^\infty_G \colon G\mathscr{T} \longrightarrow G\mathscr{S}$, where $G\mathscr{T}$ is the category of based G-spaces [19, p.14]. For a based G-CW complex X (with based attaching maps), $\Sigma^\infty_G X$ is cofibrant in $G\mathscr{S}$. In particular, the sphere G-spectrum $S_G = \Sigma^\infty_G S^0$ is cofibrant. Every G-spectrum is fibrant. However, $G\mathscr{S}$ is not symmetric monoidal under the smash product.

The suspension G-spectra $\Sigma^\infty_G Y$ are all objects of $G\mathscr{M}$, but they are almost never cofibrant there. There is a composite functor, $\mathbb{F} = \mathbb{J}\mathbb{L}$, from $G\mathscr{S}$ to $G\mathscr{M}$, and it is a Quillen left adjoint; see [21, 4.5]. For a based G-space Y, we define

$$\mathbf{S}^\infty_G Y = \mathbb{F}\Sigma^\infty_G Y.$$

We also define $\mathbf{S}_G = \mathbf{S}^\infty_G S^0$. There is a natural weak equivalence, indeed a homotopy equivalence of underlying spectra [9, I.4.6 and I.8.5]

$$\mathbf{S}^\infty_G Y \longrightarrow \Sigma^\infty_G Y$$

in $G\mathscr{M}$. Every S_G-module is fibrant. If Y is a based G-CW complex, then $\mathbf{S}^\infty_G Y$ is cofibrant and therefore bifibrant in $G\mathscr{M}$.

The category $G\mathscr{M}$ is closed symmetric monoidal with internal hom objects denoted $F_G(X,Y)$. The functor $\mathbf{S}^\infty_G$ is almost strong symmetric monoidal. There is a natural and coherent isomorphism

$$\mathbf{S}^\infty_G X \wedge \mathbf{S}^\infty_G Y \cong \mathbf{S}^\infty_G (X \wedge Y),$$

but instead of a unit isomorphism we only have the unit weak equivalence $\mathbf{S}_G \longrightarrow S_G$. The implicit trade offs just described are intrinsic to the mathematics, as was explained by Lewis [18] in the nonequivariant context; see [29] for a recent discussion. The category $G\mathscr{M}$ is enriched over the category $\mathscr{M}$ of nonequivariant S-modules, with hom objects the fixed point S-modules $F(X,Y)^G$. Here $S = \Sigma^\infty S^0$ is the nonequivariant sphere spectrum.

As in the previous section, we consider finite G-sets A, B, and C, but in this section we agree to write

$$\mathbb{A} = \mathbf{S}^\infty_G A_+, \quad \mathbb{B} = \mathbf{S}^\infty_G B_+, \quad \text{and} \quad \mathbb{C} = \mathbf{S}^\infty_G C_+.$$

Let $G\mathscr{B}$ and $\mathscr{B}_G$ denote the full subcategories of $G\mathscr{M}$ and $\mathscr{M}_G$ whose objects are the S_G-modules $\mathbb{A}$, where A runs through the finite G-sets; compare Remark 1.4. Then $G\mathscr{B}$ is enriched in $\mathscr{M}$ and $\mathscr{B}_G$ is enriched in $\mathscr{M}_G$. Thus $G\mathscr{B}$ and $\mathscr{B}_G$ are the EKMM analogues of the categories $G\mathscr{B}^{\mathrm{fib}}$ and $\mathscr{B}^{\mathrm{fib}}_G$ of orthogonal G-spectra discussed in the introduction. The $\mathbb{A}$ are bifibrant objects of $G\mathscr{M}$.

The homotopy category Ho$G\mathscr{M}$ is a good model for the stable homotopy category of G-spectra. It is equivalent to any other good model, such as Ho$G\mathscr{S}$ or Ho$G\mathscr{I}\mathscr{S}$. We start our work with duality theory in Ho$G\mathscr{M}$, but the theory can be transported from any other model. The only complete exposition is the original one in [19], which works in Ho$G\mathscr{S}$. We again refer the reader to [19, III§1] or [28, §2] for duality theory in general symmetric monoidal categories.

