We think through some formalities about change of base categories. There is a motivating example that should be followed in order to see what is going on. I'm thinking that maybe a bunch of this formal stuff now up front should be taken away and put in an appendix or two, to make it more coherent and to get on to the main points earlier than we do now in the prequel.

[CHANGE horrid $\mathscr{V}O$-notation]

1. Motivating analogy, formal definitions, and contextualization

Fix a set O of objects. We are thinking of the based orbits G/H_+ or, when G is finite, the based finite G-sets A_+. But the set might be arbitrary. We describe elementary formal structures relating categories of presheaves. But to see that the formal structure is elementary we keep our eyes on a genuinely elementary special case, which motivates the formal structure. I've just written this off the top of my head, and it needs checking.

In our special case, we take our base category $\mathscr{V}$ to be the category of modules over a commutative ring R. Think of another choice $\mathscr{W}$ as the category of modules over a commutative ring S. In general, we suppose given an adjunction

$$\mathscr{V} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}\,. \tag{1.1}$$

In our special case, if $f\colon R \longrightarrow S$ is a homomorphism of commutative rings, then we have the forgetful functor $\mathbb{U} = f^*$ that regards an S-module as an R-module by pullback of the action. The left adjoint $\mathbb{T} = f_!$ is given by extension of scalars, $M \mapsto M \otimes_R S$.

Here $f_!$ is strong symmetric monoidal and f^* is lax symmetric monoidal, and that is the situation we are interested in. We assume once and for all that $\mathbb{T}$ is strong symmetric monoidal, and it follows formally that $\mathbb{U}$ is lax symmetric monoidal.

Let $\mathscr{D}$ be a small $\mathscr{V}$-category with object set O. It is usual to think of $\mathscr{D}$ as a ring with many objects, but we prefer to think of $\mathscr{D}$ as a $\mathscr{V}O$-algebra. In our analogy, we take O to be a point, and then $\mathscr{D}$ is precisely an R-algebra A. A $\mathscr{V}$-enriched presheaf $X\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{V}$ is given by evaluation maps

$$\mathscr{D}(i,j) \otimes X_j \longrightarrow X_i$$

in $\mathscr{V}$, so in our special case it is just a right R-module. The catgory $\mathscr{V}^{\mathscr{D}}$ is just the category of right A-modules in our special case.

Let $\mathscr{V}O$-alg denote the category of $\mathscr{V}$-functors $\mathscr{D}$ and $\mathscr{V}$-natural transformations between them. Defining weak equivalences and fibrations "objectwise", that is by regarding each $\mathscr{D}(i,j)$, $i,j \in \mathscr{O}$, as specifying a functor $\mathscr{D} \longrightarrow \mathscr{V}$, we can give $\mathscr{V}O$-alg a model structure. I doubt that we will want to use that, unless forced to, since fibrant and cofibrant approximation get us away from the concrete specification of $\mathscr{V}O$-algebras $\mathscr{D}$ that is our main focus.

We have two adjunctions induced by (1.1). First, we have

$$\mathscr{V}O\text{-alg} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}O\text{-alg}. \tag{1.2}$$

Fix a homomorphism $f\colon R \longrightarrow S$ of commutative rings. We again have the forgetful functor $\mathbb{U} = f^*$ from S-algebras to R-algebras and its extension of scalars functor $A \mapsto A \otimes_R S$ from R-algebras A to S-algebras B. These are the original functors $\mathbb{T}$ and $\mathbb{U}$ on the underlying objects in $\mathscr{V}$, which have induced products because $\mathbb{U}$

and $\mathbb{T}$ are lax symmetric monoidal. The general case works the same way. For a $\mathscr{V}$-algebra A and a $\mathscr{W}$-algebra B, define $\mathbb{T}A$ to be the $\mathscr{W}$-algebra obtained by applying $\mathbb{T}$ objectwise, $(\mathbb{T}\mathscr{D})(i,j) = \mathbb{T}(\mathscr{D}(i,j))$, with composition and identities given by the assumption that $\mathbb{T}$ is lax monoidal:

$$\mathbb{T}\mathscr{D}(j,k) \otimes_{\mathscr{W}} \mathbb{T}\mathscr{D}(i,j) \longrightarrow \mathbb{T}(\mathscr{D}(j,k) \otimes_{\mathscr{V}} \mathscr{D}(i,j)) \longrightarrow \mathbb{T}\mathscr{D}(i,k)$$

and

$$\mathbb{I}_{\mathscr{W}} \longrightarrow \mathbb{T}\mathbb{I}_{\mathscr{V}} \longrightarrow \mathbb{T}\mathscr{D}(i,i).$$

Similarly, the right adjoint $\mathbb{U}$ is defined by applying $\mathbb{U}$ objectwise.

