

ABEL'S THEOREM-7. CONTINUOUS CURVES.

For a real- or complex-valued function of a real or complex variable, the notion of continuity is defined in the same way that for a real-valued function of a real variable: a function $f(x)$ is continuous at x_0 , if for all $0 < \varepsilon \in \mathbb{R}$ there is $\delta \in \mathbb{R}$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. Here $||$ means absolute value for a real value, and the norm for a complex value. All standard facts that are true for continuous functions of real variable, are also true in this more general case.

DEFINITION 1. By a *continuous curve* in the complex plane we will mean the image of a continuous map from $[0, 1]$ to \mathbb{C} . This map (a function $z(t) : [0, 1] \rightarrow \mathbb{C}$) is called the *parametric equation* of the curve.

7.1. Trace the following curves on the complex plane: **a)** $z(t) = it$ **b)** $z(t) = it^2$ **c)** $z(t) = t^2 + it$ **d)** $z(t) = \cos t + i \sin t$ **e)** $z(t) = t \cos 2\pi t + it \sin 2\pi t$
f) $z(t) = \begin{cases} 2t - 1 + i - 4it^2 & \text{if } 0 \leq x < 1/2 \\ 2t - 1 + 4ti - 2i & \text{if } 1/2 \leq x \leq 1 \end{cases}$

7.2. Write a parametric equation for a curve in the shape of the letter S.

7.3. Let $z(t) = \cos \pi t + i \sin \pi t$.

- Find all values of $\text{Arg } z(t)$ as a multi-valued function of t .
- Choose one value of $\text{Arg } z(t)$ for each t so that the resulting function is continuous and takes 0 at $t = 0$.
- Choose one value of $\text{Arg } z(t)$ for each t so that the resulting function is continuous and takes $2\pi k$ at $t = 0$.

7.4. Same question as in problem 7.3 but for the curve $z(t) = (t + 1)(\cos 3\pi t + i \sin 3\pi t)$.

THEOREM 1. Let C be a continuous curve with a parametric equation $z(t)$, that does not cross the origin. Then for any value $\phi_0 \in \text{Arg}(z(0))$ there is a unique continuous function $\phi(t)$ such that $\phi(t) \in \text{Arg}(z(t))$ and $\phi(0) = \phi_0$.

7.5. Prove the uniqueness part in the Theorem above.

7.6. In the terminology of Theorem 1, prove that the function $\psi(t) = \phi(t) - \phi(0)$ does not depend on the choice of ϕ_0 . The value $\phi(1) - \phi(0)$ is called *variation of the argument along the curve C*.

7.7. Prove that if C is closed (i.e. $z(1) = z(0)$), then the variation of the argument along C is $2\pi k$ for some $k \in \mathbb{Z}$. In this case we say that C goes k times around the origin. We can generalize this notion to any point: we say that $z(t)$ goes k times around $z_0 \in \mathbb{C}$ if a curve $\tilde{z}(t) = z(t) - z_0$ goes k times around the origin.

7.8. Let C be a closed curve with a parametric equation $z(t)$. Let C go k times around a point z_0 .

- Consider the curve \tilde{C} that differs from C by a choice of direction: $\tilde{z}(t) = z(1 - t)$. Prove that the number of times \tilde{C} goes around z_0 is $-k$.
- Consider the curve \hat{C} that differs from C by a choice of an initial point: $\hat{z}(t) = z(\{t + \alpha\})$, where $\{ \}$ is the fractional part, and α is some number. Prove that the number of times \hat{C} goes around z_0 is k .