In Ho$G\mathscr{M}$, we have isomorphisms

$$\zeta\colon \mathbb{B}\wedge D\mathbb{A} \longrightarrow F_G(\mathbb{A},\mathbb{B}) \quad \text{and} \quad \tilde{\varepsilon}\colon \mathbb{B} \longrightarrow D\mathbb{B}, \tag{6.1}$$

hence

$$\zeta(\mathrm{id}\wedge\tilde{\varepsilon})\colon \mathbb{B}\wedge\mathbb{A} \longrightarrow F_G(\mathbb{A},\mathbb{B}). \tag{6.2}$$

Therefore, composition and units

$$F_G(\mathbb{B},\mathbb{C})\wedge F_G(\mathbb{A},\mathbb{B}) \longrightarrow F_G(\mathbb{A},\mathbb{C}) \quad \text{and} \quad S_G \longrightarrow F_G(\mathbb{B},\mathbb{B}) \tag{6.3}$$

can be expressed as maps

$$\mathbb{C}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} \longrightarrow \mathbb{C}\wedge\mathbb{A} \quad \text{and} \quad S_G \longrightarrow \mathbb{B}\wedge\mathbb{B}. \tag{6.4}$$

in Ho$G\mathscr{M}$. We need to understand these maps in the category $G\mathscr{M}$, and for the last of these maps we need to take cofibrant approximation $\mathbf{S}_G \longrightarrow S_G$ into account.

For that purpose, we need to understand representative maps in $G\mathscr{M}$ for the equivalences ζ and $\tilde{\varepsilon}$ in (6.1). The first is categorical and makes sense in any closed symmetric monoidal category, so it applies to both $G\mathscr{M}$ and Ho$G\mathscr{M}$. The map ζ is just the adjoint of the composite

$$\mathbb{B}\wedge D\mathbb{A}\wedge\mathbb{A} \xrightarrow{\mathrm{id}\wedge\varepsilon} \mathbb{B}\wedge S_G \cong \mathbb{B}$$

given by the evaluation map $\varepsilon\colon F(\mathbb{A},S_G)\wedge\mathbb{A} \longrightarrow S_G$ and the unit isomorphism.

The second has a technically simple but perhaps conceptually surprising answer that seems not to have been written down before. For a finite G-set A, $\mathbb{A}$ is self-dual, and a duality is determined by appropriate maps

$$\eta\colon S_G \longrightarrow \mathbb{A}\wedge\mathbb{A} \quad \text{and} \quad \varepsilon\colon \mathbb{A}\wedge\mathbb{A} \longrightarrow S_G.$$

The equivalence $\tilde{\varepsilon}\colon \mathbb{A} \longrightarrow F(\mathbb{A},S_G) = D\mathbb{A}$ is the adjoint of ε. The surprise is that an obvious naive map of G-sets gives an appropriate choice for ε; η on the other hand requires suspension by representation spheres in its definition.

Definition 6.5. Write $S^0 = \{1\}_+$, with basepoint denoted $*$. Define a G-map $\varepsilon\colon (A\times A)_+ \longrightarrow S^0$ by letting $\varepsilon(a,a) = 1$ and $\varepsilon(a,b) = *$ if $a \neq b$. Continue to write ε for the composite of $\mathbf{S}^\infty_G\varepsilon$ with the unit weak equivalence $\mathbf{S}_G \longrightarrow S_G$. Recall that $(A\times B)_+$ can be identified naturally with $A_+\wedge B_+$. Since $\mathbf{S}^\infty_G$ is (almost) strong symmetric monoidal, we may write ε as a map

$$\mathbb{A}\wedge\mathbb{A} \longrightarrow S_G.$$

Proposition 6.6. *The adjoint of the explicit map $\varepsilon\colon \mathbb{A}\wedge\mathbb{A} \longrightarrow S_G$ is a weak equivalence $\tilde{\varepsilon}\colon \mathbb{A} \longrightarrow D\mathbb{A}$.*