To motivate the second adjunction, consider a homomorphism $f\colon R \longrightarrow S$ of commutative rings and an f-equivariant map $g\colon A \longrightarrow B$ from an R-algebra A to an S-algebra B. This means that g is a map of R-algebras from A to f^*B, so that $g(ra) = f(r)g(a)$. Equivalently, we could start with the adjoint map of S-algebras $f_!\colon A \longrightarrow B$. Then the adjunction $(\mathbb{T},\mathbb{U})$ induces an adjunction between the category of A-modules and the category of B-modules. The two most obvious special cases start with B and A, respectively, and take g to be either

$$\mathrm{id}\colon \mathbb{U}B \longrightarrow \mathbb{U}B \quad \text{or} \quad \eta\colon A \longrightarrow f^*f_!A.$$

Here $\eta\colon A \longrightarrow f^*(A \otimes_R S)$ sends a to $a \otimes 1$; categorically, it is an example of the unit of the adjunction (1.2).

In the general case, let $\mathscr{D}$ be a $\mathscr{V}$-algebra, $\mathscr{E}$ be a $\mathscr{W}$-algebra, and $\phi\colon \mathscr{D} \longrightarrow \mathbb{U}\mathscr{E}$ be a map of $\mathscr{V}$-algebras; equivalently, we could start with the adjoint, which we shall also denote by $\phi\colon \mathbb{T}\mathscr{D} \longrightarrow \mathscr{E}$. We then have induced adjunctions

$$\mathscr{V}^{\mathscr{D}} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{W}^{\mathscr{E}}. \tag{1.3}$$

For an $\mathscr{E}$-module Y, the $\mathscr{D}$-module strucure on $\mathbb{U}Y$ is given by the maps in $\mathscr{V}$

$$\mathscr{D}(i,j) \otimes_{\mathscr{V}} \mathbb{U}Y_j \xrightarrow{\phi\otimes \mathrm{id}} \mathbb{U}\mathscr{E}(i,j) \otimes_{\mathscr{V}} \mathbb{U}Y_j \longrightarrow \mathbb{U}(\mathscr{E}(i,j) \otimes_{\mathscr{W}} Y_j) \longrightarrow \mathbb{U}Y_i.$$

For a $\mathscr{D}$-module X, the $\mathscr{E}$-module $\mathbb{T}X$ can be written conceptually as $\mathbb{T}X \otimes_{\mathbb{T}\mathscr{D}} \mathbb{E}$. To make sense of this, recall that we have the represented right $\mathscr{E}$-modules $\mathbb{E}(j)$ such that $\mathbb{E}(j)_i = \mathscr{E}(i,j)$. As j-varies, these define a left $\mathscr{E}$-module $\mathbb{E}$ with values in $\mathscr{W}^{\mathscr{E}}$. We can pull this back via ϕ to obtain a left $\mathbb{T}\mathscr{D}$-module with values in $\mathscr{W}^{\mathscr{E}}$, and of course $\mathbb{T}X$ is a right $\mathbb{T}\mathscr{D}$-module. The tensor product is the coequalizer of the pair of maps

$$\coprod_{j,k} \mathbb{T}X_k \otimes_{\mathscr{W}} \mathbb{T}\mathscr{D}(j,k) \otimes_{\mathscr{W}} \mathbb{E}(j) \rightrightarrows \coprod_j \mathbb{T}X_j \otimes_{\mathscr{W}} \mathbb{E}(j) \longrightarrow \mathbb{T}X \otimes_{\mathbb{T}\mathscr{D}} \mathbb{E}.$$

The right action of $\mathscr{E}$ is inherited from the right action of $\mathscr{E}$ on $\mathbb{E}$. Again, the obvious special cases start with $\mathscr{E}$ and $\mathscr{D}$, respectively, and take ϕ to be either[1]

$$\mathrm{id}\colon \mathbb{U}\mathscr{E} \longrightarrow \mathbb{U}\mathscr{E} \quad \text{or} \quad \eta\colon \mathscr{D} \longrightarrow \mathbb{U}\mathbb{T}\mathscr{D}.$$

With the level model structures, it is evident that if the original $(\mathbb{T},\mathbb{U})$ is a Quillen adjunction, then $\mathbb{U}$ in (1.3) creates the weak equivalences and fibrations in $\mathscr{W}^{\mathscr{E}}$, so that (1.3) is again a Quillen adjunction. We want sharp hypotheses for the following metatheorem. The versions in [2, 3] require the units of both $\mathscr{V}$ and $\mathscr{W}$ to be cofibrant, for starters.

Theorem 1.4. *If $(\mathbb{T},\mathbb{U})$ in (1.1) is a Quillen equivalence and $\phi\colon \mathscr{D} \longrightarrow \mathbb{U}\mathscr{E}$ is a weak equivalence, then $(\mathbb{T},\mathbb{U})$ in (1.3) is a Quillen equivalence.*

[1] These cases are treated separately and the general case is not introduced in [2, App A].

We want the non-unital variant as well.

Now we can go further. We are interested in model categories that have presheaf category approximations, so we naturally want to consider situations generalizing (refthreead) where we have a $\mathscr{V}$-category $\mathscr{M}$, a $\mathscr{W}$-category $\mathscr{N}$, and an enriched adjunction

$$\mathscr{M} \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathscr{N}. \tag{1.5}$$

We require of enriched adjunctions that they commute with tensors and cotensors, among other things. Thus $\mathbb{T}(X \otimes V) \cong \mathbb{T}X \otimes \mathbb{T}W$.

