

ABEL'S THEOREM-4. SOLVABLE GROUPS.

DEFINITION 1. Let $K(G)$ denote the commutator subgroup of a group G . A group G is *solvable*, if the sequence $G, K(G), K(K(G)),$ etc. ends, after a finite number of terms, with the trivial group $\{e\}$.

DEFINITION 2. A group G is solvable, if there is a sequence of subgroups

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$$

such that G_i is a normal subgroup of G_{i+1} , and G_{i+1}/G_i is commutative.

4.1. It is obvious that Definition 1 implies Definition 2. Prove that Definition 2 implies Definition 1.

4.2. Prove that every subgroup and every quotient group of a solvable group is solvable.

4.3. Prove that if $N \subset G$ is a normal subgroup, and N and G/N are solvable, then G is solvable. This implies in particular that a direct product of two solvable groups is solvable.

4.4. Prove that the group of symmetries of the tetrahedron is solvable.

In fact, every group that we have considered so far is solvable. We will now come up with an example of a non-solvable group.

4.5. All rotations of a regular dodecahedron can be divided into four classes:

- (1) the identity rotation;
- (2) non-identity rotations around the axes going through centers of opposite faces;
- (3) non-identity rotations around the axes going through opposite vertices;
- (4) non-identity rotations around the axes going through middle points of opposite edges.

How many elements are there in each class?

4.6. Prove that the classes from the previous problem are the conjugacy classes in the group G of rotations of the dodecahedron. It implies that if a normal subgroup $N \subset G$ contains an element in one of these classes, it must contain the whole class.

4.7. Prove that the group of the rotations of the dodecahedron does not have any normal subgroups other than $\{e\}$ and the whole group (such a group is called a *simple* group), and hence is not solvable.