Chapter 5

Real Semisimple Lie Groups

Our study of real semisimple Lie groups and algebras is based on the theory of complex semisimple Lie groups developed in Ch. 4. This is possible because the complexification of a real semisimple Lie algebra is also semisimple (see 1.4.7). However, the correspondence between real and complex semisimple Lie algebras established with the help of the complexification is not one-to-one; any complex semisimple Lie group has at least two non-isomorphic real forms. As it turns out, to describe the real forms of a given complex semisimple Lie algebra $\mathfrak{g}$ is the same as to classify the involutive automorphisms of $\mathfrak{g}$ up to conjugacy in $\operatorname{Aut}\mathfrak{g}$. This classification is easily obtained from the results of 4.4. The global classification of real semisimple Lie groups makes use of the so-called Cartan decomposition of these groups which also plays an important role in various applications of the Lie group theory.

§ 1. Real Forms of Complex Semisimple Lie Groups and Algebras

The main goal of this section is to classify real semisimple Lie algebras. After we discuss some general properties of real forms of complex semisimple Lie groups and algebras we reduce the classification to the listing (up to conjugacy) of the involutive automorphisms of complex simple Lie algebras. The latter problem is easily solved by methods of 4.4.

1°. Real Structures and Real Forms. Recall (see 2.3.6) that the real forms of a complex Lie algebra $\mathfrak{g}$ are in a one-to-one correspondence with the involutive antilinear automorphisms of this algebra. Namely, to each real form $\mathfrak{h} \subset \mathfrak{g}$ associated is the complex conjugation $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ with respect to $\mathfrak{h}$ and to each involutive antilinear automorphism $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ associated is the real form $\mathfrak{g}^\sigma = \{x \in \mathfrak{g}\colon \sigma(x) = x\}$ of $\mathfrak{g}$. Therefore, the involutive antilinear automorphisms of a complex Lie algebra $\mathfrak{g}$ will be called *real structures* on $\mathfrak{g}$.

Problem 1. If σ is a real structure on a complex Lie algebra $\mathfrak{g}$ and $\varphi \in \operatorname{Aut}\mathfrak{g}$, then $\varphi\sigma\varphi^{-1}$ is also a real structure and $\mathfrak{g}^{\varphi\sigma\varphi^{-1}} = \varphi(\mathfrak{g}^\sigma)$. Let σ' be another real structure, then the real forms $\mathfrak{g}^\sigma$ and $\mathfrak{g}^{\sigma'}$ are isomorphic if and only if $\mathfrak{g}^{\sigma'} = \varphi(\mathfrak{g}^\sigma)$ or, equivalently, $\sigma' = \varphi\sigma\varphi^{-1}$ for some $\varphi \in \operatorname{Aut}\mathfrak{g}$.

Let G be a complex Lie group, H its real Lie subgroup (i.e. a Lie subgroup of G considered as a real Lie group). The subgroup H is a *real form* of G if

a) its tangent algebra $\mathfrak{h}$ is a real form of $\mathfrak{g}$;
b) H has a nonzero intersection with any connected component of G.

Theorem 1.3.1 implies that b) is equivalent to the identity

$$G = HG^0. \tag{1}$$

Problem 2. If G is a complex algebraic group then its real form H in the sense of 3.1.2 is also its real form in the sense of the above definition.

Problem 3. If H is a real form of a complex Lie group G then the center $Z(H)$ of H coincides with $H \cap Z(G)$.

A *real structure* on a complex Lie group G is an involutive differentiable in a real sense homomorphism $S: G \to G$, such that dS is a real structure on the tangent algebra $\mathfrak{g}$ of G. For instance, the complex conjugation of a complex algebraic group G with respect to its real form (or, which is the same, an involutive antiholomorphic automorphism of G) is a real structure on G. If S is a real structure on a connected complex Lie group G then by Problem 1.2.31 the subgroup G^S is a real form of G and its tangent algebra coincides with $\mathfrak{g}^{dS}$. For algebraic groups the similar fact was proved in Ch. 3 (Problem 3.1.10).

In what follows an involutive antiholomorphic automorphism of an algebraic group will be called an *algebraic real structure* and a real form in the sense of the theory of algrbraic groups will be called an *algebraic real form*.

Example 1. Let $T = (\mathbb{C}^*)^n$ be the n-dimensional algebraic torus. The algebraic real structure $(z_1, \ldots, z_n) \mapsto (\bar{z}_1, \ldots, \bar{z}_n)$ determines the real form $(\mathbb{R}^*)^n$ of T. Its tangent algebra is the real form $\mathfrak{t}(\mathbb{R}) = \mathbb{R}^n$ of $\mathfrak{t} = \mathbb{C}^n$ considered in 3.3.2.

Example 2. The algebraic real structure $(z_1, \ldots, z_n) \mapsto (\bar{z}_1^{-1}, \ldots, \bar{z}_n^{-1})$ determines the real form $\mathbb{T}^n = \{(z_1, \ldots, z_n): |z_1| = \cdots = |z_n| = 1\}$ of T with the tangent algebra $i\mathbb{R}^n \subset \mathbb{C}^n$.

Example 3. The algebraic real structure $A \mapsto \bar{A}$ on $\mathrm{GL}_n(\mathbb{C})$ determines the real forms $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R}) \subset \mathfrak{gl}_n(\mathbb{C})$. The same example can be given in a coordinate-free form. Let V be a finite-dimensional vector space over $\mathbb{R}$. Then on the group $\mathrm{GL}(V(\mathbb{C}))$ a real structure S is defined by the formula

$$S(A)(v) = \overline{A(\bar{v})} \qquad (v \in V(\mathbb{C})). \tag{2}$$

The corresponding real form is the subgroup of the group of linear transformations defined over $\mathbb{R}$, naturally identified with $\mathrm{GL}(V)$. The Lie algebra $\mathfrak{gl}(V)$ is embedded into $\mathfrak{gl}(V(\mathbb{C}))$ as the real form tangent to $\mathrm{GL}(V)$.

Example 4. If V is a finite-dimensional algebra over $\mathbb{R}$ then an antiautomorphism S defined by (2) transforms the group $\mathrm{Aut}(V(\mathbb{C}))$ into itself and determines an algebraic real structure there. The corresponding real form is $\mathrm{Aut}\,V$. Passing to tangent algebras we get the real form $\mathfrak{der}\,V$ of $\mathfrak{der}(V(\mathbb{C}))$ (see Example 2 in 1.2.3).

Example 4 enables us to generalize one of important properties of complex semisimple Lie algebras to real ones.

Problem 4. If $\mathfrak{g}$ is a real semisimple Lie algebra then $\mathfrak{Der}\,\mathfrak{g} = \operatorname{ad}\mathfrak{g}$ and $\operatorname{Int}\mathfrak{g} = (\operatorname{Aut}\mathfrak{g})^0$.

As we have seen in 3.1.1, any real algebraic group G is embedded as a real form in a complex algebraic group $G(\mathbb{C})$. The following example shows that for the Lie groups (even semisimple ones) the similar statement fails.

Example 5. Considering the natural transitive action of $\mathrm{SL}_2(\mathbb{R})$ in $\mathbb{R}^2\backslash\{0\}$ and applying Theorem 1.3.4 it is easy to show that $\pi_1(\mathrm{SL}_2(\mathbb{R})) \simeq \pi_1(\mathbb{R}^2\backslash\{0\}) \simeq \mathbb{Z}$. Let $G = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ be the simply connected covering for $\mathrm{SL}_2(\mathbb{R})$. Then G cannot be embedded as a real form in any complex Lie group $\hat{G}$. In fact, let $f\colon G \to \hat{G}$ be such an embedding. We may assume that the tangent algebra of $\hat{G}$ is $\mathfrak{sl}_2(\mathbb{C})$ and df is the natural embedding $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{C})$. The group $\hat{G}$ is connected and its simply connected covering is $\mathrm{SL}_2(\mathbb{C})$. Therefore f is covered by the injective homomorphism $\hat{f}\colon G \to \mathrm{SL}_2(\mathbb{C})$ such that $d\hat{f} = df$. Clearly, $\hat{f}(G) = \mathrm{SL}_2(\mathbb{R})$ which leads to contradiction.

The fact proved also implies that $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ does not admit any real algebraic group structure and cannot even be isomorphic to the identity component of an irreducible real algebraic group. Since any semisimple linear Lie algebra is algebraic (Problem 4.1.8) the group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ does not admit a faithful linear representation.

Now consider the realification of complex Lie algebras. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{g}^{\mathbb{R}}$ the same algebra considered as an algebra over $\mathbb{R}$.

In the Lie algebra $\mathfrak{g}^{\mathbb{R}}$ the multiplication by i is defined:

$$Ix = ix \qquad (x \in \mathfrak{g}^{\mathbb{R}}).$$

It is a linear transformation over $\mathbb{R}$ such that

$$I^2 = -E, \tag{3}$$

$$I[x,y] = [x,Iy] \qquad (x,y \in \mathfrak{g}^{\mathbb{R}}). \tag{4}$$

In general, given a real Lie algebra $\mathfrak{g}$ we call a *complex structure on it* a linear transformation of $\mathfrak{g}$ satisfying (3) and (4).

Problem 5. Given a real Lie algebra $\mathfrak{g}$ with a complex structure I we make $\mathfrak{g}$ into a Lie algebra $\tilde{\mathfrak{g}}$ over $\mathbb{C}$ such that $\tilde{\mathfrak{g}}^{\mathbb{R}} = \mathfrak{g}$ by setting

$$(a + bi)x = ax + bIx \qquad (a,b \in \mathbb{R}, x \in \mathfrak{g}).$$

Notice that if I is a complex structure on $\mathfrak{g}$, then so is $-I$. Therefore from each complex Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ we may construct another Lie algebra over $\mathbb{C}$ obtained from $\mathfrak{g}$ by reversing the sign of the complex structure; this Lie algebra will be denoted by $\bar{\mathfrak{g}}$. Clearly, $\mathfrak{g}^{\mathbb{R}} = \bar{\mathfrak{g}}^{\mathbb{R}}$. A homomorphism $\mathfrak{g} \to \bar{\mathfrak{g}}$ is nothing but an antilinear endomorphism of $\mathfrak{g}$. Therefore $\mathfrak{g} \simeq \bar{\mathfrak{g}}$ if and only if $\mathfrak{g}$ admits an antilinear automorphism. In particular, if $\mathfrak{g}$ possesses a real form then $\mathfrak{g} \simeq \bar{\mathfrak{g}}$.

Problem 6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\{h_i, e_i, f_i\ (i = 1, \dots, l)\}$ its canonical system of generators. Then the real subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by h_i, e_i, f_i is a real form of $\mathfrak{g}$. The corresponding real structure on $\mathfrak{g}$ transforms each of h_i, e_i, f_i into itself. Therefore, any semisimple complex Lie algebra $\mathfrak{g}$ is isomorphic to $\bar{\mathfrak{g}}$.

A real form $\mathfrak{h}$ of a semisimple complex Lie algebra $\mathfrak{g}$ constructed in Problem 6 is called a *normal* one. By Theorem 4.3.1 any two normal forms (constructed from different canonical systems of generators) are isomorphic.

For any complex Lie algebra $\mathfrak{g}$ the complex Lie algebra $\mathfrak{g}^{\mathrm{dbl}} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ will be called the *double* of $\mathfrak{g}$.

Problem 7. The transformation $\sigma\colon \mathfrak{g}^{\mathrm{dbl}} \to \mathfrak{g}^{\mathrm{dbl}}$ defined by the formula $\sigma(x, y) = (y, x)$ is a real structure on $\mathfrak{g}^{\mathrm{dbl}}$ and the map $(x, x) \mapsto x$ is an isomorphism of $(\mathfrak{g}^{\mathrm{dbl}})^{\sigma}$ onto $\mathfrak{g}^{\mathbb{R}}$. Therefore $\mathfrak{g}^{\mathbb{R}}(\mathbb{C}) \simeq \mathfrak{g}^{\mathrm{dbl}}$. Under this isomorphism $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ are sent into the eigenspaces of the operator I (extended by linearity to $\mathfrak{g}^{\mathbb{R}}(\mathbb{C})$) corresponding to the eigenvalues i and $-i$ respectively.

Problem 8. If $\mathfrak{g}$ is a semisimple complex Lie algebra then $\mathfrak{g}^{\mathbb{R}}(\mathbb{C}) \cong \mathfrak{g} \oplus \mathfrak{g}$. If $\mathfrak{h}$ is another semisimple complex Lie algebra and $\mathfrak{g}^{\mathbb{R}} \cong \mathfrak{h}^{\mathbb{R}}$ then $\mathfrak{g} = \mathfrak{h}$.

Problem 9. Let $(\cdot, \cdot)$ be the Cartan scalar product in a complex Lie algebra $\mathfrak{g}$. Then the Cartan scalar product in $\mathfrak{g}^{\mathbb{R}}$ is of the form $(x, y)^{\mathbb{R}} = 2\,\mathrm{Re}(x, y)$. If $\mathfrak{h}$ is a real form of $\mathfrak{g}$ then the restriction of $(\cdot, \cdot)$ onto $\mathfrak{h}$ coincides with the Cartan scalar product in $\mathfrak{h}$. For any antilinear automorphism γ of $\mathfrak{g}$ we have

$$(\gamma(x), \gamma(y)) = \overline{(x, y)} \qquad (x, y \in \mathfrak{g}).$$

As it was proved in 1.4.7, a real Lie algebra is semisimple if and only if so is its complexification. Now let us investigate the relation between simple noncommutative Lie algebras over $\mathbb{R}$ and $\mathbb{C}$.

Problem 10. If $\mathfrak{g}$ is a non-commutative simple Lie algebra over $\mathbb{C}$ then any real form of $\mathfrak{g}$ is simple and the Lie algebra $\mathfrak{g}^{\mathbb{R}}$ is simple.

Problem 11. If $\mathfrak{g}$ is a simple real Lie algebra then either $\mathfrak{g}(\mathbb{C})$ is simple or $\mathfrak{g}$ admits a complex structure.

Problems 10 and 11 imply

Theorem 1. *A non-commutative real Lie algebra is simple if and only if it is isomorphic to either algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is a simple complex Lie algebra, or to a real form of a simple complex Lie algebra.*

Theorem 1 and Problem 8 imply that the classification of simple real Lie algebras reduces to the classification of simple complex Lie algebras obtained in 4.3 and to the classification of non-isomorphic real forms of each of them.

2°. Real Forms of Classical Lie Groups and Algebras. In this subsection we specify several real forms of classical complex Lie groups $\mathrm{GL}(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ and their tangent algebras. Actually, as we will see in § 3, the real forms listed here exhaust up to an isomorphism all real forms of the classical complex Lie algebras. It is easy to observe that all real structures and real forms of classical groups listed below are algebraic.

Recall (see Example 3 of 1°) that $\mathrm{GL}(\mathbb{R})$ is a real form of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R})$ is a real form of $\mathfrak{gl}_n(\mathbb{C})$. The corresponding real structure on $\mathrm{GL}_n(\mathbb{C})$ is the complex conjugation: $S(A) = \bar{A}$.

Example 1. The complex conjugation $A \mapsto \bar{A}$ transforms each of the groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ into itself and determines real structures in them. Therefore the following real forms of the classical groups are defined:

$$\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C}), \mathrm{O}_n \subset \mathrm{O}_n(\mathbb{C}), \mathrm{SO}_n \subset \mathrm{SO}_n(\mathbb{C}), \mathrm{Sp}_n(\mathbb{R}) \subset \mathrm{Sp}_n(\mathbb{C})$$

The corresponding real forms of the Lie algebras are:

$$\mathfrak{sl}_n(\mathbb{R}) \subset \mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n \subset \mathfrak{so}_n(\mathbb{C}), \mathfrak{sp}_n(\mathbb{R}) \subset \mathfrak{sp}_n(\mathbb{C}).$$

The following series of examples has to do with quadratic forms. In 1.3.1° the pseudoorthogonal group $\mathrm{O}_{k,l} \subset \mathrm{GL}_{k+l}(\mathbb{R})$ of signature (k, l) preserving the quadratic form

$$x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+l}^2, \tag{5}$$

and the special pseudoorthogonal group $\mathrm{SO}_{k,l}$ had been defined.

Let $I_{k,l} = \begin{pmatrix} E_k & 0 \\ 0 & -E_l \end{pmatrix}$ be the matrix of the form (5) and let $L_{k,l} = \begin{pmatrix} E_k & 0 \\ 0 & iE_l \end{pmatrix}$. Then $L_{k,l}^2 = I_{k,l}$.

