# ABEL'S THEOREM-2. NORMAL SUBGROUPS AND QUOTIENT GROUPS.

Definition 1. For a group $G$, a subgroup $H \subset G$ and an element $x \in G$, define *the left (resp. right) coset of* $H$ *in* $G$ *generated by* $x$ to be the subset $xH = \{xh | h \in H\}$ in $G$ (resp. $Hx = \{hx | h \in H\}$).

**2.1.** According to the problem 1.23 there is a subgroup $H$ of $\mathbb{Z}_{12}$, $H \simeq \mathbb{Z}_4$. List all right cosets of $H$ in $\mathbb{Z}_{12}$.

**2.2.** List all left and right cosets of $H$ in $G$ for $H$ and $G$ being as follows:

- **a)** $G$ the group of all symmetries of an equilateral triangle, $H = \{e, r\}$, where $r$ is a reflection;
- **b)** $G$ the group of all symmetries of a square, $H = \{e, r\}$, where $r$ is a reflection with respect to a diagonal;
- **c)** $G$ the group of all symmetries of a square, $H = \{e, s\}$, where $s$ is the central symmetry.

**2.3.** Prove that every element of $G$ belongs to a unique left (resp. right) coset of $H$ in $G$, so that left (resp. right) cosets form a partition of $G$ into subsets.

**2.4.** Prove that every left (resp. right) coset of $H$ in $G$ is in bijection with $H$.

Define the *order* of a finite group as the number of elements in this group. The preceding two problems imply the following

Theorem 1. (Lagrange's Theorem) The order of a subgroup $H$ of a finite group $G$ divides the order of $G$.

**2.5.** Prove that any group of a prime order $p$ is isomorphic to the cyclic group $\mathbb{Z}_p$, and that every element in it except for the identity is a generator.

**2.6.** Suppose that $m$ divides $n$. Given a group $H$ of order $m$, construct a group $G$ of order $n$ containing $H$.

**2.7.** Suppose that $m$ divides $n$. Is it possible that a group of order $n$ does not have any subgroups of order $m$?

Definition 2. For $g \in G$ let the *inner automorphism* $\phi_g : G \to G$ be the automorphism that sends $h \in G$ to $ghg^{-1}$ (also called *conjugation by* $g$).

**2.8.** Prove that $\phi_g$ is an isomorphism of $G$ onto itself, and that the maps $\phi_g$ for all $g \in G$ form a group isomorphic to $G$.

**2.9.** Prove that in any group the orders of elements $ab$ and $ba$ are equal.

**2.10.** Consider the group of symmetries of an equilateral triangle $ABC$. Denote by $\begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$ the symmetry that takes $A$ to $B$, $B$ to $A$, and $C$ to $C$ (in this case, a reflection). Let us rename the vertices of the triangle, say, $A$ will be $\mathcal{C}$, $B$ will be $\mathcal{A}$, and $C$ will be $\mathcal{B}$. Then the above reflection takes $\mathcal{C}$ to $\mathcal{A}$, $\mathcal{A}$ to $\mathcal{C}$, and $\mathcal{B}$ to $\mathcal{B}$, and may be written down as $\begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{C} \\ \mathcal{C} & \mathcal{B} & \mathcal{A} \end{pmatrix}$. Now, if we identify letters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, with $A$, $B$, and $C$, the old notation for the symmetry of the triangle gives a map $\Phi : \{A, B, C\} \to \{A, B, C\}$, the new notation for the same symmetry gives another map $\Psi : \{A, B, C\} \to \{A, B, C\}$, and the renaming is yet another map $g : \{A, B, C\} \to \{A, B, C\}$. Prove that $\Psi = g\Phi g^{-1}$ (i.e. renaming vertices is equivalent to applying an inner automorphism to the corresponding permutation).

Definition 3. For an element $h \in G$ its *conjugacy class* in $G$ is the set of elements of $G$ of the form $ghg^{-1}$, for all $g \in G$. It is straightforward that conjugacy classes form a partition of $G$.

**2.11.** Find the partition into conjugacy classes of the group of symmetries of the
**a)** triangle **b)** tetrahedron **c)** the group of rotations of the tetrahedron.

