ABEL'S THEOREM-6. COMPLEX NUMBERS.

DEFINITION 1. The set $\mathbb{R} \times \mathbb{R}$ of pairs of real numbers with addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$ and multiplication defined by $(x_1, y_1)(x_2, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + y_1 x_2)$ is called *the field of complex numbers* and denoted by $\mathbb{C}$. The element $(1, 0)$ is the multiplicative unity and is denoted by 1; for $x \in \mathbb{R}$, we can denote the element $(x, 0)$ by x. The element $(0, 1)$ is denoted by i. Thus, for $x, y \in \mathbb{R}$, $(x, y) = x + yi$. The element $x - yi$ is called *the conjugate* of the element $z = x + yi$, and is denoted by $\overline{z}$.

6.1. Prove that the above operations are commutative and associative, and that every element $z = x + yi$ has an additive inverse $-z$, such that $z + (-z) = 0$, and every non-zero element has a multiplicative inverse z^{-1} such that $zz^{-1} = z^{-1}z = 1$. Thus, $\mathbb{C}$ with addition is a commutative group, and $\mathbb{C}^* = \mathbb{C} - \{0\}$ with multiplication is a commutative group.

6.2. Prove that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, and $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

6.3. Prove that $z_1 z_2 \in \mathbb{R}$ if and only if $z_2 = \lambda \overline{z_1}$, where $\lambda \in \mathbb{R}$.

DEFINITION 2. The field $\mathbb{C}$ may be viewed as a plane $\mathbb{R}^2$. For a complex number $z = x + yi$, define the *norm* $|z|$ to be the distance $\sqrt{x^2 + y^2}$ from the point (x, y) to the origin, and define the *argument* $\operatorname{Arg} z$, $z \neq 0$, to be the set of possible values of the angle between the x axis and the vector directed from the origin to (x, y). Then for any value $\phi \in \operatorname{Arg} z$ we have $z = |z| \cdot (\cos\phi + i \sin\phi)$. The set $\operatorname{Arg} z$ is an infinite subset of $\mathbb{R}$, and any two elements in it differ by $2\pi k$, $k \in \mathbb{Z}$, so cos and sin are the same for all values of the argument.

6.4. Prove that $|z_1 z_2| = |z_1||z_2|$, $|\overline{z}| = |z|$, and $|z_1 + z_2| \leq |z_1| + |z_2|$.

6.5. Prove that $r_1(\cos\phi_1 + i\sin\phi_1) \cdot r_2(\cos\phi_2 + i\sin\phi_2) = r_1 r_2(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2))$.

6.6. Prove that $(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi)$. Derive a formula for $\cos 3\alpha$ and $\sin 3\alpha$ in terms of $\cos\alpha$, $\sin\alpha$.

6.7. Calculate $(1 - \sqrt{3}i)^{100}/2^{100}$.

DEFINITION 3. The argument $\operatorname{Arg} z$ is a multi-valued function. Another example of a multi-valued function is $\sqrt[n]{z}$, a function that assigns to $z \in \mathbb{C}$ the set of complex solutions for the equation $x^n = z$.

6.8. Find all solutions for the following equations in the algebraic form $(x + yi)$:

a) $x^6 = 1$;
b) $x^3 = 8$;
c) $x^4 = 1 + i$.

6.9. Prove that for $z \neq 0$ the function $\sqrt[n]{z}$ takes exactly n values.