Example 2. The transformation $S(A) = I_{k,l}\bar{A}I_{k,l}$ is a real structure on the complex Lie groups $G = \mathrm{O}_{k+l}(\mathbb{C})$, $\mathrm{SO}_{k+l}(\mathbb{C})$, the corresponding real forms G^S coincide with $L_{k,l}\mathrm{O}_{k,l}L_{k,l}$ and $L_{k,l}\mathrm{SO}_{k,l}L_{k,l}^{-1}$ respectively. The cooresponding real form $L_{k,l}\mathfrak{so}_{k,l}L_{k,l}^{-1}$ of $\mathfrak{so}_{k+l}(\mathbb{C})$ consists of the matrices of the form

$$\begin{pmatrix} X & iY \\ -iY^T & Z \end{pmatrix}$$

where X, Y, Z are real matrices, X and Z of sizes $k \times k$ and $l \times l$ respectively, $X^T = -X, Z^T = -Z$.

The *pseudounitary group of signature* (k, l) is the group $\mathrm{U}_{k,l}$ of all linear transformations of $\mathbb{C}^{k+l}$ preserving the pseudohermitian quadratic form

$$|z_1|^2 + \cdots + |z_k|^2 - |z_{k+1}|^2 - \cdots - |z_{k+l}|^2.$$

In particular, $\mathrm{U}_n = \mathrm{U}_{n,0}$ is the group of *unitary* matrices (or the *unitary* group). The groups $\mathrm{SU}_{k,l} = \mathrm{U}_{k,l} \cap \mathrm{SL}_{k+l}(\mathbb{C})$ and $\mathrm{SU}_n = \mathrm{SU}_{n,0}$ are called *special pseudo-unitary* and *special unitary* groups. The corresponding tangent algebras will be denoted by $\mathfrak{u}_{k,l}$, $\mathfrak{u}_n$, $\mathfrak{su}_{k,l}$, $\mathfrak{su}_n$.

Example 3. The transformation $S(A) = I_{k,l}\bar{A}^{T-1}I_{k,l}$ defines a real structure on the complex groups $G = \mathrm{GL}_{k+l}(\mathbb{C})$, $\mathrm{SL}_{k+l}(\mathbb{C})$, the corresponding real forms G^S coincide with $U_{k,l}$ and $\mathrm{SU}_{k,l}$ respectively. To these real forms of Lie groups correspond the real forms $\mathfrak{u}_{k,l} \subset \mathfrak{gl}_{k+l}(\mathbb{C})$ and $\mathfrak{su}_{k,l} \subset \mathfrak{su}_{k+l}(\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix} X & Y \\ \bar{Y}^T & Z \end{pmatrix},$$

where $\bar{X}^T = -X$, $\bar{Z}^T = -Z$, X and Z of sizes $k \times k$ and $l \times l$ respectively, and for $\mathfrak{su}_{k,l}$ additionally satisfying $\operatorname{tr} X + \operatorname{tr} Z = 0$.

Finally, the last group of examples results from the existence of a quaternionic structure in $\mathbb{C}^{2m}$. Consider the right quaternion vector space $\mathbb{H}^m$ over the quaternion field $\mathbb{H}$. Its linear transformations are identified with $m \times m$ matrices over $\mathbb{H}$. Let $\mathrm{GL}_m(\mathbb{H})$ be the group of invertible quaternion matrices. Its tangent algebra is the Lie algebra $\mathfrak{gl}_m(\mathbb{H})$ of all quaternion matrices.

Consider $\mathbb{C}$ as a subfield of $\mathbb{H}$ generated by 1, i. Each vector $q \in \mathbb{H}^m$ uniquely presents in the form $q = z + jw$, where $z, w \in \mathbb{C}^m$. The correspondence $q \mapsto (z, w)$ is an isomorphism $\mathbb{H}^m \to \mathbb{C}^{2m}$ of vector spaces over $\mathbb{C}$ that maps qj into $(-\bar{w}, \bar{z})$. Therefore $\mathfrak{gl}_m(\mathbb{H})$ is identified by this isomorphism with a subalgebra of $\mathfrak{gl}_{2m}(\mathbb{C})$ consisting of all transformations commuting with the antilinear transformation $J\colon \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ given by $J(z, w) = (-\bar{w}, \bar{z})$. Notice that $J = S_m\tau$, where τ is the standard complex conjugation in $\mathbb{C}^{2m}$ and $S_m = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}$.

Example 4. The transformation $S(A) = JAJ^{-1} = -S_m\bar{A}S_m$ determines a real structure on the complex Lie groups $G = \mathrm{GL}_{2m}(\mathbb{C})$, $\mathrm{SL}_{2m}(\mathbb{C})$, $\mathrm{SO}_{2m}(\mathbb{C})$. The corresponding real form of $\mathrm{GL}_{2m}(\mathbb{C})$ is identified with $\mathrm{GL}_m(\mathbb{H})$. The real forms G^S of the groups $G = \mathrm{SL}_{2m}(\mathbb{C})$, $\mathrm{SO}_{2m}(\mathbb{C})$ are denoted by $\mathrm{SL}_m(\mathbb{H})$, $\mathrm{U}_m^*(\mathbb{H})$ respectively. The latter notation is chosen since $\mathrm{U}_m^*(\mathbb{H})$ is identified with the subgroup of $\mathrm{GL}_m(\mathbb{H})$ consisting of all linear transformations C of $\mathbb{H}^m$ preserving the skew-Hermitian quadratic form

$$\sum_{1 \leqslant r \leqslant m} \bar{q}_r j q_r,$$

i.e. satisfying $\bar{C}^T(jE)C = jE$. The tangent algebras of $\mathrm{SL}_m(\mathbb{H})$, $\mathrm{U}_m^*(\mathbb{H})$ are denoted by $\mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{u}_m^*(\mathbb{H})$. These Lie algebras are real forms of $\mathfrak{sl}_{2m}(\mathbb{C})$, $\mathfrak{so}_{2m}(\mathbb{C})$. The Lie algebras $\mathfrak{gl}_m(\mathbb{H})$, $\mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{u}_m^*(\mathbb{H})$ are subalgebras of $\mathfrak{gl}_{2m}(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix},$$

where $X, Y \in \mathfrak{gl}_m(\mathbb{C})$, such that $\operatorname{tr} X + \operatorname{tr} \bar{X} = 0$ for $\mathfrak{sl}_m(\mathbb{H})$ and $X^T = -X$, $Y^T = \bar{Y}$ for $\mathfrak{u}_m^*(\mathbb{H})$.

In $\mathrm{GL}_{k+l}(\mathbb{H})$, consider the subgroup $\mathrm{Sp}_{k,l}$ consisting of the transformations preserving the Hermitian quadratic form

$$|q_1|^2 + \cdots + |q_k|^2 - |q_{k+1}|^2 - \cdots - |q_{k+l}|^2. \tag{6}$$

Under the isomorphism $\mathbb{H}^{k+l} \to \mathbb{C}^{2(k+l)}$ described above the form (6) is mapped into the Hermitian quadratic form

$$\sum_{1 \leqslant i \leqslant k} |z_i|^2 - \sum_{k+1 \leqslant j \leqslant k+l} |z_j|^2 + \sum_{1 \leqslant i \leqslant k} |w_i|^2 - \sum_{k+1 \leqslant j \leqslant k+l} |w_j|^2. \tag{7}$$

Therefore $\mathrm{Sp}_{k,l}$ is identified with a subgroup of $\mathrm{GL}_{2(k+l)}(\mathbb{C})$ consisting of the matrices A such that

$$A = -S_{k+l}\bar{A}S_{k+l},\ \bar{A}^T K_{k,l} A = K_{k,l},$$

where $K_{k,l} = \begin{pmatrix} I_{k,l} & 0 \\ 0 & I_{k,l} \end{pmatrix}$ is the matrix of the form (7). These conditions imply that $A(K_{k,l}S_{k+l})A^T = K_{k,l}S_{k+l}$, i.e. $\mathrm{Sp}_{k,l}$ is contained in the complex symplectic group preserving the form with the matrix $K_{k,l}S_{k+l}$. Setting $M_{k,l} = \begin{pmatrix} L_{k,l} & 0 \\ 0 & L_{k,l} \end{pmatrix}$ (see Example 2) we see that the group $M_{k,l}\mathrm{Sp}_{k,l}M_{k,l}^{-1}$ is contained in the standard symplectic group $\mathrm{Sp}_{2(k+l)}(\mathbb{C})$ and coincides with the subgroup of all elements of the symplectic group preserving (7).

Example 5. The transformation $S(A) = K_{k,l}\bar{A}^{T^{-1}}K_{k,l}$ is a real structure on $G = \mathrm{Sp}_{2(k+l)}(\mathbb{C})$ and $G^S = M_{k,l}\mathrm{Sp}_{k,l}M_{k,l}^{-1}$. In what follows we will identify the subgroup G^S with $\mathrm{Sp}_{k,l}$. The corresponding real form $\mathfrak{sp}_{k,l} \subset \mathfrak{sp}_{2(k+l)}(\mathbb{C})$ consists of the matrices of the form

$$\begin{array}{c} \\ k \\ l \\ k \\ l \end{array}\!\!\begin{array}{c} \begin{array}{cccc} k & l & k & l \end{array} \\ \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ \bar{X}_{12}^T & X_{22} & X_{14}^T & X_{24} \\ -\bar{X}_{13} & \bar{X}_{14} & \bar{X}_{11} & -\bar{X}_{12} \\ \bar{X}_{14}^T & -\bar{X}_{24} & -X_{12}^T & \bar{X}_{22} \end{pmatrix} \end{array}$$

where $\bar{X}_{11}^T = -X_{11}$, $\bar{X}_{22}^T = -X_{22}$, $X_{13}^T = X_{13}$, $X_{24}^T = X_{24}$.

In particular, the group $\mathrm{Sp}_{m,0}$ coincides with the group $\mathrm{Sp}_m = \mathrm{GL}_m(\mathbb{H}) \cap \mathrm{U}_{2m}$ of unitary quaternion matrices (see Exercise 1.1.3) and its tangent algebra $\mathfrak{sp}_{m,0}$ coincides with the Lie algebra $\mathfrak{sp}_m = \mathfrak{gl}_m(\mathbb{H}) \cap \mathfrak{u}_{2m}$ (here $M_{k,l} = E$).

3°. The Compact Real Form. In this section we will show that each connected semisimple complex Lie group has a compact real form. This will enable us to establish a one-to-one correspondence between the reductive complex algebraic groups and compact real Lie groups.

A finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is called *compact* if there exists a positive definite invariant scalar product in $\mathfrak{g}$. Clearly, any subalgebra of a compact Lie algebra is compact.

Problem 12. The tangent algebra of any compact Lie group is compact.

Problem 13. The Cartan scalar product on a compact Lie algebra is always negative semi-definite. A real Lie algebra is semisimple compact if and only if its Cartan scalar product is negative definite.

Problem 14. For a compact Lie algebra $\mathfrak{g}$ the derived algebra $\mathfrak{g}'$ is semisimple and $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$.

Problem 15. For any compact Lie algebra $\mathfrak{g}$ there exists a connected compact Lie group G with the tangent algebra $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then we may take $G = \operatorname{Int} \mathfrak{g}$.

Now let $\mathfrak{g}$ be an arbitrary complex Lie algebra, σ a real structure on $\mathfrak{g}$. Define the Hermitian form on $\mathfrak{g}$ by setting

$$h_\sigma(x, y) = -(x, \sigma(y)), \tag{8}$$

where $(\cdot, \cdot)$ is the Cartan scalar product.

Problem 16. The form h_σ is invariant with respect to $\operatorname{ad} \mathfrak{g}^\sigma$, i.e.

$$h_\sigma([z, x], y) + h_\sigma(x, [z, y]) = 0 \qquad (x, y \in \mathfrak{g}, z \in \mathfrak{g}^\sigma).$$

The restriction of the form $-h_\sigma$ onto $\mathfrak{g}^\sigma$ coincides with the Cartan scalar product in $\mathfrak{g}^\sigma$.

Problem 17. If $\gamma \in \operatorname{Aut} \mathfrak{g}$ is an automorphism commuting with σ then

$$h_\sigma(\gamma x, \gamma y) = h_\sigma(x, y) \qquad (x, y \in \mathfrak{g}).$$

Now assume that G is a connected complex semisimple Lie group, $\mathfrak{g}$ its tangent algebra, S a real structure on G such that $\sigma = dS$.

Problem 18. The following conditions are equivalent:
a) G^S is compact;
b) the Lie algebra $\mathfrak{g}^\sigma$ is compact;
c) the Hermitian form h_σ is positive definite.

Fix a maximal torus $T \subset G$ and a base $\{\alpha_1, \ldots, \alpha_l\}$ of the root system Δ_G with respect to T. Consider the canonical system of generators $\{h_i, e_i, f_i : i = 1, \ldots, l\}$ of $\mathfrak{g}$ defined in 4.3.2. As it is known, $\{-\alpha_1, \ldots, -\alpha_l\}$ is also a base. The system $\{-h_i, -f_i, -e_i : i = 1, \ldots, l\}$ is the canonical system of generators associated with this base. By Theorem 4.3.1 there exists a unique automorphism μ of $\mathfrak{g}$ such that

$$\mu(h_i) = -h_i, \quad \mu(e_i) = -f_i, \quad \mu(f_i) = -e_i \qquad (i = 1,\dots,l).$$

We have $\mu^2 = \mathrm{id}$.

Problem 19. There exists a unique antilinear automorphism σ of $\mathfrak{g}$ such that

$$\sigma(h_i) = -h_i, \quad \sigma(e_i) = -f_i, \quad \sigma(f_i) = -e_i \qquad (i = 1,\dots,l).$$

This automorphism is involutive, i.e. σ is a real structure on $\mathfrak{g}$.

Problem 20. There exists a real structure S on G such that $dS = \sigma$.

Problem 21. The subspaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ $(\alpha, \beta \in \Delta_G, \alpha \neq \beta)$ are orthogonal with respect to h_σ. The subspace $\mathfrak{t}$ is orthogonal to any $\mathfrak{g}_\sigma$, $\alpha \in \Delta_G$.

Problem 22. The Hermitian form h_σ is positive definite on $\mathfrak{t}$ and on any $\mathfrak{g}_{\alpha_i}$ $(i = 1,\dots,l)$.

Let $G^{(i)} = G^{(\alpha_i)}$ be the simple three-dimensional (complex) subgroup of G corresponding to a simple root α_i. It is the image of $\mathrm{SL}_2(\mathbb{C})$ under the homomorphism $F_i = F_{\alpha_i}$ (see 4.1.6°).

Problem 23. We have $F_i(\bar{g}^{T-1}) = S(F_i(g))$ $(g \in \mathrm{SL}_2(\mathbb{C}))$.

Problem 24. Each element of the Weyl group of G with respect to T is induced by an element of $N(T) \cap G^S$.

Problem 25. The Hermitian form h_σ is positive definite on $\mathfrak{g}$.

Problems 18, 20 and 25 imply the following.

Theorem 2. *Any connected semisimple complex Lie group G has a compact real form. The tangent algebra of this form is a compact real form of the tangent algebra $\mathfrak{g}$ of G.*

Problem 26. A compact Lie algebra admitting a complex structure is commutative.

Problem 27. A complex Lie algebra is simple if and only if it has a simple compact real form.

As it will be shown in 4°, a compact real form of a semisimple complex Lie algebra is unique up to an inner automorphism of this algebra.

Example. The following real forms of classical groups and their tangent algebras are compact: $U_n \subset \mathrm{GL}_n(\mathbb{C})$, $\mathrm{SU}_n \subset \mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n \subset \mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n \subset \mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n \subset \mathrm{Sp}_{2n}(\mathbb{C})$; $\mathfrak{u}_n \subset \mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{su}_n \subset \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n \subset \mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_n \subset \mathfrak{sp}_{2n}(\mathbb{C})$.

4°. Real Forms and Involutive Automorphisms. Let $\mathfrak{g}$ be a complex Lie algebra. Consider the problem of classifying the real forms of $\mathfrak{g}$ up to an isomorphism. By Problem 1 the classes of isomorphic real forms are in one-to-one correspondence with the involutive antilinear automorphisms considered up to conjugacy in $\operatorname{Aut}\mathfrak{g}$. In this section we will show that for a semisimple Lie algebra $\mathfrak{g}$ the antilinear automorphisms in this classification can be replaced by the linear ones.

