ABEL'S THEOREM - 9. RIEMANN SURFACES.

Example 1. Consider the multi-valued function $\tau(x) : \mathbb{R}_{\geq 0} \to \mathbb{R}$, that assigns to x the set of solutions for $t^2 = x$. It takes two values for every x except $x = 0$, where it takes only one value. It can, however, be viewed as a combination of two single-valued continuous functions $\tau_1(x) = \sqrt{x}$ and $\tau_2(x) = -\sqrt{x}$, which are called the *branches* of τ. For all $x \neq 0$ and one of the values of τ at x, we can tell which branch of τ this value belongs to. Another approach to this situation will be considering the curve C given by the equation $x = y^2$ in the half plane $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Then we can define a continuous function σ on C by $\sigma(x, y) = y$, which will induce the function τ on $\mathbb{R}_{\geq 0}$ in the following sense: the projection $(x, y) \mapsto x$ defines a map $p : C \to \mathbb{R}_{\geq 0}$, and $\tau(x)$ is the set of values of σ on the set $p^{-1}(x)$ of the preimages of x.

Example 2. The above construction for displaying multivalued functions in terms of single-valued functions works quite generally. Consider another multi-valued function $\mu(z) = \operatorname{Re}\sqrt{z} : \mathbb{C} \to \mathbb{R}$. Then we can consider the surface $S \subset \mathbb{C} \times \mathbb{R}$ given by the parametric equation $S = \{(w^2, \operatorname{Re} w) | w \in \mathbb{C}\}$.

9.1. Sketch the surface S in the three-dimensional space $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$.

The function μ is induced by the single-valued continuous function ν on S, defined by $\nu(z, y) = y$. Here "induced" means the same thing as in the previous example: the values of μ on z are the values of ν on $p^{-1}(z)$, where $p : S \to \mathbb{C}$ is the projection.

Our goal is to construct such surfaces S for certain multi-valued functions of a complex variable.

Definition 1. Let $\mu(z)$ be a multi-valued function from $\mathbb{C}$ to $\mathbb{C}$. By the *Riemann surface* associated to μ we will mean a surface S with a map $p : S \to \mathbb{C}$ and a continuous function $\nu : S \to \mathbb{C}$ such that the set of values of μ at $z \in \mathbb{C}$ is the set of values of ν on the points $p^{-1}(z)$. Such a surface is unique in some sense (see problem 9.11).

Example 3. The Riemann surface for $\mu(z) = \sqrt{z}$ is the surface $S \subset \mathbb{C} \times \mathbb{C}$ given by the equation $z = w^2$, with $p : (z, w) \mapsto z$ and $\nu(z, w) = w$.

In the following S is assumed to be the Riemann surface of $\sqrt{z}$, and p, μ, ν are supposed to be as in the example 3. Let us investigate the geometry of S.

9.2. For $a \in \mathbb{C}$, let the domain $D_a \subset \mathbb{C}$ be $\mathbb{C} - \mathbb{R}_{\geq 0}a$, the complex plane $\mathbb{C}$ without the ray generated by a. Prove that for any $a \in \mathbb{C}$ the preimage of D_a under p consists of two distinct components S_1^a, S_2^a, and for each of them $p_i = p|_{S_i^a} : S_i^a \to D_a$ is bijective. The composition $\nu \circ p_i^{-1}$ for $i = 1, 2$ then gives two well-defined continuous functions on D_a. Those are called *branches* of $\sqrt{z}$ on D_a.

9.3. Let $C = z(t)$ be a continuous curve in $\mathbb{C}$ that does not hit 0. Prove that for any choice of a starting point $s_0 \in S$, such that $p(s_0) = z(0)$, there is a unique continuous function $s(t)\colon [0, 1] \to S$ such that $s(0) = s_0$ and $p(s(t)) = z(t)$.

9.4. Let $C = z(t)$ be a continuous curve in $\mathbb{C}$ that does not hit 0, and let $p^{-1}(z(0)) = \{s_0, r\}$. Assume that C is closed, so $z(0) = z(1)$. Let $s(t)$ be the function from problem 9.3, such that $s(0) = s_0$. Prove that $s(1) = s_0$ if and only if C goes around zero an even number of times, otherwise $s(1) = r$.

9.5. In the notation of problem 9.2, let $a = 1$, $S_1 = S_1^1$, $S_2 = S_2^1$, so that $\nu_1(-1) = i$, $\nu_2(-1) = -i$. Both S_1 and S_2 are in bijection with $\mathbb{C} - \mathbb{R}_{\geq 0}$. Let $C = z(t) = 1 + (1-2t)i$, and let $s_0 \in S_1$ be such that $p(s_0) = z(0) = 1+i$. Let $s(t)$ be as in problem 9.3. Find $\nu(s(0))$, $\nu(s(1))$. Is $s(1)$ in S_1 or S_2?