Definition 4. A subgroup $H$ of a group $G$ is a *normal subgroup*, if for all elements $g \in G$ we have $gHg^{-1} = H$ as a set, i.e. $\forall h \in H, g \in G \; ghg^{-1} \in H$.

**2.12.** Which subgroups of the group of symmetries of the
**a)** triangle **b)** tetrahedron **c)** the group of rotations of the tetrahedron are normal?

**2.13.** Prove that if $H$ is the only subgroup of $G$ of a given order, then $H$ is normal.

**2.14.** Prove that any subgroup of the *center* of the group $G$ (the set of all elements that commute with all elements of $G$, $Z(G) = \{z \in G | \forall g \in G \; zg = gz\}$) is a normal subgroup. In particular, if the group $G$ is *commutative*, i.e. all elements of $G$ commute, then every subgroup of $G$ is normal.

**2.15.** Prove that the intersection of any number of normal subgroups is a normal subgroup.

**2.16.** Prove that if $H_1 \subset G_1$ and $H_2 \subset G_2$ are normal subgroups, then $H_1 \times H_2$ is a normal subgroup in $G_1 \times G_2$.

Theorem 2. A subgroup $H \subset G$ is normal if and only if right and left cosets of $H$ in $G$ coincide.

**2.17.** Prove Theorem 2.

**2.18.** Prove that a subgroup $H \subset G$ of order $|G|/2$ is normal.

Example 1. It can happen that $K \subset H \subset G$, $K$ is a normal subgroup of $H$, $H$ is a normal subgroup of $G$, but $K$ is not a normal subgroup of $G$. Let $G$ be the group of symmetries of a square, $H$ be the group generated by two reflections $r_1$, $r_2$ with respect to the diagonals, and $K$ be the group generated by $r_1$. Then $|G| = 8$, $|H| = 4$, $|K| = 2$, so by problem 2.18, $H$ is normal in $G$ and $K$ is normal in $H$, but one can see that $K$ is not normal in $G$.

**2.19.** Let $N$ be a normal subgroup of $G$. Prove that if $x_1$, $x_2$ belong to the same coset $X$ of $N$ in $G$, and $y_1$, $y_2$ belong to the same coset $Y$ of $N$ in $G$, then $x_1y_1$ and $x_2y_2$ also belong to the same coset of $N$ in $G$.

Definition 5. Let $N$ be a normal subgroup of $G$. By Theorem 2, left and right cosets of $N$ coincide, so we can consider the set of cosets $S$. By problem 2.19, the product of any two elements taken from the same two cosets $X, Y \in S$ belongs to the same coset, so this coset can be associated to the pair $X, Y$, giving a binary operation on $S$.

**2.20.** Prove that $S$ with this binary operation is a group. It is called the *quotient group of $G$ by $N$*, and denoted $G/N$.

**2.21.** Let $G$ be the group of symmetries of the square, and let $N$ be its subgroup generated by the central symmetry. Prove that $N$ is normal and find the quotient group.

**2.22.** Find quotient groups for all normal subgroups of the group of symmetries of the tetrahedron.

**2.23.** Prove that if $N_i \subset G_i$, $i = 1, 2$ is a normal subgroup, then $(G_1 \times G_2)/(N_1 \times N_2) \simeq (G_1/N_1) \times (G_2 \times N_2)$.

Definition 6. Let $G$ be a group. For two elements $a, b \in G$ the element $aba^{-1}b^{-1}$ is called *the commutator* of $a, b$. The subgroup of $G$ generated by commutators of all elements is called *the commutator subgroup* of $G$. Note that the commutator subgroup may contain elements that are not commutators of any pair of elements themselves, but that are products of several commutators.

**2.24.** Prove that the commutator subgroup is a normal subgroup.

**2.25.** Find the commutator subgroup of the group of symmetries of the tetrahedron.

**2.26.** Prove that the commutator subgroup of a group $G$ is trivial (i.e. consists only of the identity) if and only if $G$ is commutative.

**2.27.** Prove that the quotient group by the commutator subgroup is commutative.

**2.28.** Prove that if the quotient group $G/N$ is commutative, then $N$ contains the commutator subgroup of $G$.