Let σ and τ be two real structures on a Lie algebra $\mathfrak{g}$. The real forms $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ are said to be *compatible* if $\sigma\tau = \tau\sigma$.

Problem 28. The following conditions are equivalent:

a) $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ are compatible;

b) $\tau(\mathfrak{g}^\sigma) = \mathfrak{g}^\sigma$;

c) $\sigma(\mathfrak{g}^\tau) = \mathfrak{g}^\tau$;

d) $\mathfrak{g}^\sigma = \mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \oplus \mathfrak{g}^\sigma \cap (i\mathfrak{g}^\tau)$; (9)

e) $\mathfrak{g}^\tau = \mathfrak{g}^\tau \cap \mathfrak{g}^\sigma \oplus \mathfrak{g}^\tau \cap (i\mathfrak{g}^\sigma)$; (10)

f) the automorphism $\theta = \sigma\tau$ of $\mathfrak{g}$ is involutive.

Notice that if σ and τ are compatible then θ transforms $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ into themselves, hence $\theta|\mathfrak{g}^\sigma = \tau|\mathfrak{g}^\sigma$ and $\theta|\mathfrak{g}^\tau = \sigma|\mathfrak{g}^\tau$. Clearly, (9) and (10) coincide with the decompositions of $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ into the eigenspaces of θ corresponding to the eigenvalues 1 and -1.

Example. All real forms of the classical groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ listed in 2° are compatible with their compact real forms U_n, SU_n, O_n, SO_n, Sp_n, respectively.

Problem 29. Two compact real forms of a semisimple complex Lie algebra are compatible if and only if they coincide.

Our next goal is to prove the following.

Theorem 3. *Any two compact real forms of a semisimple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *are conjugate. Any real form of* $\mathfrak{g}$ *is compatible with a compact form. If a real form* $\mathfrak{h}$ *is compatible with two compact real forms* $\mathfrak{u}_1$ *and* $\mathfrak{u}_2$, *then there exists an automorphism* $\varphi \in \operatorname{Int}\mathfrak{g}$, *such that* $\varphi(\mathfrak{u}_1) = \mathfrak{u}_2$ *and* $\varphi(\mathfrak{h}) = \mathfrak{h}$.

Let us fix a compact form $\mathfrak{u}$ existing thanks to Theorem 2 and let τ be the corresponding structure on $\mathfrak{g}$. Let σ be an arbitrary real structure on $\mathfrak{g}$. We wish to show that the real forms $\mathfrak{g}^\sigma$ and $\mathfrak{u}$ can be made compatible by applying an inner automorphism of $\mathfrak{g}$ to one of these forms.

Consider the automorphism $\theta = \sigma\tau$ and a positive definite Hermitian form h_τ on $\mathfrak{g}$ defined by (8).

Problem 30. The operator θ is self-adjoint with respect to the form h_τ, i.e. $h_\tau(\theta x, y) = h_\tau(x, \theta y)$ $(x, y \in \mathfrak{g})$.

This implies that $p = \theta^2$ is a positive definite self-adjoint operator.

Problem 31. Let $\mathbf{E}$ be a finite-dimensional Euclidean or Hermitian space, $S(\mathbf{E})$ the space of all its self-adjoint linear operators and $P(\mathbf{E}) \subset S(\mathbf{E})$ the open set of positive definite operators. Then exp bijectively maps $S(\mathbf{E})$ onto $P(\mathbf{E})$.

Let $\log = \exp^{-1}\colon P(\mathbf{E}) \to S(\mathbf{E})$. For $p \in P(\mathbf{E})$ and $t \in \mathbb{R}$ set $p^t = \exp(t \log p)$.

Problem 32. If $G \subset \mathrm{GL}(\mathbf{E})$ is a real algebraic group and $p \in G \cap P(\mathbf{E})$, then $p^t \in G$ for all $t \in \mathbb{R}$ and $\log p$ belongs to the tangent algebra $\mathfrak{g}$ of G. Therefore, exp bijectively maps $\mathfrak{g} \cap S(\mathbf{E})$ onto $G \cap P(\mathbf{E})$.

Applying Problem 32 to the element $p = \theta^2$ of $\operatorname{Aut}\mathfrak{g}$ we get a one-parameter subgroup $p^t (t \in \mathbb{R})$ in $\operatorname{Aut}\mathfrak{g}$ consisting of positive definite self-adjoint (with respect to h_τ) operators such that $p^1 = p$. By Corollary of Theorem 4.4.1 $p^t \in \operatorname{Int}\mathfrak{g}$.

Problem 33. We have $\sigma p^t \sigma = \tau p^t \tau = p^{-t}$.

Problem 34. The automorphism $\varphi = p^{1/4}$ satisfies $\sigma(\varphi\tau\varphi^{-1}) = (\varphi\tau\varphi^{-1})\sigma$. Therefore $\mathfrak{g}^\sigma$ is compatible with the compact real form $\varphi(\mathfrak{u})$. If a real structure φ on $\mathfrak{g}$ commutes with σ and τ then ψ commutes with φ as well.

Problems 28 and 34 immediately imply Theorem 3.

Theorems 2, 3 and Problem 27 imply

Corollary. *The map* $\mathfrak{g} \mapsto \mathfrak{g}(\mathbb{C})$ *determines the bijection between the classes of isomorphic compact semisimple Lie algebras and the classes of isomorphic complex semisimple Lie algebras assigning to a simple compact Lie algebra a simple complex Lie algebra and vice versa.*

Theorem 3 enables us to establish a correspondence between the real forms of a semisimple complex Lie algebra $\mathfrak{g}$ and its involutive automorphisms. Namely, let σ be a real structure on $\mathfrak{g}$. By Theorem 3 there exists a compact real structure τ commuting with σ. Then $\theta = \sigma\tau$ is an involutive automorphism of $\mathfrak{g}$. If τ_1 is another compact real structure commuting with σ, then, as easily follows from Theorem 3, the automorphisms θ and $\theta = \sigma\tau_1$ are conjugate in $\operatorname{Aut}\mathfrak{g}$. Therefore there is a map assigning to each real structure (or a real form) in $\mathfrak{g}$ a class of conjugate involutive automorphisms of $\mathfrak{g}$.

Theorem 4. *The constructed map defines a bijection of the set of isomorphism classes of real forms of* $\mathfrak{g}$ *onto the set of classes of conjugate involutive automorphisms of* $\mathfrak{g}$.

To prove this theorem let θ be an involutive automorphism of $\mathfrak{g}$. Making use of Theorem 2 choose a compact real structure τ on $\mathfrak{g}$. Then $q = (\theta\tau)^2$ is an automorphism of $\mathfrak{g}$.

Problem 35. The automorphism q is a positive definite self-adjoint operator with respect to the Hermitian form h_τ.

Problem 36. There exists a compact real structure τ_1 commuting with θ. This structure is determined up to conjugacy by an automorphism of $\mathfrak{g}$ commuting with θ.

As it follows from Problem 36, $\theta = \sigma\tau_1$, where σ is a real structure commuting with τ_1. This makes transparent the surjectivity of the map constructed above.

It is clear that two real structures which are conjugate by an automorphism define the same class of involutive automorphisms. Let us prove that the converse is also true. Let σ_i $(i = 1, 2)$ be two real structures, τ_i a compact real structure commuting with σ_i, $\theta_i = \sigma_i\tau_i$. Let $\theta_2 = \varphi\theta_1\varphi^{-1}$, where $\varphi \in \operatorname{Aut}\mathfrak{g}$. Since τ_1 and τ_2 are conjugate, we may assume that $\tau_1 = \tau_2 = \tau$. Then the structures τ and $\varphi^{-1}\tau\varphi$ commute with θ_1. By Problem 36 $\varphi^{-1}\tau\varphi = \psi\tau\psi^{-1}$, where $\psi \in \operatorname{Aut}\mathfrak{g}$ and $\psi\theta_1 = \theta_1\psi$. Clearly, $\sigma_2 = \omega\sigma_1\omega^{-1}$ for $\omega = \varphi\psi$. Theorem 4 is proved. □

It is useful to indicate an explicit construction of the real form $\mathfrak{h}$ of $\mathfrak{g}$ corresponding to an involutive automorphism $\theta \in \operatorname{Aut}\mathfrak{g}$. For this it is convenient to fix a compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Problem 36 implies that replacing θ by a conjugate automorphism we may assume that $\theta(\mathfrak{u}) = \mathfrak{u}$. Let

$$\mathfrak{u} = \mathfrak{u}(1) \oplus \mathfrak{u}(-1)$$

be the decomposition of $\mathfrak{u}$ into the eigenspaces of θ corresponding to the eigenvalues 1 and -1.

Problem 37. The real form $\mathfrak{h}$ of $\mathfrak{g}$ corresponding to the class of θ by Theorem 4 is of the form

$$\mathfrak{h} = \mathfrak{u}(1) \oplus i\mathfrak{u}(-1). \tag{11}$$

In particular, to the identity automorphism $\theta = \mathrm{id}$ the class of compact real forms of $\mathfrak{g}$ corresponds.

5°. Involutive Automorphisms of Complex Simple Lie Algebras. Here we describe the classes of conjugate involutive automorphisms of complex simple Lie algebras with the help of the method of 4.4°. Let $\mathfrak{g}$ be a non-commutative complex simple Lie algebra of type L_n. It suffices to consider non-identical involutive automorphisms $\theta \in \operatorname{Aut}\mathfrak{g}$, i.e. automorphisms θ of order 2. By Theorem 4.4.8 and Problem 4.4.57 the classes of conjugate in $\operatorname{Aut}\mathfrak{g}$ automorphisms of order 2 are in one-to-one correspondence with the considered up to an isomorphism Kac diagrams of types $L_n^{(k)}$ whose numerical labels u_j are of the form $u_j = s_j/2$, where s_j ($j = 0, 1, \ldots, l$) are non-negative integers, relatively prime and satisfying

$$k \sum_{0 \leqslant j \leqslant l} n_j s_j = 2. \tag{12}$$

Here $n_0, n_1, \ldots, n_l$ are relatively prime positive integers listed in Table 6. It follows from (12) that $k = 1$ or 2.

Problem 38. Kac diagrams satisfying (12) belong to one of the following three types:

I) $k = 1$; $u_i = 0$ for all i except some $i = p$; $u_p = 1/2$; $a_p = 2$;

II) $k = 1$; $u_i = 0$ for all i except some $i = p, q, p \neq q$; $u_p = u_q = 1/2$; $a_p = a_p = 1$;

III) $k = 2$; $u_i = 0$ for all i except some $i = p$; $u_p = 1/2$; $a_p = 1$.

In case II we may assume that $q = 0$ if we consider Kac diagrams up to an isomorphism.

Making use of Problem 38 and Table 6 it is not difficult to list all up to isomorphism Kac diagrams satisfying (12). The results are given in Table 7 (in case II we assume that $q = 0$). Problem 4.4.61 helps also to determine the type of the corresponding subalgebras $\mathfrak{g}^\theta$ (note that $\mathfrak{g}^\theta$ is semisimple in cases I and III and has a one-dimensional center in case II).

Problem 39. Let θ_1, θ_2 be involutive automorphisms of a simple noncommutative Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Then $\mathfrak{g}^{\theta_1} \cong \mathfrak{g}^{\theta_2}$ if and only if θ_1 and θ_2 are conjugate in $\operatorname{Aut}\mathfrak{g}$.

As an application, let us explicitly describe the classes of conjugate involutive automorphisms of simple classical complex Lie algebras. We make use of notation of 2°.

Theorem 5. *The following automorphisms θ of simple classical complex Lie algebras $\mathfrak{g}$ form the complete system of representatives of classes of conjugate involutive automorphisms (for $\theta \neq \mathrm{id}$ the type of the corresponding Kac diagram is indicated, see Problem* 39):

1) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}),\ n \geqslant 2$	
a) $\theta(X) = -X^T$	III
b) $\theta(X) = -\operatorname{Ad} S_m(X^T),\ n = 2m$	III
c) $\theta = \operatorname{Ad} I_{p,n-p}\ (p = 0, 1, \ldots, [n/2])$	II for $p > 0$
2) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C}),\ n = 3$ or $n \geqslant 5$	
a) $\theta = \operatorname{Ad} I_{p,n-p}\ (p = 0, 1, \ldots, [n/2])$	I and II for $p \neq 0, 2$; II for $p = 2$
b) $\theta = \operatorname{Ad} S_m,\ n = 2m$	III
3) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}),\ n = 2m \geqslant 2$	
a) $\theta = \operatorname{Ad} S_m$	II
b) $\theta = \operatorname{Ad} K_{p,m-p}\ (p = 0, 1, \ldots, [m/2])$	I for $p > 0$

Problem 40. Prove this theorem.

6°. Classification of Real Simple Lie Algebras. The results of 4° and 5° enable us to list up to an isomorphism all real forms of non-commutative complex simple Lie algebras. For the classical Lie algebras this list is given by the following theorem.

Theorem 6. *Any real form of a clasical simple complex Lie algebra $\mathfrak{g}$ is isomorphic to exactly one of the following real forms $\mathfrak{h} \subset \mathfrak{g}$:*

1) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}),\ n \geqslant 2$
    a) $\mathfrak{h} = \mathfrak{sl}_n(\mathbb{R})$
    b) $\mathfrak{h} = \mathfrak{sl}_m(\mathbb{H}),\ n = 2m$
    c) $\mathfrak{h} = \mathfrak{su}_{p,n-p}\ (p = 0, 1, \ldots, [n/2])$
2) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C}),\ n = 3$ or $n \geqslant 5$
    a) $\mathfrak{h} = \mathfrak{so}_{p,n-p}\ (p = 0, 1, \ldots, [n/2])$
    b) $\mathfrak{h} = \mathfrak{u}_m^*(\mathbb{H}),\ n = 2m$
3) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}),\ n = 2m \geqslant 2$
    a) $\mathfrak{h} = \mathfrak{sp}_n(\mathbb{R}),\ n = 2m$
    b) $\mathfrak{h} = \mathfrak{sp}_{p,m-p}\ (p = 0, 1, \ldots, [m/2])$.

Problem 41. Prove this theorem.

Noncompact real forms of the exceptional simple complex Lie algebras are listed in Tables 7 and 9.

Theorems 1, 6 and Problem 8 imply the following final result of classification of real simple Lie algebras.

Theorem 7. *Non-commutative real simple Lie algebras are exhausted up to an isomorphism by the real forms $\mathfrak{h}$ listed in Theorem 6, by the real forms* *of the exceptional simple complex Lie algebras and by the Lie algebras* $\mathfrak{g}^{\mathbb{R}}$, *where* $\mathfrak{g}$ *are different non-commutative complex simple Lie algebras.*

Notice that Theorem 7 completely solves the classification problem for an arbitrary semisimple Lie algebra over $\mathbb{R}$ since by Theorem 4.1.3 any semisimple Lie algebra uniquely decomposes into the direct sum of non-commutative simple algebras.

