ABEL'S THEOREM-3. HOMOMORPHISMS.

Definition 1. Let G and H be two groups. A map $f : G \to H$ is a *homomorphism* of groups, if for all elements $g_1, g_2 \in G$ we have $f(g_1 g_2) = f(g_1)f(g_2)$. The *kernel* of a homomorphism $f : G \to H$ is the preimage of e_H. The *image* of a homomorphism is the image of G. The kernel is a subset of G, the image is a subset of H.

3.1. Let $f : G \to H$ be a homomorphism. Prove that $f(e_G) = e_H$, and for all $g \in G$ $f(g^{-1}) = (f(g))^{-1}$. What can be said about the order of $f(g)$, if the order of g is n?

3.2. Let $f : G \to H$ be a surjective homomorphism. Prove that if G is commutative, then H is commutative. Is it true that if H is commutative, then G is commutative?

3.3. Prove that the kernel of a homomorphism $f : G \to H$ is **a)** a subgroup **b)** a normal subgroup of G.

3.4. Prove that for any normal subgroup $N \subset G$, the map $\phi : G \to G/N$ that maps an element $g \in G$ into its coset $gN = Ng$, is a homomorphism with kernel N. It is called *the natural*, or *canonical homomorphism* of G onto G/N. Thus, any normal subgroup is a kernel of some homomorphism.

Theorem 1. For any surjective homomorphism $f : G \to H$ with kernel N there is an isomorphism $\psi : G/N \to H$ such that $f = \psi \circ \phi$, where $\phi : G \to G/N$ is the canonical homomorphism. The map ψ maps $X \in G/N$, which is, a coset of N in G, to $f(x)$, where x is any element of the coset X.

3.5. Prove Theorem 1 (prove that ψ is well-defined, a homomorphism, surjective and injective).

3.6. Let R be the (infinite) group of rotations of the plane around the origin. Let $\mathbb{Z}_n \subset R$ be the group of rotations generated by the rotation by $2\pi/n$. Prove that $\mathbb{Z}_n \subset R$ is normal and find the quotient group.

3.7. Give examples of the following situations:

- **a)** N_i is a normal subgroup of G_i, $i = 1, 2$. $N_1 \simeq N_2$, $G_1/N_1 \simeq G_2/N_2$, but G_1 and G_2 are not isomorphic.
- **b)** N_1, N_2 are normal subgroups of G. N_1 and N_2 are isomorphic, but G/N_1 and G/N_2 are not isomorphic.
- **c)** N_1, N_2 are normal subgroups of G. G/N_1 and G/N_2 are isomorphic, but N_1 and N_2 are not isomorphic.

3.8. Let $f : G \to H$ be a homomorphism. Prove that the image of a subgroup of G (resp. preimage of a subgroup of H) is a subgroup of H (resp. of G).

3.9. Prove that the preimage of a normal subgroup under any homomorphism is a normal subgroup.

3.10. Prove that the image of a normal subgroup under a surjective homomorphism is a normal subgroup. Observe that surjectivity of the homomorphism here is a sufficient, but not a necessary condition (for the image of any normal subgroup to be normal).