Proof. This could be proven from scratch by writing down an appropriate map η and proving the required triangle identities, but in fact this is a special case of equivariant Atiyah duality. Regarding our finite G-set A as a smooth closed G-manifold of dimension 0, we specialize the explicit description of the duality maps η and ε given in [19, p. 152]. We may embed the G-set A as the basis elements of the real representation $V = \mathbb{R}[A]$ of G generated by A and use this embedding to obtain V-duality maps η and ε. The normal bundle of the embedding is of course just $A\times V$, and its Thom complex is $A_+\wedge S^V$. We obtain a tubular embedding by sending (a,v), $a\in A$ and $v\in V$, to $a+\rho(|v|)v$ where $\rho\colon [0,\infty) \longrightarrow [0,\delta)$ is a homeomorphism with $\delta < 1/2$. This is a G-map since $|gv| = |v|$ for all g and v. Applying the Pontryagin-Thom construction, we obtain a G-map $t\colon S^V \longrightarrow A_+\wedge S^V$, which is an equivariant pinch map $S^V \longrightarrow \vee_{s\in S} S^V \cong A_+ \wedge S^V$. The diagonal map on A induces the Thom diagonal $\Delta\colon A_+ \wedge S^V \longrightarrow A_+ \wedge S_+ \wedge S^V$ and

$$\eta\colon S^V \longrightarrow A_+ \wedge A_+ \wedge S^V$$

is the composite $\Delta \circ t$. Smashing with the negative sphere S_G-module S^{-V}, which in formal terms means applying the cotensor $\odot$ between based G-spaces and S_G-modules, we obtain

$$\eta\colon S_G \longleftarrow \mathbf{S}_G \longrightarrow \mathbf{S}_G^\infty (A\times A)_+ \cong \mathbb{A}\wedge\mathbb{A}. \tag{6.7}$$

While simple and canonical, this map only gives a direct arrow after passing to the homotopy category Ho$G\mathscr{M}$.

Letting s be the zero section of ν, we have the composite embedding

$$A \xrightarrow{\Delta} A\times A \xrightarrow{s\times \mathrm{id}} (A\times V)\times A \cong A\times A\times V.$$

The normal bundle of this embedding is $A\times V$, and we may view

$$\Delta\times\mathrm{id}\colon A\times V \longrightarrow A\times A\times V$$

as giving a big tubular neighborhood. The Pontryagin-Thom map here is obtained by smashing the map $(A\times A)_+ \longrightarrow A_+$ that sends (a,a) to a and sends (a,b) to $*$ if $a\neq b$ with the identity map of S^V. Composing with the map induced by the projection $A_+ \longrightarrow S^0$ that sends A to 1, this gives the composite

$$\varepsilon\wedge\mathrm{id}\colon (A\times A)_+\wedge S^V \longrightarrow S^0\wedge S^V \cong S^V.$$

Since we have faithfully transcribed the duality maps from [19], we have displayed V-duality maps. The resulting spectrum level duality map can be obtained by smashing with S^{-V}, but it is obtained more directly by applying the functor $\mathbf{S}_G^\infty$ to $\varepsilon\colon (A\times A)_+ \longrightarrow S^0$ and composing with the unit weak equivalence $\mathbf{S}_G \longrightarrow S_G$. □

Recall the standard fact of duality theory that for any S_G-modules X and Y, where X or Y is dualizable, the natural map $DY\wedge Z \longrightarrow F_G(Y,X)$ is a weak equivalence.

Theorem 6.8. *Under the equivalences (6.1) and (6.2), the composition (6.3) is equivalent to the composition (6.4), and the latter is obtained by applying the functor* $\mathbf{S}_G^\infty$ *to the map of finite G-sets*

$$\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}\colon C_+\wedge (B\times B)_+\wedge A_+ \longrightarrow C_+\wedge A_+. \tag{6.9}$$

The unit $S_G \longrightarrow F_G(\mathbb{A},\mathbb{A})$ is represented by the (formal) composite

$$S_G \xrightarrow{\eta} \mathbb{A}\wedge\mathbb{A} \xrightarrow{\mathrm{id}\wedge\tilde{\varepsilon}} \mathbb{A}\wedge D\mathbb{A} \xrightarrow{\zeta} F_G(\mathbb{A},\mathbb{A}).$$

Proof. We have the adjoint $\tilde{\varepsilon}\colon \mathbb{B}\longrightarrow D\mathbb{B}$ and also the counit $\varepsilon\colon D\mathbb{B}\wedge\mathbb{B}\longrightarrow\mathbb{B}$ of the $(\wedge, F_G)$ adjunction in $G\mathscr{M}$. Formal properties of adjunctions give the following commutative diagram in $G\mathscr{M}$.