Exercises

1) Let $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, where $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{E, -E\}$, and H be the subgroup of G consisting of the pairs $(\pi(X), \bar{X})$, where $\pi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ is the natural projection. Then H is a real form of G which is not of the form G^S, where S is a real structure in G (and not even an open subgroup of a group of the form G^S). In particular, H is not an algebraic real form.
2) Let S be a real structure on a complex algebraic torus T. Then there exists an isomorphism $T \simeq (\mathbb{C}^*)^n$ such that in appropriate coordinates S is expressed in the following form

$$S(x_1, \ldots, z_n) = (\bar{z}_1, \ldots, \bar{z}_p, \bar{z}_{p+q+1}, \bar{z}_{p+1}, \ldots, \bar{z}_{p+2q}, \bar{z}_{p+q}, \bar{z}_{p+2q+1}^{-1}, \ldots, \bar{z}_n^{-1}).$$

In particular, any real structure S on T is algebraic.
3) Any real structure on a connected complex reductive algebraic group is algebraic.
4) A real semisimple Lie group G with a finite number of connected components admits a faithful linear representation if and only if G admits an embedding as a real form in a complex Lie group.
5) The groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{E, -E\}$ are the only (up to an isomorphism) connected Lie groups with the tangent algebra $\mathfrak{sl}_2(\mathbb{R})$ admitting a faithful linear representation.
6) The center of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ (see Example 1.5) is infinite and isomorphic to $\mathbb{Z}$.
7) Let $G = (\mathbb{T} \times \widetilde{\mathrm{SL}}_2(\mathbb{R}))/\langle (t, z) \rangle$, where $t \in \mathbb{T}$, be an element of infinite order and z a generator of $Z(\widetilde{\mathrm{SL}}_2(\mathbb{R}))$. Then the commutator group G' is not a Lie subgroup of G.
8) Let G be a Lie group, $\mathfrak{h}$ a semisimple subalgebra of its tangent algebra $\mathfrak{g}$. If G is simply connected or if the simply connected Lie group with the tangent algebra $\mathfrak{h}$ has a finite center then there is a connected Lie subgroup H of G with the tangent algebra $\mathfrak{h}$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. As follows from Example 4, formula (2) determined an algebraic real structure on the irreducible algebraic group $\mathrm{Int}(\mathfrak{g}(\mathbb{C}))$. The corresponding algebraic real form

$$\mathrm{Int}(\mathfrak{g}(\mathbb{C}))(\mathbb{R}) = \mathrm{Int}(\mathfrak{g}(\mathbb{C})) \cap \mathrm{Aut}\,\mathfrak{g}$$

is called the group of *quasi-inner automorphisms* of $\mathfrak{g}$; denote it $\mathrm{Q\,Int}\,\mathfrak{g}$. Clearly,

$(\mathrm{Q}\,\mathrm{Int}\,\mathfrak{g})^\circ = \mathrm{Int}\,\mathfrak{g}$. The group $\mathrm{Int}\,\mathfrak{g}$ is an algebraic linear group (over $\mathbb{R}$) if and only if $\mathrm{Int}\,\mathfrak{g} = \mathrm{Q}\,\mathrm{Int}\,\mathfrak{g}$.

9) If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, $(n \geqslant 2)$ then $\mathrm{Q}\,\mathrm{Int}\,\mathfrak{g}$ consists of two connected components for even n and coincides with $\mathrm{Int}\,\mathfrak{g}$ for odd n.
10) If $\mathfrak{g} = \mathfrak{so}_{p,q}$, where $p > 0$, $q > 0$, then the number of connected components of $\mathrm{Q}\,\mathrm{Int}\,\mathfrak{g}$ can be found from the following table:

$p+q$ odd	p, q even, $p \neq q$	$p = q$ even	p, q odd, $p \neq q$	$p = q$ odd
2	2	4	1	2

11) The connected simple Lie group $\mathrm{PSL}_2(\mathbb{R}) \simeq O^0_{1,2} \simeq \mathrm{Int}\,\mathfrak{so}_{1,2}$ has no real algebraic group structure.
12) The linear group $\mathrm{Int}(\mathfrak{sl}_3(\mathbb{R}))$ is algebraic (see Exercise 9). The adjoint representation $\mathrm{Ad}\colon \mathrm{SL}_3(\mathbb{R}) \to \mathrm{Int}(\mathfrak{sl}_3(\mathbb{R}))$ is a polynomial isomorphism of Lie groups but it is not a real algebraic group isomorphism.
13) The real algebraic groups $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{Int}(\mathfrak{sl}_3(\mathbb{R}))$ are not isomorphic. Therefore on the connected simple Lie group $\mathrm{SL}_3(\mathbb{R})$ there are at least two non-isomorphic real algebraic group structures.
14) Let $\mathfrak{g}$ be a semisimple complex Lie algebra. A real form of $\mathfrak{g} \oplus \mathfrak{g}$ corresponding by Theorem 4 to the automorphism $\theta\colon (x, y) \mapsto (y, x)$ $(x, y \in \mathfrak{g})$ is isomorphic to $\mathfrak{g}^{\mathbb{R}}$.
15) There are the following isomorphisms between the classical real Lie algebras of different series (see 2°):

$$\begin{array}{ll}
\mathfrak{so}_3 \simeq \mathfrak{su}_2 \simeq \mathfrak{sp}_1, & \mathfrak{so}_6 \simeq \mathfrak{su}_4, \\
\mathfrak{so}_{1,2} \simeq \mathfrak{su}_{1,1} \simeq \mathfrak{sl}_2(\mathbb{R}) \simeq \mathfrak{sp}_2(\mathbb{R}), & \mathfrak{so}_{1,5} \simeq \mathfrak{sl}_2(\mathbb{H}), \\
\mathfrak{so}_4 \simeq \mathfrak{su}_2 \oplus \mathfrak{su}_2, & \mathfrak{so}_{2,4} \simeq \mathfrak{su}_{2,2}, \\
\mathfrak{so}_{1,3} \simeq \mathfrak{sl}_2(\mathbb{C})^{\mathbb{R}}, & \mathfrak{so}_{3,3} \simeq \mathfrak{sl}_4(\mathbb{R}), \\
\mathfrak{so}_{2,2} \simeq \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}), & \mathfrak{u}_2^*(\mathbb{H}) \simeq \mathfrak{su}_2 \oplus \mathfrak{sl}_2(\mathbb{R}), \\
\mathfrak{so}_5 \simeq \mathfrak{sp}_2, & \mathfrak{u}_3^*(\mathbb{H}) \simeq \mathfrak{su}_{1,3}, \\
\mathfrak{so}_{1,4} \simeq \mathfrak{sp}_{1,1}, & \mathfrak{u}_4^*(\mathbb{H}) \simeq \mathfrak{so}_{2,6}. \\
\mathfrak{so}_{2,3} \simeq \mathfrak{sp}_4(\mathbb{R}), &
\end{array}$$

Let $\mathfrak{g}$ be a real Lie algebra, $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ its finite-dimensional real linear representation. Then ρ extends to a complex representation $\rho(\mathbb{C})\colon \mathfrak{g} \to \mathfrak{gl}(V(\mathbb{C}))$.

16) If ρ is irreducible then $\rho(\mathbb{C})$ is irreducible if and only if there is no complex structure on V (i.e. no operator I satisfying (3)) commuting with all $\rho(x)$, $x \in \mathfrak{g}$.
17) If ρ is irreducible and complex, i.e. V admits a complex structure I commuting with ρ, then $\rho(\mathbb{C}) \sim \rho + \bar{\rho}$ (as representations over $\mathbb{C}$), where $\bar{\rho}$ is the representation ρ considered in the space $\bar{V}$ with the complex structure $-I$.

Hints to Problems

1. Notice that any isomorphism of real forms of a complex Lie algebra extends to an automorphism of this algebra.
2. Make use of the identity $\bar{H} = G$ (in Zariski topology) and the fact that the connected components of G coincide with its irreducible components (see Theorem 3.3.1).
3. If $z \in Z(H)$, then $\operatorname{Ad} z = E$ in $\mathfrak{h}$ and therefore in $\mathfrak{g} = \mathfrak{h}(\mathbb{C})$. Next, apply Theorem 1.2.4 and formula (1).
4. Make use of Corollary of Theorem 4.4.1.
6. Show that there exists a unique antilinear automorphism of $\hat{\mathfrak{g}}$ (see 4.3.2°), fixing $\hat{h}_i$, $\hat{e}_i$, $\hat{f}_i$. Clearly, this automorphism maps $\mathfrak{m}$ into itself and therefore induces an antilinear automorphism σ of $\mathfrak{g}$ fixing h_i, e_i, f_i. Clearly, $\sigma^2 = \mathrm{id}$ and $\mathfrak{h} \subset \mathfrak{g}^\sigma$. Since the complex linear span of $\mathfrak{h}$ coincides with $\mathfrak{g}$, we have $\mathfrak{h} = \mathfrak{g}^\sigma$.
8. To prove the second statement make use of Theorem 4.1.3.
10. If $\mathfrak{a}$ is a non-zero ideal of $\mathfrak{g}^{\mathbb{R}}$, then the complex linear span of $\mathfrak{a}$ in $\mathfrak{g}$ coincides with $\mathfrak{g}$. Therefore the ideal $\mathfrak{b} \subset \mathfrak{g}^{\mathbb{R}}$ complementary to $\mathfrak{a}$ must belong to the center of $\mathfrak{g}$ implying $\mathfrak{b} = 0$.
11. Deduce from the simplicity of $\mathfrak{g}$ that if $\mathfrak{a} \neq 0$ is a proper ideal of $\mathfrak{g}(\mathbb{C})$, then $\mathfrak{g}(\mathbb{C}) = \mathfrak{a} \oplus \bar{\mathfrak{a}}$. Next, define the transformation $I\colon \mathfrak{g} \to \mathfrak{g}$ by the formula $Ix = iy - i\bar{y}$ for $x = y + \bar{y} \in \mathfrak{g}$, $y \in \mathfrak{a}$, and prove that I is a complex structure on $\mathfrak{g}$.
12. Follows from Theorem 3.4.2.
13. Make use of the fact that in an orthonormal basis of a compact Lie algebra $\mathfrak{g}$ all operators $\operatorname{ad} x$ ($x \in \mathfrak{g}$) are expressed by skew-symmetric matrices.
14. Problem 4.1.7 implies that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$. With the help of Problem 4.1.2 it is easy to deduce that any commutative ideal of $\mathfrak{g}$ is contained in $\mathfrak{z}(\mathfrak{g})$. This implies that $\mathfrak{g}'$ is semisimple (see Problem 1.4.13).
15. Make use of Problem 4. The compactness of $\operatorname{Int} \mathfrak{g}$ follows from its closedness in $\operatorname{Aut} \mathfrak{g}$ and the compactness of $\operatorname{Aut} \mathfrak{g}$ (thanks to Problem 13).
18. The implication a) $\Rightarrow$ b) follows from Problem 12, the equivalence b) $\Leftrightarrow$ c) from Problem 13. To prove the implication c) $\Rightarrow$ a) consider the finite-sheeted covering $\operatorname{Ad}\colon G \to \operatorname{Ad} G = \hat{G}$. On $\hat{G}$, a real structure $\hat{S}(\operatorname{Ad} g) = S(\operatorname{Ad} g)S^{-1} = \operatorname{Ad} S(g)$ is defined such that $\operatorname{Ad}(G^S) = \hat{G}^{\hat{S}}$. Therefore, the subgroup $\operatorname{Ad}(G^S)$ is closed in $\mathrm{GL}(\mathfrak{g})$. On the other hand, by Problem 17 $\operatorname{Ad}(G^S)$ is contained in the compact group of all operators unitary with respect to h_σ. Hence $\operatorname{Ad}(G^S)$ and G^S are compact.

19. Set $\sigma = \sigma_0 \mu = \mu \sigma_0$, where σ_0 is the real structure determining the normal real form (see Problem 6).
20. By Theorem 1.2.6 the statement holds if G is simply connected. It follows from Problem 4.3.47 that S acts as the identity on $Z(G)$. Therefore, a real structure with the differential σ is defined on any group of the form G/N, where N is a subgroup of $Z(G)$.
24. It suffices to prove this for the generators r_{α_i} $(i = 1, \dots, l)$. But by Problem 4.1.37 r_{α_i} is induced by the element $n_{\alpha_i} = F_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in N(T)$. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}_2$, then $n_{\alpha_i} \in G^S$ by Problem 23.
25. By Theorem 1.2.6, Problems 1.1.24 and 24 any root subspace $\mathfrak{g}_\alpha$ is transformed into the subspace $\mathfrak{g}_{\alpha_i}$ corresponding to a simple root α_i by an appropriate automorphism $\operatorname{Ad} g$, where $g \in N(T) \cap G^S$. Therefore Problems 22 and 17 imply that h_σ is positive definite on $\mathfrak{t}$ and on each subspace $\mathfrak{g}_\alpha$. Then apply Problem 21.
26. The complex structure I transforms $\mathfrak{g}'$ into itself and induces there a self-adjoint linear transformation. If $\mathfrak{g}' \neq 0$ then this contradicts the fact that the characteristic roots of I are $\pm i$.
29. Let σ, τ be real structures on $\mathfrak{g}$ defining its compatible compact real forms and $\theta = \sigma\tau$. Problem 18 implies that $(\theta x, x) < 0$ for all $x \in \mathfrak{g}^\sigma$. It follows from Problem 28 that $\theta x = x (x \in \mathfrak{g}^\sigma)$, whence $\theta = \mathrm{id}$ and $\mathfrak{g}^\sigma = \mathfrak{g}^\tau$.
31. Let $X \in S(\mathbb{E})$ and $\mathbb{E} = \bigoplus_{1 \leqslant i \leqslant \tau} \mathbb{E}_{\lambda_i}$ be the decomposition of $\mathbb{E}$ into the orthogonal sum of eigenspaces with respect to X. Then $\mathbb{E}_{\lambda_i}$ is the eigenspace of $\exp X$ corresponding to the eigenvalue $e^{\lambda_i} > 0$. Therefore, $\exp X \in P(\mathbb{E})$. Conversely, if $A \in P(\mathbb{E})$ and $\mathbb{E} = \bigoplus_{1 \leqslant i \leqslant j} \tilde{\mathbb{E}}_{\mu_i}$ is the corresponding eigenspace decomposition then define $\log A \in S(\mathbb{E})$ setting $(\log A)|\tilde{\mathbb{E}}_{\mu_i} = (\log \mu_i)\mathbb{E}$. It is easy to verify that the map $\log \colon P(\mathbb{E}) \to S(\mathbb{E})$ is inverse to exp.
32. Let us prove that $p^t \in G$ for all $t \in \mathbb{R}$. Let us express the linear operators in $\mathbb{E}$ by matrices in an orthonormal basis. We may assume that $\log p$ is a diagonal matrix with the real diagonal elements $a_1, \dots, a_n$. If F is a polynomial function on the space of all the matrices vanishing on G and $\tilde{F}$ the restriction of F onto the subspace of diagonal matrices then $\tilde{F}(e^{ka_1}, \dots, e^{ka_n}) = 0$ for all $k \in \mathbb{Z}$ since $p^k \in G$. If $\varphi(t) = \tilde{F}(e^{ka_1}, \dots, e^{ka_n})$ does not vanish identically then it is of the form $\varphi(t) = \sum_i c_i e^{tb_i}$, where $c_i \neq 0$ and $b_1 > b_2 > \cdots$ are real numbers. Clearly, the absolute value of $c_1 e^{tb_1}$ for $t = k$ grows as $k \to \infty$ faster than the absolute value of the sum of other terms. This leads to contradiction.
36. Set $\tau_1 = q^{1/4} \tau q^{-1/4}$ (cf. Problem 34). The proof of the second assertion is similar to that of the corresponding assertion of Theorem 3.
39. In one direction the statement is obvious, in the other direction it follows from the obtained classification (see Table 7).
40. Make use of Problem 39.
41. Make use of Theorem 4, Example 2 from 4° and Theorem 5.

§ 2. Compact Lie Groups and Reductive Algebraic Groups

The main goal of this section is to establish a one-to-one correspondence between the compact Lie groups and the reductive complex algebraic groups and also between homomorphisms of compact and reductive groups. In the language of category theory this means that there is an equivalence between the categories of compact Lie groups and reductive complex algebraic groups. An important corollary is the theorem on complete reducibility of linear representations of semisimple Lie algebras. An essential role in the theory developed here is played by the theorem on polar decomposition which we prove in the real setting having in mind its different applications. One of them is the proof of the connectedness of the set of real points of a simply connected complex semisimple Lie group G, defined over $\mathbb{R}$.

1°. Polar Decomposition. In linear algebra the theorem on polar decomposition of a linear operator in a finite-dimensional Euclidean or Hermitian space $\mathbf{E}$ is well-known: any element $A \in \mathrm{GL}(\mathbf{E})$ uniquely presents in the form $A = XY$, where X is an orthogonal (or unitary) operator and Y is a positive definite self-adjoint operator. In this subsection we distinguish a class of algebraic linear groups for which a similar theorem holds. In the complex case all algebraic groups possessing a compact real form belong to this class (we shall see later that these algebraic groups are exactly the reductive ones).

At first we want to refine the above theorem on polar decomposition for the group $\mathrm{GL}(\mathbf{E})$. Set $K = \mathrm{O}(\mathbf{E})$ (respectively $\mathrm{U}(\mathbf{E})$). Consider the map $\varphi\colon K \times S(\mathbf{E}) \to \mathrm{GL}(\mathbf{E})$ defined by

$$\varphi(k, y) = k \exp y. \tag{1}$$

The uniqueness of the polar decomposition and Problem 1.31 imply that φ is bijective. Actually, the following lemma holds.