This is essentially the context of Dugger and Shipley [2, §3§4]. Their notion of an adjoint module relates the adjunctions (1.1). and (1.5). There goal is to apply the notion of quasi-equivalence to this conetext, for example allowing quotation of Theorem 1.4 more generally than it obviously applies. The starting point for what Dugger and Shipley do is [2, 3.2], which reads as follows with the notations above.

Proposition 1.6. *There are canonical bijections relating enriched binatural transformations of the following types, where $X \in \mathscr{M}$, $Y, Z \in \mathscr{N}$, $V \in \mathscr{V}$ and $W \in \mathscr{W}$.*

(i) *$\mathscr{V}$-transformations* $\mathbb{U}\underline{\mathscr{N}}(\mathbb{T}X, Y) \longrightarrow \underline{\mathscr{M}}(X, \mathbb{U}Y)$
(ii) *$\mathscr{W}$-transformations* $\mathbb{T}(X \odot V) \longrightarrow \mathbb{T}X \odot \mathbb{T}V$
(iii) *$\mathscr{V}$-transformations* $\mathbb{U}Y \odot \mathbb{U}W \longrightarrow \mathbb{U}(Y \odot W)$
(iv) *$\mathscr{V}$-transformations* $\mathbb{U}\underline{\mathscr{N}}(Y, Z) \longrightarrow \underline{\mathscr{M}}(\mathbb{U}Y, \mathbb{U}Z)$

Notation 1.7. For a transformation ω of the first sort, we continue to write ω for the corresponding transformation of the other three sorts.

Definition 1.8. The adjunction $(\mathbb{T}, \mathbb{U})$ of (1.5) is an adjoint module over the adjunction $(\mathbb{T}, \mathbb{U})$ of (1.1) if there is such a transformation ω that satisfies the equivalent conditions of Propositions 1.9 and 1.10. It is a Quillen adjoint module if, further, the equivalent conditions of Proposition 1.11 are satisfied.

Proposition 1.9. *Dugger and Shipley* [2, 3.6]

Proposition 1.10. *Dugger and Shipley* [2, 3.7]

Proposition 1.11. *Dugger and Shipley* [2, 3.5]

There is a kind of functoriality. If $(\mathbb{T}, \mathbb{U})$ of (1.5) is a $\mathscr{V}$-Quillen adjunction, it is a Quillen adjoint module over $(\mathrm{id}_{\mathscr{V}}, \mathrm{id}_{\mathscr{V}})$. There is an analogue for composition that reads as one would expect [2, 3.10, 3.11].

The relevance to quasi-equivalences in their work read as follows [2, 4.1, 4.2]. I'm ignoring terminology from [1] that I haven't waded through and that may implicitly include hypotheses, and I'm not at all sure I have translated correctly. We should check this out. With Theorem 1.4 it promises to show directly that in our context the $\mathscr{B}$ in the EKMM world pull back to something quasi-equivalent in the orthogonal world, thus maybe leading to a direct proof that $\mathscr{I}\mathscr{S}^{\mathscr{B}}$ is equivalent to $\mathscr{I}\mathscr{S}^{\mathbb{N}^{\#}\mathscr{B}}$. Or it might be a starting point of transferring all relevant presheaf categories $\mathscr{M}^{\mathscr{E}}$ to presheaf categories $\mathscr{I}\mathscr{S}^{\mathbb{N}^{\#}\mathscr{E}}$.

Theorem 1.12. *Let $(\mathbb{T}, \mathbb{S})$ specify a Quillen adjoint module $(\mathscr{M}, \mathscr{N})$ over a Quillen equivalence $(\mathbb{T}, \mathbb{U})$ between $\mathscr{V}$ and $\mathscr{W}$. Let $\mathscr{E}$ be a small full subcategory of $\mathscr{N}$. Then $\mathbb{U}\mathscr{E}$ is quasi-equivalent to the small full subcategory $\mathscr{D}$ of $\mathscr{M}$ with objects the $\mathbb{Q}\mathbb{U}\mathbb{R}Y$, $Y \in \mathscr{E}$ where $\mathbb{Q}$ and $\mathbb{R}$ are cofibrant approximation in $\mathscr{M}$ and fibrant approximation in $\mathscr{N}$; $\mathbb{R}$ can be deleted if $\mathbb{U}$ preserves all weak equivalences.*

Corollary 1.13. *With the hypotheses of the theorem, let $\mathscr{D}$ be a small full subcategory of $\mathscr{M}$. Then $\mathscr{D}$ is quasi-equivalent to $\mathbb{U}\mathscr{E}$, where $\mathscr{E}$ is the small full subcategory of $\mathscr{N}$ with objects the $\mathbb{RTQ}X$ for $X \in \mathscr{D}$.*

References

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[3] S. Schwede and B. Shipley. Equivalences of stable model categories. Algebr. Geom. Topol. 3(2003), 287–334.