$$\begin{array}{ccc}
\mathbb{C}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} & \xrightarrow{\mathbf{S}^\infty_G(\mathrm{id}\wedge\varepsilon\wedge\mathrm{id})} & \mathbb{C}\wedge\mathbb{A} \\
{\scriptstyle \mathrm{id}\wedge\tilde{\varepsilon}\wedge\mathrm{id}\wedge\tilde{\varepsilon}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge\tilde{\varepsilon}} \\
\mathbb{C}\wedge D\mathbb{B}\wedge\mathbb{B}\wedge D\mathbb{A} & \xrightarrow{\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}} & \mathbb{C}\wedge D\mathbb{A} \\
{\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \zeta} \\
F(\mathbb{B},\mathbb{C})\wedge F(\mathbb{A},\mathbb{B}) & \xrightarrow[\circ]{} & F(\mathbb{A},\mathbb{C})
\end{array}$$

At the bottom, we do not know that the function S_G-modules or their smash product are cofibrant, but all objects at the top are cofibrant and thus bifibrant. In general, to compute the smash product of G-spectra X and Y in the homotopy category, we should take the smash product of cofibrant approximations QX and QY of X and Y. When all objects are fibrant, to compute a map $X \wedge Y \longrightarrow Z$ in the homotopy category, we should represent it by a map $QX \wedge QY \longrightarrow QZ$ and take its homotopy class. As the diagram displays, that is precisely what we are doing. The unit statement is a formal consequence of the definition of dualizability [19, III.1.1]. □

Remark 6.10. Let $\mathscr{C}_G$ denote the non-unital $\mathscr{M}_G$-category whose objects are the $\mathbb{A} = \mathbf{S}^\infty_G A_+$, where A is a finite G-set. The hom objects in $G\mathscr{M}$ are the S_G-modules

$$\underline{\mathscr{C}}_G(\mathbb{A},\mathbb{B}) = \mathbb{B}\wedge\mathbb{A} \cong \mathbf{S}^\infty_G(B\times A)_+.$$

Composition is given by applying $\mathbf{S}^\infty_G$ to the composition specified in (6.9). The composition is associative, but this is a non-unital category because it does not have point-set level unit maps. The maps (6.7) give unit maps in the homotopy category of $G\mathscr{M}$. Remembering that the unit object S_G is not cofibrant in $G\mathscr{M}$, it makes sense to regard the well-defined point-set level maps $\mathbf{S}_G \longrightarrow \mathbf{S}^\infty_G(A\times A)_+$ in (6.7) as cofibrant approximations of unit maps. For finite G-sets A, B, and C, composition with these maps (for B) on the left and right specifies weak self-equivalences of $\underline{\mathscr{C}}_G(\mathbb{A},\mathbb{B})$ and $\underline{\mathscr{C}}_G(\mathbb{B},\mathbb{C})$. For the first, this means that the composite

$$\mathbf{S}_G \wedge \mathbf{S}^\infty_G(B\times A)_+ \longrightarrow \mathbf{S}^\infty_G(B\times B)_+ \wedge \mathbf{S}^\infty_G(B\times A)_+ \longrightarrow \mathbf{S}^\infty_G(B\times A)_+$$

is a weak equivalence, and similarly for the second.

7. Categorical constructions and proofs for §3

The classifying space BG of a topological group is the classifying space of G regarded as a topological category with a single object. Parametrizing with a space X leads to a familiar construction (with several names) of a category whose classifying space can be identified as the Borel construction $B(*,G,X) = EG\times_G X$, where $EG = B(*,G,G)$. We give a based variant (as in [9, p. 180]).

Definition 7.1. Let Y be a based G space. Define a based topological category $\mathscr{F}(G,Y)$ as follows. The object space is Y and the morphism space is $G_+\wedge Y$. The map $I\colon Y \longrightarrow G_+\wedge Y$ sends y to (e,y). The maps S and T send (g,y) to y and to gy, respectively. The domain of composition, $(G_+\wedge Y)\wedge_Y(G_+\wedge Y)$ can be identified with $(G\times G)_+\wedge Y$, and composition sends (h,g,y) to (hg,y).