Lemma 1. *The map* $\varphi\colon K \times S(\mathbf{E}) \to \mathrm{GL}(\mathbf{E})$ *given by* (1) *is a diffeomorphism.*

Proof. Show that the map $d_{(k_0, y_0)}\varphi$ is injective for all $k_0 \in K$, $y_0 \in S(\mathbf{E})$. Using the left translation by k_0 we reduce the proof to the case $k_0 = e$. The tangent algebra $\mathfrak{k}$ or K consists of all skew-symmetric (skew-Hermitian) operators. It is easy to see that

$d_{(e, y_0)}\varphi(x, y) = x \exp y_0 + (d_{y_0}\exp)y \ (x \in \mathfrak{k}, y \in S(\mathbf{E}))$.

Set $p_0 = \exp y_0$, $z = (d_{y_0}\exp)y$. Suppose $d_{(e, y_0)}\varphi(x, y) = xp_0 + z = 0$. Then $p_0^{-1/2} x p_0^{1/2} = -p_0^{-1/2} z p_0^{-1/2}$; the right-hand side of this identity is, clearly, a self-adjoint operator, but on the left we have an operator whose characteristic roots are purely imaginary. Hence, $x = z = 0$. Therefore, we have to prove that $y = 0$, i.e. the injectivity of $d_{y_0}\exp$.

Consider the curves $g(t) = y_0 + ty$ and $z(t) = \exp y(t)$ and differentiate the identity $y(t)z(t) = z(t)y(t)$ with respect to t. Since $z = 0$, we have $yp_0 = p_0 y$. Since

y_0 and p_0 have the same eigenspaces, $yy_0 = y_0 y$. It follows from Problem 1.2.27 that $z(t) = p_0 \exp ty$. Hence, $p_0 y = 0$ and $y = 0$. □

For an arbitrary $g \in \mathfrak{gl}(\mathbb{E})$ denote by g^* its adjoint operator. A linear group $G \subset \mathrm{GL}(\mathbb{E})$ is called *self-adjoint* if $g^* \in G$ for any $g \in G$.

Theorem 1. *Let* $\mathbb{E}$ *be a finite-dimensional Euclidean (Hermitian) vector space,* $G \subset \mathrm{GL}(\mathbb{E})$ *a self-adjoint algebraic (real or complex) group,* $K = G \cap \mathrm{O}(\mathbb{E})$ *(resp.* $G \cap \mathrm{U}(E)$*) and* $P = G \cap P(\mathbb{E})$*. Then*

$$G = KP, \tag{2}$$

each element $g \in G$ *being uniquely presented in the form* $g = kp$*, where* $k \in K, p \in P$*. More precisely, denote* $\mathfrak{p} = \mathfrak{g} \cap S(\mathbb{E})$*, then the map* $\varphi\colon K \times \mathfrak{p} \to G$ *defined by* (1) *is a diffeomorphism. For any* $g \in G$ *we have*

$$gPg^* = P. \tag{3}$$

Proof. Formula (2) is proved by a trick well known in the linear algebra. If $g \in G$ then $q = g^*g \in P$. Problem 1.32 implies that $p = q^{1/2} \in P$. Clearly, $k = gp^{-1}$ is an orthogonal (unitary) operator, whence $k \in K$ and $g = kp$. It follows from Lemma 1 that φ is a diffeomorphism. Formula (3) is obvious. □

The decomposition (2) is called the *polar decomposition* of a self-adjoint algebraic linear group G.

Corollary 1. *A self-adjoint algebraic linear group* G *is diffeomorphic to* $K \times \mathbb{R}^m$*, where* K *is the compact subgroup defined in Theorem* 1 *and* $m = \dim \mathfrak{p}$*. In particular,* G *is connected if and only if so is* K*, and in this case* $\pi_1(G) \simeq \pi_1(K)$*.*

Problem 1 (Corollary 2). Under the assumptions of Theorem 1

$$Z(G) = (Z(G) \cap K) \times (Z(G) \cap P),$$

and $Z(G) \cap P \simeq \mathbb{R}^s$ for some $s \geqslant 0$. If G is semisimple then $Z(G) \subset K$.

Problem 2 (Corollary 3). Under the same assumptions $L \cap P = \{e\}$ for any compact subgroup $L \subset G$. In particular, K is a maximal compact subgroup of G (i.e. is not contained in any larger compact subgroup of G).

Now we may consider a special case which is convenient to formulate as a separate theorem because it is important in what follows.

Theorem 2. *Let* $G \subset \mathrm{GL}(V)$ *be a complex algebraic linear group with a compact real form* K *and* $\mathfrak{p} = i\mathfrak{k}$*. The map* $\varphi\colon K \times \mathfrak{p} \to G$ *defined by* (1) *is a diffeomorphism of real manifolds. A real form* K *is an algebraic one.*

Proof. Make V into a Hermitean space $\mathbb{E}$ fixing a positive definite Hermitian form in it invariant with respect to K (see Theorem 3.4.2). Then $\mathfrak{k}$ consists of skew-Hermitian operators and $\mathfrak{p} = i\mathfrak{k}$ consists of self-adjoint operators so that $\mathfrak{p} = \mathfrak{g} \cap S(\mathbb{E})$.

Problem 3. G is self-adjoint.

Problem 3 implies that Theorem 1 is applicable to G, where the role of K is played by $K_1 = G \cap \mathrm{U}(\mathbf{E})$.

Problem 4. K_1 coincides with K.

Therefore it only remains to prove the last statement of Theorem 2. Consider the automorphism $S\colon g \mapsto (g^*)^{-1}$ of G. Clearly, S is an algebraic real structure on G and by Problem 4 $K = G^S$. □

Corollary 1. *Under the assumptions of Theorem* 2 *G is diffeomorphic to $K \times \mathbb{R}^m$, where $m = \dim_{\mathbb{C}} G$.*

Problems 1 and 1.3 imply

Corollary 2. *Under the assumptions of Theorem* 2

$$Z(G) = Z(K) \times (Z(G) \cap P).$$

If G is semisimple then $Z(G) = Z(K)$.

Corollary 3. *Under the assumptions of Theorem* 2

$$N(K) = K \times (Z(G) \cap P).$$

If G is semisimple then $N(K) = K$.

Proof. Clearly, $N(K) = K(N(K) \cap P)$. If $g \in N(K) \cap P$ then the uniqueness of the polar decomposition and (3) imply that $g \in Z(K)$. Since $\mathfrak{g} = \mathfrak{k}(\mathbb{C})$, then $\operatorname{Ad} g = E$. One easily deduces that $gpg^{-1} = p$ for all $p \in P$, whence $g \in Z(G)$. □

Let us apply the polar decomposition to the proof of the following statement.

Theorem 3. *Let S be a real structure on a simply connected complex semisimple Lie group G. Then the real form G^S is algebraic and connected.*

Proof. Set $\sigma = dS$. Let us show that there exists a compact real form K of G such that the corresponding real form $\mathfrak{k}$ of $\mathfrak{g}$ is compatible with $\mathfrak{g}^\sigma$. By Problem 1.33 there exists on $\mathfrak{g}$ a real structure τ commuting with σ such that $\mathfrak{g}^\tau$ is compact. By Theorem 1.2.6 there exists an automorphism T of G (considered as a real Lie group) such that $\tau = dT$.

Clearly, T is a real structure in G commuting with S. Thanks to Problem 1.18 the real form $K = G^T$ is compact.

By Theorem 3.3.4 the involutive automorphism $\Theta = TS$ of G is polynomial. Therefore the algebraicity of the real structure T (Theorem 2) implies that S is also an algebraic real structure.

As in the proof of Theorem 2, we may assume that $G \subset \mathrm{GL}(\mathbf{E})$, where $\mathbf{E}$ is a Hermitian vector space, whose scalar product is K-invariant. Moreover, $T(g) = (g^*)^{-1}$ and G is a self-adjoint algebraic linear group. Since T commutes with S and Θ, the groups G^S and G^Θ are also self-adjoint. Clearly, the compact parts $G^S \cap K$ and $G^\Theta \cap K$ of the polar decompositions coincide. By Theorem 4.4.9 G^Θ

is connected. Applying Corollary 1 of Theorem 1 we derive from here that the subgroup $G^{\Theta} \cap K = G^S \cap K$ is connected and therefore so is G^S. □

2°. Lie Groups with Compact Tangent Algebras. By Problem 1.15 each compact Lie algebra is isomorphic to the tangent algebra of a compact Lie group. However, a non-compact Lie group can have a compact tangent algebra: the simplest example is the additive group $\mathbb{R}$. In this subsection we will study the structure of Lie groups with a finite number of connected components whose tangent algebra is compact. First consider connected groups. Recall (see Problem 1.14) that a compact Lie algebra $\mathfrak{k}$ presents in the form $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}'$, where $\mathfrak{z}$ is the center of $\mathfrak{k}$ and the derived algebra $\mathfrak{k}'$ is a semisimple compact Lie algebra.

Problem 5. Any simply connected Lie group K with a compact semisimple tangent algebra is isomorphic to a compact real form of a simply connected complex semisimple Lie group.

Problem 5 implies that a simply connected (hence an arbitrary connected) semisimple Lie group with a compact tangent algebra is compact and therefore has a finite center.

Problem 6. Any connected compact Lie group K has a finite-sheeted covering $Z \times L \to K$, where Z is a compact torus and L is a simply connected semisimple compact Lie group.

Problem 7. Any connected compact Lie group K is isomorphic to an algebraic real form of a connected complex reductive algebraic group. In particular, K admits a faithful linear representation.

Problem 7 implies the following theorem describing the structure of connected compact Lie groups.

Theorem 4. *Let K be a connected compact Lie group. Then K' is a connected semisimple compact Lie subgroup of K and K admits the locally direct decomposition $K = ZK'$, where $Z = \operatorname{Rad} K$ is the compact torus coinciding with the identity component $Z(K)^0$ of the center of K.*

Problem 8. Prove this theorem.

Now pass to arbitrary connected Lie groups with compact tangent algebras. The simplest class of these groups are connected commutative groups. Recall (see Proposition 1, 2, 3) that any connected commutative group G presents in the form $G = A \times B$, where $A \simeq \mathbb{R}^p$ is a vector group and $B \simeq \mathbb{T}^q$ a compact torus.

Problem 9. B is the largest compact subgroup of the connected commutative group G, i.e. contains all compact subgroups of this group, and therefore is uniquely defined. For A one can take any subgroup of the form $\exp \mathfrak{a}$, where $\mathfrak{a}$ is a subspace of the tangent algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{b}$ is the tangent algebra of B.

A and B are called the *non-compact* and *compact* parts of the connected commutative group G respectively.

Theorem 5. *Let G be a connected Lie group with a compact tangent algebra and A and B the non-compact and compact parts of $Z(G)^0$. Then $G = A \times K$, where $K = BG'$ is a compact Lie subgroup. K is the largest compact subgroup of G.*

To prove this theorem we will need the following

Problem 10. Let $\pi: G \to G_0$ be a finite-sheeted covering and G_0 satisfy Theorem 5. Then G also satisfies Theorem 5.

Now let G be a connected Lie group with a compact tangent algebra. Let us construct a finite-sheeted covering $G \to G_0$ satisfying the conditions of Problem 10. Let $\pi: \tilde{G} \to G$ be a simply connected covering of G. Clearly, $\tilde{G} = \tilde{Z} \times \tilde{G}'$, where $\tilde{Z}$ is a vector group, $\tilde{G}'$ a semisimple compact Lie group (see Problem 6). Set

$$N = \operatorname{Ker} \pi, \quad N_0 = NZ(\tilde{G}'), \quad G_0 = \tilde{G}/N_0.$$

Problem 11. $N_0 = N_1 \times Z(\tilde{G}')$, where N_1 is a discrete subgroup of $\tilde{Z}$ and $G_0 = \tilde{Z}/N_1 \times \tilde{G}'/Z(\tilde{G}')$. There exists a finite-sheeted covering $\pi_0: G \to G_0$.

Since $\tilde{G}/Z(\tilde{G}')$ is compact, G_0 satisfies Theorem 5. By Problem 10 so does G. □

Now we can prove the main result of this subsection.

Theorem 6. *Let G be a Lie group with a finite number of connected components and a compact tangent algebra and $Z = Z(G^0)^0$. We can choose a non-compact part A of Z which is a normal subgroup of G. For any such a choice of A we have $G = A \rtimes K$, $G^0 = A \times K^0$, where K is a compact Lie subgroup.*

Let $\mathfrak{b} \subset \mathfrak{z}$ be the tangent algebras of the compact part B of Z and Z itself, respectively. Clearly, the automorphisms $a(g) (g \in G)$ transform Z into itself. By Problem 9 B is also mapped into itself by all the $a(g)$. Therefore $\mathfrak{z}$ and $\mathfrak{b}$ are invariant with respect to the adjoint representation of G.

Problem 12. In $\mathfrak{z}$, there exists a subspace $\mathfrak{a}$ invariant with respect to $\operatorname{Ad} G$ such that $\mathfrak{z} = \mathfrak{a} \oplus \mathfrak{b}$.

Problems 9 and 12 imply the existence of a subgroup $A \subset G$ described in Theorem 6. Applying Theorem 5 to G^0 we get $G^0 = A \times K_0$, where K_0 is a compact Lie subgroup. To finish the proof of Theorem 6 we need the following.

Lemma 2. *Let G be a Lie group with a normal vector Lie subgroup A of finite index. Then $G = A \rtimes L$, where L is a finite subgroup.*

Proof. Let $L_0 = G/A$, $\pi: G \to L_0$ the natural homomorphism. It suffices to construct a homomorphism $\varphi: L_0 \to G$ such that $\pi\varphi = \mathrm{id}$; then $G = A \rtimes L$, where $L = \varphi(L_0)$. Choose a map $\psi: L_0 \to G$ such that $\pi\psi = \mathrm{id}$ and seek φ in the form

$$\varphi(x) = h(x)\psi(x) \qquad (x \in L_0), \tag{4}$$

where $h: L_0 \to A$ is a map. Observe that

$$\varphi(x)\psi(y) = f(x, y)\psi(xy) \qquad (x, y \in L_0) \tag{5}$$

where $f(x, y) \in A$. The condition $\varphi(xy) = \varphi(x)\varphi(y)$ is equivalent to the following identity relating h with the map $f\colon L_0 \times L_0 \to A$:

$$f(x, y) = \psi(x)h(y)^{-1}\psi(x)^{-1}h(x)^{-1}h(y) \qquad (x, y \in L_0) \tag{6}$$

We will express the group operation in A additively. As follows from Problem 1.2.26 any automorphism of the vector group A is a linear transformation. Therefore the formula

$$R(g) = a(g)|A \qquad (g \in G) \tag{7}$$

determines a linear representation $R\colon G \to \mathrm{GL}(A)$. Since $A \subset \operatorname{Ker} R$, there arises a linear representation $R_0\colon L_0 \to \mathrm{GL}(V)$ such that $R = R_0\pi$. Formula (6) takes the form

$$f(x, y) = h(xy) - h(x) - R_0(x)h(y) \qquad (x, y \in L_0) \tag{8}$$

Thus, it suffices to choose a map $h\colon L_0 \to A$ satisfying (8) with f defined by (5); then (4) defines the desired homomorphism φ.

Problem 13. For any $x, y, z \in L_0$ we have

$$f(x, yz) + R_0(x)f(y, z) = f(xy, z) + f(x, y).$$

Problem 14. The map $h\colon L_0 \to A$ defined by the formula

$$h(x) = -\frac{1}{|L_0|}\sum_{y \in L_0} f(x, y),$$

satisfies (8).

Therefore Lemma 2 is proved. □

Problem 15. Prove Theorem 6.

A subgroup K of a Lie group G is a *maximal compact subgroup* of G if K is compact and is not contained in any larger compact subgroup of G. We will not assume that K is a Lie subgroup. (This is automatically so since K is closed in G (see 1.2.9°; this fact will not be used though).) Any automorphism of G permutes its maximal compact subgroups.

The following theorem shows that the subgroup K mentioned in Theorem 5 is maximal compact in G and is unique up to conjugacy.

Theorem 7. *Let $G = A \rtimes K$, where A is a vector group, K a compact Lie group. Then K is a maximal compact subgroup of G. For any compact subgroup $K_1 \subset G$ there exists $a \in A$ such that $aK_1a^{-1} \subset K$ and if K_1 is a maximal compact subgroup this inclusion is actually an equality.*

Before proving this theorem make several general remarks on semidirect products of Lie groups. Let $G = A \rtimes K$, where A is a vector group. Then the automorphisms $a(g)|A$ $(g \in G)$ are linear transformations of the space A (see the proof of Lemma 2). Therefore formula (7) defines linear representation $R\colon G \to \mathrm{GL}(A)$. Now, consider the vector space A as an affine space. Then we may define a natural affine G-action on A:

Problem 16. There exists a unique affine action $\tilde{R}\colon G \to \mathrm{GL}(A)$ such that $\tilde{R}(a) = t_a$ $(a \in A)$ and $\tilde{R}(k) = R(k)$ $(k \in K)$. This action contains all translations and in particular it is transitive on A. The subgroup K is the stabilizer of $0 \in A$.