9.6. In the notation of problem 9.2, let $a = 1$, $S_1 = S_1^1$, $S_2 = S_2^1$. Draw two copies of $\mathbb{C} - \mathbb{R}_{\geq 0}$ representing S_1 and S_2. Mark the intersections with S_i^{-1+i}, $i = 1, 2$.

Now we see that we can think of S as the surface glued from two sheets, that are copies of $\mathbb{C}$ cut along $\mathbb{R}_{\geq 0}$: the upper side of each cut is connected to the lower side of the cut on the other sheet. Every curve $C \subset \mathbb{C}$ can be lifted to a curve $\widehat{C} \subset S$. If C is a circle around 0, then $\widehat{C}$ is not closed, and its start point and its end point lie over the same point in $\mathbb{C}$, but on different sheets. On the other hand, for any point $z_0 \neq 0$ there exists a sufficiently small circle around z_0 such that its lift to S is closed. Such points as zero in this example are called *branching* points. Going around a branching point in $\mathbb{C}$ may get us into a different sheet in S.

9.7. Describe the Riemann surface for the function $\sqrt[3]{z}$.

9.8. Let $P(z)$ be a polynomial. Consider the function $\mu(z) = \sqrt[n]{P(z)}$. Prove that z_0 is a branching point of μ if and only if $P(z_0) = 0$.

9.9. Consider the function $\mu(z) = \sqrt{z(z-1)}$. Prove that μ has two distinct branches on the complex plane with two cuts $\mathbb{C} - (\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1})$. It means that its Riemann surface S can be constructed as the union of two sheets S_1 and S_2, each sheet isomorphic to $\mathbb{C} - (\mathbb{R}_{\leq 0} \cup \mathbb{R}_{\geq 1})$, glued along the cuts.

9.10. Consider the same function $\mu(z) = \sqrt{z(z-1)}$. Prove that μ has two distinct branches on the complex plane with one cut $\mathbb{C} - [0, 1]$. It means that its Riemann surface S can be constructed as the union of two sheets S_1 and S_2, each sheet isomorphic to $\mathbb{C} - [0, 1]$, glued along the cut.

9.11. Establish a bijection between Riemann surfaces constructed in problems 9.9 and 9.10.

9.12. In fact, the configuration of the cuts that we use to construct the Riemann surface does not matter. What matters is what happens at the branching points. Let $S_1, \ldots, S_n$ be the n sheets of a Riemann surface for a function μ. For each branching point z_0, let going counterclockwise along a small circle around z_0 bring us from S_i to $S_{\sigma(i)}$ for some function $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Prove that σ is bijective.

Example 4. Consider the function $\sqrt{z^2}$. It has two values for $z \neq 0$, and one value for $z = 0$, however, for every circle around zero we can choose a continuous branch of $\sqrt{z^2}$ on the whole circle. In other words, every circle around zero can be lifted to a closed circle on the Riemann surface of $\sqrt{z^2}$. We will still consider such point a branching point, but the permutation of sheets attached to it will be identity.

9.13. Let $\mu = \sqrt{z} + \sqrt{z-1}$.

- **a)** Find the number of values of μ at a generic point (i.e. not a branching point).
- **b)** Find the branching points of μ.
- **c)** Describe the Riemann surface (i.e. enumerate the sheets and find corresponding permutations for all branching points).

9.14. Let $\mu = \sqrt[3]{\sqrt{z} + 1}$.

a) Find the number of values of μ at a generic point.
b) Find the branching points of μ.
c) Describe the corresponding Riemann surface.

9.15. Let $\mu = \sqrt{\sqrt{z} - \sqrt{z-1}}$.
a) Find the number of values of μ at a generic point.
b) Find the branching points of μ.
c) Describe the corresponding Riemann surface.

9.16. Let $\mu = \sqrt{\sqrt{z} + \sqrt{z-1}}$.
a) Find the number of values of μ at a generic point.
b) Find the branching points of μ.
c) Describe the corresponding Riemann surface.

Definition 2. Let S be the Riemann surface of a function μ, represented by a set of n sheets and a permutation σ_i attached to every branching point z_i of μ. The subgroup of the symmetric group S_n generated by $\sigma_1, \dots, \sigma_k$ is called the *monodromy group* of S.

9.17. Let μ be a multi-valued function, S its Riemann surface, and G its monodromy group. Let $C = z(t)$ be a closed curve in $\mathbb{C}$ that does not hit any of the branching points of μ. Let the preimage of $z(0)$ be $r_1, \dots, r_n$. For any of r_i, there is a unique continuous function $s(t) : [0,1] \to S$ such that $s(0) = r_i$, and $p(s(t)) = z(t)$ (cf. problem 9.3). Then $s(1) = r_{\sigma(i)}$, where σ is some permutation of $r_1, \dots, r_n$ (cf. problem 9.12). Prove that $\sigma \in G$.

9.18. Find the monodromy groups for the Riemann surfaces in problems 9.13-9.16.