Proof of Proposition 3.11. We may identify $EG_+ \wedge_G Y$ with the geometric realization $B(S^0, G_+, Y)$ of the evident simplicial based space, defined using smash products rather than cartesian products, whose space of q-simplices is $(G^q)_+ \wedge Y$. But that simplicial space is exactly the nerve of the based topological category $\mathscr{F}(G, Y)$. $\square$

We next recall a well-known construction $\mathscr{E} \int G$ for a group G and then generalize it to a construction $\mathscr{E} \int \mathscr{F}$ for a based category $\mathscr{F}$. Both G and $\mathscr{F}$ may be topological. Recall that a group Π has a translation category $\tilde{\Pi}$ with object set Π and a morphism $\tau\sigma^{-1} \colon \sigma \longrightarrow \tau$ for each pair of elements $\sigma, \tau \in \Pi$. There is an evident free right action of Π on this category. Every object of $\tilde{\Pi}$ is initial and terminal, so that $B\tilde{\Pi}$ is contractible, and the right action of Π is free. Therefore $B\tilde{\Pi}$ is a model for the universal principal Π-bundle $E\Pi$.

Now take $\Pi = \Sigma_n$ and let it act from the left on the category G^n. Then the orbit category $\tilde{\Pi} \times_\Pi G^n$ is just the wreath product $\Pi \int G$.

Definition 7.2. The objects of $\mathscr{E} \int G$ are the $\underline{n}$. The wreath product $\Sigma_n \int G$ is the space of morphisms $\underline{n} \longrightarrow \underline{n}$, and there are no morphisms $\underline{m} \longrightarrow \underline{n}$ for $m \neq n$.

Equivalently we can start with the disjoint union of the categories $\tilde{\Sigma}_n \times G^n$ and see that passage to orbits under the evident action of Σ_n gives $\Sigma_n \int G$. Commuting this observation with the classifying space functor B gives the following result.

Lemma 7.3. *The classifying space of $\mathscr{E} \int G$ is $\coprod_n E\Sigma_n \times_{\Sigma_n} BG^n$, where $E\Sigma_n$ is taken to be $B\tilde{\Sigma}_n$.*

We replace G by a based topological category $\mathscr{F}$ and mimic the construction to define $\mathscr{E} \int \mathscr{F}$. However, we must be careful to arrange basepoint identifications. Where we just used passage to orbits at each level n above, we must now allow identifications where n varies. The basepoint of $\mathscr{F}$ gives a map from the trivial category $*$ to $\mathscr{F}$. Define Λ to be the category whose objects are the $\underline{n}$ and whose morphisms are the injections $\underline{m} \longrightarrow \underline{n}$. Quotienting using injections and not just isomorphisms builds in basepoint identifications, as formalized for example in [5].

Definition 7.4. Interpret $\tilde{\mathscr{E}}$ as a contravariant functor from Λ to categories using that $\tilde{\mathscr{E}}(0)$ is the trivial category (as in [23, 2.3]) and define $\mathscr{F}^*$ as a covariant functor from Λ to categories, where the functor $\mathscr{F}^*$ takes $\underline{n}$ to the n-fold cartesian product $\mathscr{F}^n$. Insertion of $* \longrightarrow \mathscr{F}$ in successive positions gives inclusions $\mathscr{F}^{n-1} \longrightarrow \mathscr{F}^n$ associated to the ordered injections $\underline{n-1} \longrightarrow \underline{n}$. Define $\mathscr{E} \int \mathscr{F}$ to be the tensor product of functors $\tilde{\mathscr{E}} \otimes_\Lambda \mathscr{F}^n$. This coend is defined exactly as in the original passage from operads to monads of [23, 2.4].

Proof of Proposition 3.12. The based classifying space construction commutes with the cotensor $\otimes_\Lambda$, and $\mathbf{O}Y$ as defined in [23, 2.4] is $B\tilde{E} \otimes_\Lambda Y^*$. The desired isomorphism between $B(\mathscr{E} \int \mathscr{F})$ and $\mathbf{O}B\mathscr{F}$ follows since the functor B transforms coends of categories to coends of spaces. $\square$

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