Since the stabilizers of any two points are conjugate for a transitive action of a group, Problem 16 implies that the subgroup $K_1 \subset G = A \rtimes K$ is conjugate to a subgroup contained in K if and only if A contains a point fixed under $\tilde{R}(K_1)$. An element $a \in G$ such that $aK_1a^{-1} \subset K$ may be assumed to belong to A.

Proof of Theorem 7. Since A does not contain non-trivial compact subgroups, K is a maximal compact subgroup of $G = A \rtimes K$. The conjugacy follows from the above remarks and the existence of a fixed point for any affine action of a compact group (Theorem 3.4.1). □

3°. Compact Real Forms of Reductive Algebraic Groups. In this subsection we will generalize Theorem 1.2 on the existence of a compact real form of a connected complex semisimple Lie group to arbitrary reductive algebraic groups. Besides, we will prove the conjugacy of compact real forms. The main results are formulated as follows:

Theorem 8. *Any reductive complex algebraic group possesses an algebraic compact real form.*

Theorem 9. *Any two compact real forms of a reductive complex algebraic group G are transformed into each other by an automorphism of the form $a(g)$, where $g \in G^0$.*

Proof of Theorem 8. Let G be a reductive complex algebraic group, $H = (G^0)'$, $Z = \operatorname{Rad} G = Z(G^0)^0$. In a connected semisimple Lie group H choose a compact real form L (see Theorem 1.2) which is connected thanks to Corollary 1 of Theorem 2 and let $U = N(L)$. Applying Corollary 3 of Theorem 2 to H and L and using the decomposition $G^0 = ZH$, we get $U \cap G^0 = ZL$. In particular, the group $U \cap G^0$ is connected implying $U^0 = U \cap G^0 = ZL$ and $\mathfrak{u} = \mathfrak{z} \oplus \mathfrak{l}$. Therefore the tangent algebra of U is compact.

Problem 17. $G = HU$, $G/G^0 \simeq U/U^0$.

Thus, U has a finite number of connected components. In the tangent algebra $\mathfrak{z}$ of the torus Z, consider the real form $\mathfrak{z}(\mathbb{R})$ defined in 3.3.2° and set $A = \exp \mathfrak{z}(\mathbb{R})$, $B = \exp(i\mathfrak{z}(\mathbb{R}))$. Then $Z = A \times B$, A being the non-compact and B the compact parts of Z (see Example 2 in 1.1°). Since $\mathfrak{z}(\mathbb{R})$ is stable under all automorphisms of Z and Z is a normal subgroup of G, A is also a normal subgroup of G. Applying to U Theorem 5 we see that $U = A \rtimes K$, where $K \subset U$ is a compact subgroup such that $K^0 = BL$.

Problem 18. The subgroup K is a real form of G.

The algebraicity of the real form K follows from Theorem 2. Therefore Theorem 8 is proved. □

Proof of Theorem 9. Let K be a compact real form of G constructed in the proof of Theorem 6 and K_1 another compact real form of G. Let σ be a real structure on $\mathfrak{g}$ such that $\mathfrak{k}_1 = \mathfrak{g}^\sigma$. Then σ transforms the center $\mathfrak{z}$ and the derived algebra $\mathfrak{h}$ of $\mathfrak{g}$ into themselves and induces on each of these subalgebras a real structure. We have $\mathfrak{k}_1 = \mathfrak{z}^\sigma \oplus \mathfrak{h}^\sigma$. Since $K_1 \cap Z$ is compact, it is contained in B so that $\mathfrak{z}^\sigma = \mathfrak{k}_1 \cap \mathfrak{z} \subset i\mathfrak{z}(\mathbb{R})$ implying $\mathfrak{z}^\sigma = i\mathfrak{z}(\mathbb{R})$ and $K_1 \cap Z = B$. Further, $\mathfrak{h}^\sigma$ is a compact real form of $\mathfrak{h}$. Applying Theorem 1.3 we may assume that $\mathfrak{h}^\sigma = \mathfrak{l}$. Then $\mathfrak{k}_1 = \mathfrak{k}$, hence $K_1^0 = BL$. Therefore, $K_1 \subset N(BL) = N(L) = U$.

Problem 19. There exists $a \in A$, such that $aK_1a^{-1} = K$.

Thus Theorem 9 is proved. □

4°. Linearity of Compact Lie Groups. Thanks to Problem 7 any connected compact Lie group admits a faithful linear representation. Now let us extend this statement to arbitrary compact Lie groups. Therefore we will prove

Theorem 10. *Any compact Lie group admits a faithful linear representation.*

Let G be a Lie group. A differentiable function $f\colon G \to \mathbb{C}$ is said to be *representative* if the functions $r_*(g)f (g \in G)$ determined by (3.1.3) generate a finite-dimensional subspace of the space $C^\infty(G)$ of all differentiable complex functions on G. For instance, if G is a complex algebraic group then all polynomial functions on G are representative (see Theorem 3.1.9). Denote by A_G the set of all representative functions on G.

Problem 20. A_G is a subalgebra of $C^\infty(G)$ and coincides with the linear span of matrix elements of all finite-dimensional complex linear representations of G.

Lemma 3. *If G is a compact Lie group then for any $g \in G$, $g \neq e$, there exists $f \in A_G$ such that $f(g) \neq f(e)$.*

Proof. If $g \notin G^0$ then we may take for f the function which vanishes on G^0 and equals 1 on all the other connected components of G; clearly, its orbit with respect to right translations is contained in the finite-dimensional space of all functions which are constant on connected components. Let $a \in G^0$. Since G^0 admits a faithful representation thanks to Problem 7, there exists a matrix element of this representation $f_0 \in A_{G^0}$ such that $f_0(g) \neq f_0(e)$. Let us extend f_0 to a function f on G setting $f(x) = 0$, if $x \in G \backslash G^0$. Clearly, the linear span L_f of the orbit of f under right translations by elements $g \in G^0$ is finite-dimensional. Furthermore, if g and g' belong to the same component of G then $r_*(g)L_f = r_*(g')L_f$. Therefore the orbit of f under right translations is contained in $\sum_g r_*(g)L_f$, where g runs through the set of representatives of the connected components of G. Hence, $f \in A_G$ and Lemma 3 is proved. □

Problem 21. Any strictly descending chain of Lie subgroups in a compact Lie group is finite.

Proof of Theorem 10. Let R_1 be a linear representation of a compact Lie group G. If $\operatorname{Ker} R_1 \neq \{e\}$ then choose some $g \in \operatorname{Ker} R_1$, $g \neq e$. By Lemma 3 and Problem 20 there exists a representation S of G such that a matrix element f of this representation satisfies $f(g) \neq f(e)$. Then $g \notin \operatorname{Ker} S$. Setting $R_2 = R_1 + S$ we have strict inclusion $\operatorname{Ker} R_1 \supset \operatorname{Ker} R_2$. If $\operatorname{Ker} R_2 \neq \{e\}$ then we similarly construct a representation R_3 with the strict inclusion $\operatorname{Ker} R_2 \supset \operatorname{Ker} R_3$, etc. Due to Problem 21 this process terminates and we get a faithful representation. □

5°. Correspondence Between Compact Lie Groups and Reductive Algebraic Groups. In this subsection we will show that the complexification of real algebraic groups leads to a one-to-one correspondence between compact Lie groups (considered up to a differentiable isomorphism) and reductive complex algebraic groups (considered up to a polynomial isomorphism).

Let K be a compact Lie group. By Theorem 10 K admits a faithful linear representation which may be considered as a real one. Therefore Theorem 3.4.5 implies that K possesses a real algebraic group structure. This structure a priori depends on the choice of a faithful representation though actually it is unique as it will follow from our future arguments.

Consider the complexification $K(\mathbb{C})$ of a compact real algebraic group K.

Problem 22. The algebraic group $K(\mathbb{C})$ is reductive.

Now we wish to prove that the algebraic group $K(\mathbb{C})$ does not depend (up to an isomorphism) on the choice of the algebraic group structure on K. This is a consequence of the following

Theorem 11. *Let K_1, K_2 be compact real algebraic groups. Then any differentiable homomorphism $\varphi: K_1 \to K_2$ uniquely extends to a polynomial homomorphism $\varphi(\mathbb{C}): K_1(\mathbb{C}) \to K_2(\mathbb{C})$. If $\psi: K_2 \to K_3$ is another differentiable homomorphism of compact real algebraic groups then*

$$(\psi\varphi)(\mathbb{C}) = \psi(\mathbb{C})\varphi(\mathbb{C}). \tag{9}$$

Corollary. *Under the assumptions of Theorem* 10 *any differentiable isomorphism $\varphi: K_1 \to K_2$ extends to a polynomial isomorphism $\varphi(\mathbb{C}): K_1(\mathbb{C}) \to K_2(\mathbb{C})$ and is a polynomial isomorphism itself.*

Therefore the group $K(\mathbb{C})$ and the algebraic structure on the compact Lie group K are uniquely defined.

Let us precede the proof of Theorem 11 by the following

Problem 23. If under the conditions of Theorem 11 the extending homomorphism $\varphi(\mathbb{C})$ exists and the homomorphism $d\varphi$ is injective then $\operatorname{Ker} \varphi(\mathbb{C}) = \operatorname{Ker} \varphi \subset K_1$.

Proof of Theorem 11. Let $G_i = K_i(\mathbb{C})$ $(i = 1,2)$. Then $G_1 \times G_2 = (K_1 \times K_2)(\mathbb{C})$. Let π_i be the projection $G_1 \times G_2 \to G_i$ onto the i-th component. Consider the graph $\Gamma = \{(k, \varphi(k))\colon k \in K_1\}$ of φ which is a compact Lie subgroup of $K_1 \times K_2$. By Theorem 3.4.5 Γ is an algebraic subgroup. Clearly, $\pi_1\colon \Gamma \to K_1$ is a polynomial and bijective homomorphism. Consider an algebraic subgroup $\Gamma(\mathbb{C}) \subset G_1 \times G_2$. The projection $\pi_1\colon \Gamma(\mathbb{C}) \to G_1$ extends $\pi_1\colon \Gamma \to K_1$ and therefore is injective by Problem 23. Theorem 3.1.6 implies that this is a polynomial isomorphism of $\Gamma(\mathbb{C})$ onto G_1. The homomorphism $\varphi(\mathbb{C}) = \pi_2\pi_1^{-1}\colon G_1 \to G_2$ is the desired extension.

The uniqueness of the extension $\varphi(\mathbb{C})$ follows from the fact that K_1 is dense in G_1 in Zariski topology and the relation (9) follows from the uniqueness. □

Now let us state the final result.

Theorem 12. *On any compact Lie subgroup K there exists a unique real algebraic group structure and the complex algebraic group* $K(\mathbb{C})$ *is reductive. Any reductive complex algebraic group possesses an algebraic compact real form. Two compact Lie groups are isomorphic (as Lie groups or as algebraic groups over* $\mathbb{R}$*) if and only if the corresponding reductive algebraic groups over* $\mathbb{C}$ *are isomorphic.*

Proof of this theorem follows from Corollary of Theorem 11, Problem 22, Theorems 8 and 9.

Problem 24 (Corollary). Any compact subgroup L of a compact Lie group K is an algebraic subgroup in K. In $K(\mathbb{C})$, there exists a unique algebraic subgroup containing L as a real form and isomorphic to $L(\mathbb{C})$; its intersection with K coincides with L.

6°. Complete Reducibility of Linear Representations. In this subsection we will prove that a complex algebraic linear group is completely reducible if and only if it is reductive. The proof is based on the complete reducibility of compact linear groups proved in 3.4. Furthermore, the completely reducible real algebraic linear groups are real forms of complex reductive groups. In particular, it turns out that any linear representation of a real semisimple Lie algebra is completely reducible. This method of the proof of complete reducibility of semisimple linear groups due to H. Weyl [49] is often called the *unitary trick*. All considered linear groups and linear representations act in finite-dimensional vector spaces over $\mathbb{C}$ or $\mathbb{R}$.

First discuss some general questions having to do with the definition of complete reducibility (see 3.4.2°). A linear group $G \subset \mathrm{GL}(V)$, where V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ is *completely reducible* if V splits into the direct sum of irreducible G-invariant subspaces or, equivalently (see Problem 3.4.2), if for any G-invariant subspace $V_1 \subset V$ there exists a G-invariant direct complement. In this setting it clearly suffices to verify the latter property for the irreducible subspaces V_1. A completely reducible linear group G determines a completely reducible linear group in any G-invariant subspace of V.

Problem 25. Let G be a linear group in a vector space V over R. Consider it as a subgroup of $\mathrm{GL}(V(\mathbb{C}))$ making use of the natural embedding $\mathrm{GL}(V) \to$

$\mathrm{GL}(V(\mathbb{C}))$. The group G is completely reducible in V if and only if so it is in $V(\mathbb{C})$.

Problem 26. A linear group G in a vector space V over $\mathbb{C}$ is completely reducible if and only if G is completely reducible (over $\mathbb{R}$) in $V^{\mathbb{R}}$.

Problem 27. A linear group G in a vector space V over $\mathbb{C}$ or $\mathbb{R}$ is completely reducible if and only if so is its algebraic closure $G^a \subset \mathrm{GL}(V)$.

A real algebraic group G is *reductive* if its complexification $G(\mathbb{C})$ is a reductive complex algebraic group. For instance the compact and semisimple real algebraic groups are reductive.

Theorem 13. *A reductive (complex or real) linear algebraic group is completely reducible.*

Proof. A reductive complex algebraic group G is an algebraic closure of a compact subgroup (see Theorem 7) which is completely reducible thanks to Corollary of Theorem 3.4.2. By Problem 27 G is also completely reducible. If G is a real reductive linear algebraic group in a real vector space V then $G(\mathbb{C})$ is a complex reductive group in $V(\mathbb{C})$. Therefore due to Problems 25 and 27 G is completely reducible over $\mathbb{R}$. Now if a real reductive group G acts in a complex space then its complete reducibility follows from Problem 26. □

Let us point out several corollaries for linear representations. Recall that a linear representation of a group (or of a Lie algebra) is called *completely reducible* if its image is a completely reducible linear group (linear Lie algebra). This is equivalent to the existence in the space of the representation of a complementary invariant subspace for any invariant subspace.

Since the image of a reductive algebraic group under a linear representation is reductive (see Problem 4.1.22), Theorem 13 implies

Corollary 1. *A linear representation of a reductive complex algebraic group is completely reducible.*

Corollary 2 (Problem 28). *If G is a semisimple real Lie group with a finite number of connected components then any linear representation of G over $\mathbb{C}$ or $\mathbb{R}$ is completely reducible.*

Problem 29. Let G be a connected Lie group, R its linear representation. The representation R is completely reducible if and only if so is the representation dR of the tangent algebra $\mathfrak{g}$.

Problem 29 and Theorem 13 imply

Corollary 3. *A linear representation of a complex or real semisimple Lie algebra is completely reducible.*

Note some applications of this corollary.

Problem 30. Let $\mathfrak{g}$ be a complex or real Lie algebra. If $\mathfrak{rad}\,\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$, then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$, the derived algebra being semisimple.

Problem 31. If a connected complex algebraic group G contains a normal subgroup T which is a torus, then $T \subset Z(G)$. A complex algebraic group is reductive if and only if its radical is a torus.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Corollary 3 implies that any finite-dimensional linear representation ρ of $\mathfrak{g}$ is equivalent to the sum $\rho_1 + \cdots + \rho_s$ of irreducible representations ρ_i which are determined uniquely up to an isomorphism. The representations ρ_i are called the *irreducible components* of ρ.

Corollary 4. *A linear representation of a semisimple complex Lie algebra is determined up to an isomorphism by the system of its highest (or lowest) weights their multiplicities (the dimensions of the corresponding weight subspaces) counted.*

Now we prove a theorem converse to Theorem 13.

Theorem 14. *Any completely reducible complex or real algebraic linear group is reductive.*

Proof. Thanks to Problems 25 and 27 the real case is reduced to the complex one. Let $G \subset \mathrm{GL}(V)$ be a completely reducible complex algebraic group. As we see from Problem 31, it suffices to show that $\operatorname{Rad} G$ is a torus.

