ABEL'S THEOREM-5. PERMUTATIONS.

Definition 1. A *permutation* of degree n is a transformation of a finite set of numbers $\{1, 2, \dots, n\}$. All permutations of degree n form a group which is called *the symmetric group of degree n* and denoted by S_n. A *transposition* is a permutation that interchanges two elements and leaves the rest in place. A *cycle* is a permutation that permutes some elements cyclically and leaves the rest in place. Two cycles are called *independent*, if the sets of elements they permute do not intersect.

5.1. Prove that every permutation is a product of pairwise independent cycles.

There are two kinds of notation for permutations: one keeps track of images of all numbers: $\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ means that 1 is taken to i_1, etc. The other reflects the structure of the permutation as a product of pairwise independent cycles, writing $(i_1 i_2 \dots i_k)$ for a cycle that permutes $i_1, i_2, \dots, i_k$ cyclically (i_1 is taken to i_2, etc., and i_k is taken to i_1), and $(i_1 i_2 \dots i_k)(j_1 j_2 \dots k_m)$ for a product of two independent cycles.

5.2. Compute products of permutations:
a) $(123)(234)$, $(234)(123)$; **b)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$.

5.3. Find the inverses of permutations: **a)** $(123)(45)$ **b)** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$.

5.4. Prove that any cycle (and hence any permutation) is a product of some transpositions.

5.5. Prove that any transposition (and hence any permutation) is a product of some transpositions from the following (artificially chosen) set:
a) $(12), (23), (34), \dots, (n-1, n)$; **b)** $(12), (13), (14), \dots, (1n)$.

5.6. For a permutation $\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ define an *inversion* as a pair i_k, i_m, where $k < m$ and $i_k > i_m$. For example, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$ has the following inversions: $(2, 1)$, $(4, 3)$, $(5, 3)$. Prove that if a permutation σ has an even number of inversions, then $\sigma\tau$, where τ is a transposition, has odd number of inversions, and vice versa.

5.7. Prove that all permutations are divided into two groups: permutations that are products of an even number of transpositions, and permutations that are products of an odd number of transpositions. Those are called *even* and *odd* permutations respectively.

5.8. Find the parity of a cycle of order n (determine whether it is even or odd).

5.9. Prove that all even permutations of degree n form a group. This group is called *the alternating group of degree n*, and denoted A_n.

5.10. Prove that the numbers of even and odd permutations of degree n are the same.

5.11. Prove that A_n is a normal subgroup of S_n.

5.12. List all conjugacy classes in S_2, S_3, S_4, S_5. Prove that S_5 is not solvable using an argument similar to one in problems 4.5-4.7 (in fact, the group of rotations of a dodecahedron is isomorphic to A_5, which makes it the same argument, but you cannot use this fact unless you prove it).

5.13. Prove that S_4 is solvable.