

ABEL'S THEOREM-8. BASIC THEOREM OF ALGEBRA.

8.1. Let $C = z(t)$ be a closed curve with variation of argument ϕ . What is the variation of argument along the curve $z^2(t)$?

8.2. Let $C = z(t)$ be a closed curve that goes k_1, k_2, k_3, k_4 times around $0, 1, i, -i$ respectively. How many times does the curve $f(z(t))$ go around the origin if $f(z)$ is

a) $z^2 - z$ **b)** $z^2 + 1$ **c)** $(z + iz)^4$ **d)** $z^3 - z^2 + z - 1$?

DEFINITION 1. We say that a curve $z_1(t)$ is continuously deformed into a curve $z_2(t)$ if there is a continuous map $\gamma(t, s) : [0, 1] \times [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(t, 0) = z_1(t)$ and $\gamma(t, 1) = z_2(t)$. We also say that the map γ is a *homotopy* between the curves $z_1(t)$ and $z_2(t)$, and that the curves are *homotopic*.

We are going to take the next theorem for granted. It can be proved using compactness of $[0, 1]$, for both variables t and s .

THEOREM 1. Let C_1 and C_2 be two homotopic closed curves. Let there be a homotopy $\gamma(t, s)$ between them such that $\gamma(t, s)$ is a closed curve for every s and $\gamma(t, s) \neq 0$ for all t, s . In other words, C_1 can be continuously deformed into C_2 while never hitting the origin. Then the variations of the argument along C_1 and C_2 are equal.

8.3. Let C be the curve $z(t)$, and let $|w(t)| < |z(t)|$ for all t . Prove that the curve $z(t) + w(t)$ is homotopic to C , and, moreover, there is a homotopy $\gamma(t, s)$ between them such that the curve $\gamma(t, s)$ is closed for every s and $\gamma(t, s) \neq 0$ for all t, s .

8.4. (The Lady with the Dog) Let C_1 be a curve with parametric equation $z(t)$, and let $w(t)$ be such that $|w(t)| < |z(t)|$ for all t , and $w(0) = w(1)$, so that the curve $C_2 = z(t) + w(t)$ is also closed. Prove that the variations of the argument along the curves C_1 and C_2 are equal.

8.5. Prove that for a polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ such that $a_0 \neq 0$ **a)** there is such A that for $|z| > A$ we have $|a_n z^n| > |a_{n-1} z^{n-1} + \dots + a_1 z + a_0|$, and **b)** there is such δ that for $|z| < \delta$ we have $|a_n z^n + \dots + a_1 z| < |a_0|$.

8.6. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial, $a_0 \neq 0$. Consider the curves $\Gamma_r = P(C_r)$, where $C_r = r \cos 2\pi t + ir \sin 2\pi t$ is the circle of radius r , for various r . Prove that for $r < \delta$ the variation of the argument along Γ_r is 0, and for $r > A$ it is n , where δ and A are as in the previous problem.

8.7. (Basic Theorem of Algebra) Prove that every polynomial over \mathbb{C} has a root.

8.8. (Bezout's Theorem) Prove that if z_0 is a root of the polynomial $P(z)$, then $P(z)$ is divisible by $(z - z_0)$ (you can use long division of polynomials: for every $p(z), q(z)$ we have $p(z) = q(z)k(z) + r(z)$, where the degree of $r(z)$ is less than the degree of $q(z)$).

8.9. Prove that every polynomial over \mathbb{C} is a product of linear polynomials:

$$P(z) = (z - z_1)^{k_1} (z - z_2)^{k_2} \dots (z - z_m)^{k_m}.$$