By Lie's theorem (see 1.4.5°) $\operatorname{Rad} G$ possesses weight vectors in V. Denote by $\lambda_1, \ldots, \lambda_p$ the complete set of distinct weights of $\operatorname{Rad} G$ in V and by V_{λ_i} the corresponding weight subspaces. Then the subspace $V' = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_p}$ is invariant with respect to G. Therefore $V = V' \oplus V''$, where V'' is another invariant subspace. If $V'' \neq 0$, then by Lie's theorem $\operatorname{Rad} G$ possesses a weight vector in V'' which is impossible. Thus, $V = V'$. It follows that $\operatorname{Rad} G$ is a torus (see Problem 3.2.17). □

7°. Maximal Tori in Compact Lie Groups. In this subsection we consider connected compact Lie groups and their generalization—connected Lie groups with compact tangent algebras. We will study some properties of maximal connected commutative subgroups of these groups similar to the properties of maximal tori in complex algebraic groups. The term "torus" means a compact torus, i.e. a Lie group isomorphic to $\mathbb{T}^n$. Recall that any connected compact commutative Lie group is a torus (see Proposition 1.2.3).

Let K be a compact Lie group.

Problem 32. Any maximal connected commutative subgroup A of K is a torus. The tangent algebra $\mathfrak{a}$ of A is a maximal commutative subalgebra of Lie algebra $\mathfrak{k}$ and $A = \exp \mathfrak{a}$. Conversely, for any maximal commutative subalgebra $\mathfrak{a} \subset \mathfrak{k}$ the subgroup $A = \exp \mathfrak{a} \subset K$ is a maximal connected commutative subgroup with the tangent algebra $\mathfrak{a}$.

A maximal connected commutative subgroup of a compact Lie group K is called a *maximal torus* of K.

Problem 33. A compact subgroup A of K is a (maximal) torus if and only if $A(\mathbb{C})$ is a (maximal) algebraic torus of $K(\mathbb{C})$.

Problem 34. A maximal torus A of a connected compact Lie group K coincides with its centralizer in K. The subgroup A contains $Z(K)$ and is maximal among commutative (not necessarily connected) subgroups of K.

Theorem 15. *Any two maximal tori of a compact Lie group K are conjugate.*

Proof. Let A_1, A_2 be maximal tori of K. By Problem 33 $A_1(\mathbb{C})$ and $A_2(\mathbb{C})$ are maximal algebraic tori in $K(\mathbb{C})$. Therefore (see Problem 3.2.24), there exists $g \in K(\mathbb{C})$ such that $gA_1(\mathbb{C})g^{-1} = A_2(\mathbb{C})$. Since A_1 and A_2 are the largest compact subgroups of $A_1(\mathbb{C})$ and $A_2(\mathbb{C})$, then $gA_1g^{-1} = A_2$. Since $K(\mathbb{C})$ can be considered as a linear group, we have the polar decomposition $K(\mathbb{C}) = KP$, where $P = \exp(i\mathfrak{k})$ (see Theorem 2). Let $g = kp$, where $k \in K$, $p \in P$. Set $l = pap^{-1}$. Then $l \in K$ for any $a \in A_1$ implying $a^{-1}pa = a^{-1}lp$. It follows from (3) and the uniqueness of the polar decomposition that $a^{-1}pa = p$. Therefore $pap^{-1} = a$ for any $a \in A_1$, hence $A_2 = kA_1k^{-1}$. □

Now consider a more general situation, when K is a connected Lie group whose tangent algebra $\mathfrak{k}$ is compact. By Theorem 4 we have the direct product decomposition $K = L \times C$, where $L \supset K'$ is the largest compact subgroup of K, $C \simeq \mathbb{R}^p$ the non-compact part of the commutative group $Z(K)^0$.

Theorem 16. *If K is a connected Lie group with a compact tangent algebra $\mathfrak{k}$ then any maximal connected commutative subgroup A in K is of the form $A = (A \cap L) \times C$, where $A \cap L$ is a maximal torus of L. The subgroup A coincides with its centralizer and, in particular, contains $Z(K)$. All maximal connected commutative subgroups of K are conjugate. The map* $\exp\colon \mathfrak{k} \to K$ *defines a one-to-one correspondence between the maximal commutative subalgebras of $\mathfrak{k}$ and the maximal connected commutative subgroups of K.*

Problem 35. Prove this theorem.

Exercises

1) Let $\mathbf{E}$ be a finite-dimensional Euclidean (or Hermitian) space, G a subgroup of $\mathrm{GL}(\mathbf{E})$, $K = G \cap \mathrm{O}(\mathbf{E})$ (or $G \cap \mathrm{U}(\mathbf{E})$), $P = G \cap P(\mathbf{E})$. If $G = KP$ then G is a self-adjoint linear group.
2) Let $G \subset \mathrm{GL}(V)$ be a reductive algebraic complex linear group, K its compact real form and S an algebraic real structure in G such that $S(K) = K$. In V, introduce a Hermitian K-invariant scalar product. Then the linear group $H = G^S$ is self-adjoint.
3) Let G be a connected reductive algebraic group over $\mathbb{C}$, H its algebraic real form. Then there exists a compact real form of G such that the corresponding real form of $\mathfrak{g}$ is compatible with $\mathfrak{h}$.
4) An irreducible reductive real algebraic group G is diffeomorphic to $L \times \mathbb{R}^s$, where L is a maximal compact subgroup of G.
5) A reductive real algebraic group consists of a finite number of connected components (in the usual topology).

6) Real algebraic linear groups $G \subset \mathrm{GL}_n(k)$, where $k = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ listed in Examples 1.2.1–1.2.5 are self-adjoint with respect to the standard scalar product in $\mathbb{R}^n$ (the standard Hermitian products in $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively). Find the corresponding polar decompositions $G = KP$ (i.e. determine K, the subalgebra $\mathfrak{k}$ and the subspace $\mathfrak{p}$ or $\mathfrak{g}$).
7) The groups $\mathrm{U}_{k,l}$, $\mathrm{SU}_{k,l}$, $\mathrm{GL}_m(\mathbb{H})$, $\mathrm{SL}_m(\mathbb{H})$, $\mathrm{U}_m^*(\mathbb{H})$, $\mathrm{Sp}_{k,l}$, are connected.
8) The fundamental groups of the classical groups (except those studied in 1.3°) are of the following form:

$$\begin{aligned}
&\pi_1(\mathrm{U}_n) \simeq \pi_1(\mathrm{Sp}_{2n}(\mathbb{R})) \simeq \pi_1(\mathrm{U}_m^*(\mathbb{H})) \cong \mathbb{Z};\\
&\pi_1(\mathrm{U}_{k,l}) \simeq \mathbb{Z} \oplus \mathbb{Z} \qquad (k, l > 0);\\
&\pi_1(\mathrm{SU}_{k,l}) \simeq \mathbb{Z} \qquad (k, l > 0);\\
&\pi_1(\mathrm{SL}_n(\mathbb{R})) \simeq \mathbb{Z}_2 \qquad (n \geqslant 3);
\end{aligned}$$

$\pi_1(\mathrm{O}_{k,l}^0)$ are contained in the table:

k, l	$k, l > 2$	$k = 1, l > 2$	$k = 2, l > 2$	$k = l = 2$	$k = 1, l = 2$
$\pi_1(\mathrm{O}_{k,l}^0)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$

9) Let $\mathbb{E}$ be a Euclidean (or Hermitian) space and let $g \in \mathrm{GL}(\mathbb{E})$ and $a \in \mathrm{O}(\mathbb{E})$ (resp. $\mathrm{U}(\mathbb{E})$) be such that $gag^{-1} \in \mathrm{O}(\mathbb{E})(\mathrm{U}(\mathbb{E}))$. Then in the polar decomposition $g = kp$, where $k \in \mathrm{O}(\mathbb{E})$ $(\mathrm{U}(\mathbb{E}))$, $p \in P(\mathbb{E})$, the factor p satisfies $ap = pa$.
10) Each element of a connected compact Lie group is contained in a maximal torus.
11) The center of a connected compact Lie group coincides with the intersection of all of its maximal tori.
12) Let A be a connected closed commutative subgroup of a connected compact Lie group K. Then the centralizer $Z(A)$ of A in K is connected.
13) Let K be a simply connected compact Lie group and $\Theta \in \operatorname{Aut} K$. Then K^Θ is connected.
14) Let K be a compact Lie group. The algebra of polynomial functions $\mathbb{R}[K]$ on K considered as a real algegraic group coincides with the algebra of real representative functions.
15) Let G be a reductive algebraic complex group. The algebra of polynomial functions $\mathbb{C}[G]$ coincides with the algebra of holomorphic representative functions A_G^h. If K is a compact real form of G then the restriction map determines an isomorphism $A_G^h \to A_K$.
16) A compact real algebraic group is irreducible if and only if it is connected (in the usual topology).
17) Let ρ be a linear representation of a semisimple complex Lie algebra. Let us represent its decomposition into irreducible components in the form

$$\rho = \rho_1 + \cdots + \rho_s + \rho_1^* + \cdots + \rho_s^* + \rho_{s+1} + \cdots + \rho_{s+1},$$

where $\rho_i \not\sim \rho_j^*$ for $i, j > s$ and $i \neq j$. The representation ρ is self-adjoint if and only if so are all ρ_i $(i > s)$. Moreover, ρ is orthogonal (symplectic) if and only if so are all ρ_i $(i > s)$.

A complex or real Lie algebra $\mathfrak{g}$ is called *reductive*, if $\mathfrak{rad}\,\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$.

18) A Lie algebra is reductive if and only if its adjoint representation is completely reducible.

19) If an arbitrary finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is completely reducible then $\mathfrak{g}$ is semisimple.

Hints to Problems

1. Apply Theorem 1 to $Z(G)$. It follows from Problem 1.31 that $Z(G) \cap P$ is a Lie subgroup of G isomorphic to $\mathbb{R}^s$, $s \geqslant 0$.
2. If $p \in P$ and $p \neq e$ then $\{p^s = \exp(s \log p)\colon s = 1, 2, \ldots\}$ is an infinite discrete sequence. Therefore p cannot belong to any compact subgroup of G.
3. First verify that $x^* \in \mathfrak{g}$ for any $x \in \mathfrak{g}$. Since $S\colon g \mapsto g^{*-1}$ is an automorphism of $\mathrm{GL}(\mathbb{E})$ (as a real Lie group) and $(dS)x = -x^*$, then $S(G^0) = G^0$. Since $G = KG^0$ and K consists of unitary operators, this implies the statement of the problem.
4. By Theorem 1 $G = K_1 P$ with $K \subset K_1$ and $K^0 = K_1^0$ since K and K_1 have the same tangent algebra. Since K is a real form of G, we have $G = KG^0 = K(K_1^0 P) = KP$ which easily implies that $K_1 = K$.
5. Let $\mathfrak{k}$ be a compact semisimple Lie algebra and let G be a simply connected semisimple algebraic group over $\mathbb{C}$ with the tangent algebra $\mathfrak{k}(\mathbb{C})$ existing thanks to Theorem 4.3.6. By Corollary 1 of Theorem 2 the compact real form K of G is a simply connected Lie group with the tangent algebra $\mathfrak{k}$.
6. Let $Z = Z(K)^0$ and let L be a simply connected Lie group with the tangent algebra $\mathfrak{k}'$. The group L is compact thanks to Problem 5. There exists a covering $\pi\colon \mathfrak{z} \times L \to K$ such that $\pi|\mathfrak{z} = \exp\colon \mathfrak{z} \to Z$. Clearly, $\Gamma = \operatorname{Ker} \exp \subset \operatorname{Ker} \pi$. Therefore there exists a covering $\pi'\colon Z \times L \to K$ such that $\pi'(\exp \times id) = \pi$. The kernel $\operatorname{Ker} \pi' \simeq \operatorname{Ker} \pi / \Gamma$ is finite since so is $Z(L)$.
7. Consider the covering $\pi'\colon \tilde{K} = Z \times L \to K$ from Problem 6. Problem 5 and Example 2 of 1.1° imply that $\tilde{K}$ is isomorphic to a compact form of a connected complex reductive algebraic group $\tilde{G}$. Let $N = \operatorname{Ker} \pi'$, then $N \subset Z(\tilde{G})$ by Problem 1.3 and K is isomorphic to a real form of the reductive group $\tilde{G}/N$.
8. By Problem 7 we may assume that K is a linear group. Then K' is a Lie subgroup since $\mathfrak{k}'$ is algebraic. The decomposition $K = ZK'$ follows from Problem 4.1.21.

10. Let $G_0 = A_0 \times K_0$ be a decomposition satisfying the conditions of Theorem 4. Prove that $A = \pi^{-1}(A_0)^0$, $K = \pi^{-1}(K_0)^0$ and $G = A \times K$.

12. Consider the representation of the compact group G/Z in $\mathfrak{z}$ induced by the adjoint representation and make use of Corollary of Theorem 3.4.2.

15. Since K_0 is a maximal compact subgroup of G^0, then K_0 is normal in G. The group $\hat{G} = G/K_0$ contains a normal Lie subgroup of finite index, $\hat{A}$, isomorphic to A. By Lemma 2 $\hat{G} = \hat{A} \rtimes L$, where L is a finite subgroup. Then the preimage K of L with respect to the natural homomorphism $G \to \hat{G}$ is the desired subgroup.
17. Consider the G-action on the set of compact real forms of $\mathfrak{h}$ determined by the adjoint representation. The subgroup $H \subset G$ acts on this set transitively (Theorem 1.3) and U is the stabilizer of $\mathfrak{l}$. This implies that $G = HU$.
18. The identity $G = KG^0$ follows from Problem 17.
19. Make use of Theorem 7.
20. In 3.1.6° we have actually proved that the matrix elements of any representation belong to A_G. Conversely, let $f \in A_G$, $f \neq 0$, and let V be the linear span of $\{r_*(g)f\colon g \in G\}$. In V, choose a basis $f_1 = f, f_2, \ldots, f_n$ and let a_{ij} be the matrix elements of the representation $r_*\colon g \mapsto r_*(g)$ of G in the space V with respect to this basis. Then
$$f(g) = \sum_{1 \leqslant k \leqslant n} a_{ki}(g^{-1}) f_k(e),$$
i.e. f is linearly expressed in terms of the functions $b_{ik}(g) = a_{ki}(g^{-1})$, the matrix elements of the representation $(r_*)^*$.
22. Let $K \subset \mathrm{GL}(V)$ be a compact real linear group. Theorem 3.4.2 implies that the scalar product (4.1.2) is negative definite on the tangent algebra $\mathfrak{k}$. Therefore a similar scalar product in $\mathfrak{sl}(V(\mathbb{C}))$ is non-degenerate on $\mathfrak{k}(\mathbb{C})$. The reductivity of $K(\mathbb{C})$ follows from Theorem 4.1.2.
23. Let $\mathfrak{p}_j = i\mathfrak{k}_j$, $P_j = \exp \mathfrak{p}_j$ $(j = 1, 2)$. Then $d\varphi(\mathbb{C})(\mathfrak{p}_1) \subset \mathfrak{p}_2$ and therefore $\varphi(\mathbb{C})(P_1) \subset P_2$. Let $N = \operatorname{Ker} \varphi(\mathbb{C})$. The uniqueness of the polar decomposition (32) implies that if $g = kp \in N$, where $k \in K_1$, $p \in P_1$, then $k, p \in N$. It is clear from Problem 1.31 that $p = e$ and $g = k \in \operatorname{Ker} \varphi$.
24. The algebraicity of L follows from Theorem 3.4.5. If $\varphi\colon L \to K$ is an embedding then $\varphi(\mathbb{C})$ is injective by Problem 23. The subgroup $\varphi(\mathbb{C})(L(\mathbb{C}))$ is the desired one.
25. Let G be completely reducible in V and let $W_1 \subset V(\mathbb{C})$ be an irreducible G-invariant subspace. Then $V_1 = (W_1 + \overline{W_1}) \cap V$ is a G-invariant subspace of V such that $V_1(\mathbb{C}) = W_1 + \overline{W_1}$ and either $W_1 \cap \overline{W_1} = 0$ or $W_1 = \overline{W_1}$. If V_2 is a G-invariant complement to V_1 in V then the G-invariant complement to W_1 in V is either $\overline{W_1} \oplus V_2(\mathbb{C})$ or $V_2(\mathbb{C})$, respectively. Conversely, let G be completely reducible in $V(\mathbb{C})$, let V_1 be an irreducible G-invariant subspace in V and W_2 the G-invariant complement to $V_1(\mathbb{C})$ in $V(\mathbb{C})$. Then $V = V_1 \oplus V_2$, where $V_2 = \{x + \bar{x}\colon x \in W_2\}$.
26. Let us embed G in $\mathrm{GL}(V^{\mathbb{R}}(\mathbb{C}))$ as in Problem 25 and let us extend the complex structure operator I from V onto $V^{\mathbb{R}}(\mathbb{C})$ (cf. 1.1°). Then $V^{\mathbb{R}}(\mathbb{C}) = V_i \oplus V_{-i}$, where $V_{\pm i}$ are eigenspaces of I corresponding to eigenvalues $\pm i$. The subspaces $V_{\pm i}$ are invariant with respect to G, the projections $V = V^{\mathbb{R}} \to V_i$ and $V = V^{\mathbb{R}} \to V_{-i}$ commute with the G-action and are an isomorphism and

an antilinear isomorphism of complex vector spaces respectively. This implies that G is completely reducible in V if it is completely reducible in $V^{\mathbb{R}}(\mathbb{C})$. Now apply Problem 25.

27. First prove that G and G^a have the same invariant subspaces.
28. The image G_1 of G under a linear representation is a semisimple linear group (see Problem 4.1.16) and $(G_1^a)^0 = G_1^0$. Therefore, the statement follows from Theorem 13 and Problem 27.
29. Make use of Problem 1.2.19.
30. Consider the representation of the semisimple Lie algebra $\mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$ in $\mathfrak{g}$ induced by the adjoint representation.
31. Let G be an algebraic subgroup of $\mathrm{GL}(V)$. Consider the weight decomposition $V = \bigoplus_{1 \leqslant i \leqslant p} V_{\lambda_i}$ of V with respect to T. Each $g \in G$ permutes the subspaces V_{λ_i}, thereby a homomorphism $G \to S_p$ is defined. Its kernel is a closed subgroup of a finite index in G and, therefore, coincides with G. Thus, all the V_{λ_i}'s are G-invariant, whence $T \subset Z(G)$.
32. Note that for any connected commutative subgroup $A \subset K$ the closure $\bar{A}$ is a compact connected commutative subgroup, hence a torus.
33. If A is a torus then the reductive group $A(\mathbb{C})$ is connected (e.g. by Corollary 1 of Theorem 2) and commutative, i.e. is an algebraic torus. Conversely, if $A(\mathbb{C})$ is an algebraic torus then the compact commutative group A is connected thanks to the same Corollary.
34. Pass to the maximal algebraic torus $A(\mathbb{C}) \subset K(\mathbb{C})$ and apply Theorem 4.2.5.
35. If A is a maximal connected commutative subgroup of K, then AC is also a connected commutative subgroup, hence $A = AC \supset C$. Therefore $A = (A \cap L) \times C$, where $A \cap L$ is a maximal connected commutative subgroup of L. The other statements of the theorem follow from Problems 32, 34 and Theorem 15.

§ 3. Cartan Decomposition

In this section we will study the so-called Cartan decomposition of a real semisimple Lie group. It is an analogue of the polar decomposition considered in 2.1° and for semisimple algebraic groups these decompositions coincide. The Cartan decomposition leads to an important theorem on conjugacy of maximal compact subgroups of any real semisimple Lie group with a finite number of connected components. It also enables us to give a global classification of connected semisimple Lie groups.

1°. Cartan Decomposition of a Semisimple Lie Algebra. Let $\mathfrak{g}$ be a real semisimple Lie algebra, $(\cdot, \cdot)$ the Cartan scalar product in $\mathfrak{g}$. A decomposition of $\mathfrak{g}$ into the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{1}$$

is a *Cartan decomposition* if

1) the map $\theta: x + y \mapsto x - y$ $(x \in \mathfrak{k}, y \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}$;
2) the bilinear form

$$b_\theta(z, y) = -(x, \theta y) \tag{2}$$

is positive definite on $\mathfrak{g}$.

Note that $\theta^2 = \mathrm{id}$, therefore b_θ is a symmetric bilinear form.

Problem 1. Condition 1) is equivalent to the following condition:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{3}$$

Problem 2. If 1) holds then $(x, y) = 0$ for $x \in \mathfrak{k}$, $y \in \mathfrak{p}$ and 2) is equivalent to the following condition:

$$(x, x) < 0 \quad \text{for} \quad x \in \mathfrak{k}, \quad x \neq 0; \quad (y, y) > 0 \quad \text{for} \quad y \in \mathfrak{p}, \quad y \neq 0. \tag{4}$$

Therefore the decomposition (1) is a Cartan one if and only if (3) and (4) hold.

Example. If $\mathfrak{u}$ is a compact real form of a semisimple complex Lie algebra $\mathfrak{g}$ then the decomposition

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus I\mathfrak{u} \tag{5}$$

ia a Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$. Here $\theta = \tau$ is the real structure corresponding to the real form $\mathfrak{u}$ and the scalar product b_θ coincides with h_τ (see Theorem 1.2).

We will now describe Cartan decompositions of an arbitrary real semisimple Lie algebra $\mathfrak{g}$. For this consider the complex semisimple Lie algebra $\mathfrak{g}(\mathbb{C})$. Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}(\mathbb{C})$ compatible with $\mathfrak{g}$. By Problem 1.28

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where} \quad \mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}, \quad \mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{u}). \tag{6}$$

Problem 3. The decomposition (6) is a Cartan one and $\theta = \sigma\tau$, where σ and τ are the real structures corresponding to the real forms $\mathfrak{g}$ and $\mathfrak{u}$. Conversely, any Cartan decomposition (1) is of the form (6) for a compact real form $\mathfrak{u} = \mathfrak{k} \oplus (i\mathfrak{p})$ compatible with $\mathfrak{g}$.

Therefore we have established a one-to-one correspondence between Cartan decompositions of $\mathfrak{g}$ and compact real forms of $\mathfrak{g}(\mathbb{C})$ compatible with $\mathfrak{g}$. Note that any automorphism of $\mathfrak{g}$ transforms a Cartan decomposition into a Cartan decomposition.

Problem 3 and Theorem 1.3 imply

Theorem 1. *Any real semisimple Lie algebra* $\mathfrak{g}$ *possesses a Cartan decomposition. Any two Cartan decompositions of* $\mathfrak{g}$ *are transformed into each other by an inner automorphism.*

Now we will establish certain proporties of Cartan decompositions. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. It is clear from (3) that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is an invariant subspace with respect to $\operatorname{ad}\mathfrak{k}$, where ad is the adjoint representation of $\mathfrak{g}$. The subspace $\mathfrak{p}$ is called the *Cartan subspace* of $\mathfrak{g}$.

Let us consider $\mathfrak{g}$ as a Euclidean space with the scalar product b_θ given by formula (2).

Problem 4. We have $\operatorname{ad}\theta(x) = -(\operatorname{ad} x)^*$ for any $x \in \mathfrak{g}$. In particular, the operator $\operatorname{ad} x$ is symmetric if and only if $x \in \mathfrak{p}$ and skew symmetric if and only if $x \in \mathfrak{k}$.

Problem 5. Let $\mathfrak{g} = \bigoplus_{1 \leqslant i \leqslant s} \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple ideals, and let, $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ $(i = 1, \dots, s)$ be their Cartan decompositions. Then $\mathfrak{k} = \bigoplus_{1 \leqslant i \leqslant s} \mathfrak{k}_i$ and $\mathfrak{p} = \bigoplus_{1 \leqslant i \leqslant s} \mathfrak{p}_i$ determine a Cartan decomposition of $\mathfrak{g}$ and any Cartan decomposition of this algebra can be obtained in this way.

Problem 6. A Lie algebra $\mathfrak{g}$ is compact if and only if $\mathfrak{k} = \mathfrak{g}$ and $\mathfrak{p} = 0$.

2°. Cartan Decomposition of a Semisimple Lie Group. Let G be a real semisimple Lie group (not necessarily connected) and let a Cartan decomposition (1) of its tangent algebra be given. In this section we will prove the existence of the corresponding global decomposition $G = KP$, where K is a Lie subgroup of G with the tangent algebra $\mathfrak{k}$ and $P = \exp\mathfrak{p}$. This decomposition described in Theorem 2 will be called a *Cartan decomposition* of G.

Denote by θ the involutive automorphism of $\mathfrak{g}$ corresponding to the decomposition (1) and consider $\mathfrak{g}$ as a Euclidean space with the scalar product b_θ defined by formula (2).

Problem 7. For any $a \in \operatorname{Aut}\mathfrak{g}$ we have $\theta a \theta^{-1} = (a^*)^{-1}$. In particular, $\operatorname{Aut}\mathfrak{g}$ is a self-adjoint linear group.

Theorem 2. *Let G be a real semisimple Lie group and let a Cartan decomposition (1) of its tangent algebra be given. Set $K = \{g \in G\colon \operatorname{Ad} g \in \mathcal{O}(\mathfrak{g})\}$, $P = \exp\mathfrak{p}$. Then $G = KP$ and every element $g \in G$ uniquely presents in the form $g = kp$, where $k \in K$, $p \in P$. The map $\varphi\colon K \times \mathfrak{p} \to G$ given by the formula*

$$\varphi(k, y) = k \exp y \qquad (k \in K, y \in \mathfrak{p})$$

is a diffeomorphism. The map $\Theta\colon kp \mapsto kp^{-1}$ is an automorphism of G.

Proof. It follows from Problem 9 and Theorem 2.1 that $\operatorname{Aut}\mathfrak{g}$ admits the polar decomposition $\operatorname{Aut}\mathfrak{g} = \hat{K}\hat{P}$, where $\hat{K} = (\operatorname{Aut}\mathfrak{g}) \cap \mathrm{O}(\mathfrak{g})$, $\hat{P} = (\operatorname{Aut}\mathfrak{g}) \cap P(\mathfrak{g})$. By Problem 1.4 the tangent algebra of $\operatorname{Aut}\mathfrak{g}$ is $\operatorname{ad}\mathfrak{g}$ and it is clear from Problem 4 that $(\operatorname{ad}\mathfrak{g}) \cap S(\mathfrak{g}) = \operatorname{ad}\mathfrak{p}$. Therefore $\hat{P} = \exp\operatorname{ad}\mathfrak{g}$ (see Theorem 2.1).

It follows from the commutative diagram

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "\exp"] \arrow[d, "\operatorname{ad}"'] & P \arrow[d, "\operatorname{Ad}"] \\
\operatorname{ad}\mathfrak{p} \arrow[r, "\exp"] & \hat{P}
\end{tikzcd} \tag{7}$$

that $\hat{P} = \operatorname{Ad} P$ and the maps $\exp\colon \mathfrak{p} \to P$ and $\operatorname{Ad}\colon P \to \hat{P}$ are one-to-one. If $g \in G$ then $\operatorname{Ad} g = \hat{k}\hat{p}$ where $\hat{k} \in \hat{K}$, $\hat{p} \in \hat{P}$. Since $\hat{p} = \operatorname{Ad} p$, where $p \in P$, then $\operatorname{Ad}(gp^{-1}) = \hat{k} \in \mathcal{O}(\mathfrak{g})$ implying $gp^{-1} = k \in K$ and $g = kp$. If there is another decomposition $g = k'p'$, where $k' \in K$, $p' \in P$, then $(\operatorname{Ad} k)(\operatorname{Ad} p) = (\operatorname{Ad} k')(\operatorname{Ad} p')$ which thanks to the uniqueness of the polar decomposition implies $\operatorname{Ad} p = \operatorname{Ad} p'$. Therefore $p = p'$ and hence $k = k'$. This also implies that φ is bijective.

Since the diagram

$$\begin{tikzcd}
K \times \mathfrak{p} \arrow[r, "\varphi"] \arrow[d, "\operatorname{Ad} \times \operatorname{ad}"'] & G \arrow[d, "\operatorname{Ad}"] \\
\hat{K} \times \hat{\mathfrak{p}} \arrow[r, "\hat{\varphi}"] & \operatorname{Aut} \mathfrak{g}
\end{tikzcd}$$

where $\hat{\varphi}$ determines the polar decomposition of $\operatorname{Aut} \mathfrak{g}$, commutes, $d_{(k,y)}\varphi$ is injective for any $k \in K$, $y \in \mathfrak{p}$. In fact, $\hat{\varphi}$ is a diffeomorphism by Theorem 2.1 and the differential of the left-hand column map is injective. Therefore, φ is a diffeomorphism.

Presenting $g \in G$ in the form $g = kp$, where $k \in K$, $p \in P$, we get

$$\operatorname{Ad} \Theta(g) = (\operatorname{Ad} k)(\operatorname{Ad} p)^{-1} = ((\operatorname{Ad} g)^*)^{-1}.$$

Therefore $(\operatorname{Ad})\Theta$ is a homomorphism and $\operatorname{Ad}(\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1}) = \mathrm{id}$ for any $g_1, g_2 \in G$, hence

$$\Psi(g_1, g_2) = \Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1} \in \operatorname{Ker} \operatorname{Ad}.$$

The subgroup $\operatorname{Ker}\operatorname{Ad}$ is discrete since $(\operatorname{Ker}\operatorname{Ad}) \cap G^0 = Z(G^0)$ (see Problem 1.2.17). Therefore $\Psi(g_1, g_2)$ depends only on the connected components of G to which the elements g_1, g_2 belong. Since $P \subset G^0$, an element of K is contained in each connected component of the group $G = KP$. But $\psi(g_1, g_2) = \mathrm{id}$ for g_1, $g_2 \in K$, hence $\psi(g_1, g_2) = \mathrm{id}$ for all $g_1, g_2 \in K$. □

Corollary 1. *G is diffeomorphic to $K \times \mathbb{R}^m$, where $m = \dim \mathfrak{p}$.*

Problem 8 (Corollary 2). K coincides with the subgroup $G^\Theta = \{g \in G\colon \Theta(g) = g\}$; its tangent algebra is $\mathfrak{k}$.

Problem 9 (Corollary 3). K coincides with $N(K^0)$.

Problem 10 (Corollary 4). The Cartan decomposition of G^0 corresponding to decomposition (1) is of the form $G^0 = K^0P$, where $K^0 = K \cap G^0$ and $K/K^0 \simeq G/G^0$.

The definition of K and Corollary 4 imply

Corollary 5. $Z(G) \subset Z(K)$, $Z(G^0) \subset Z(K^0)$.

Problem 11 (Corollary 6). K is compact if and only if G has a finite number of connected components and $Z(G^0)$ is finite.

Proof of Theorem 2 (see (7)) also implies

Corollary 7. *The map* $\mathrm{Ad}\colon P \to \hat{P} = \mathrm{Aut}\,\mathfrak{g} \cap P(\mathfrak{g})$ *is a diffeomorphism.*

Remarks. 1) Let $G \subset GL(V)$ be a complex semisimple algebraic linear group, K its compact real form. Then the Cartan decomposition of G corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus (i\mathfrak{k})$ of its tangent algebra (see Example of 1°) coincides with the polar decomposition described in Theorem 2.2. In fact, these decompositions are defined by the same set $P = \exp(i\mathfrak{k})$, and Corollary 3 of Theorem 2 implies that K coincides with the subgroup from the Cartan decomposition.

2) Let $G \subset \mathrm{GL}(V)$, where V is a vector space over $\mathbb{R}$, be a real semisimple linear Lie group. Then $Z(G^0)$ is finite since it is contained in the center of the connected semisimple complex algebraic group $(G^0)^a \subset \mathrm{GL}(V(\mathbb{C}))$. Therefore if G has a finite number of connected components then the subgroup K of Theorem 2 is compact (Corollary 6).

3) If the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is semisimple and G has a finite number of connected components then K is compact by Problem 2.5 and Corollary 4 of Theorem 2. If $\mathfrak{k}$ is not semisimple then by Corollary 1 of Theorem 2 applied to a simply connected group G the subgroup K is also simply connected, hence is not compact. The simplest example of such a group is $G = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ (see Example 5 of 1.1°). Here $\mathfrak{k} = \mathfrak{sl}_2$, $K \simeq \mathbb{R}$, therefore by Corollary 1 G is diffeomorphic to $\mathbb{R}^3$.

4) Let $G = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm E\}$ and $\pi\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ the natural homomorphism. If $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2 \cdot P$ is a Cartan decomposition, then $\mathrm{PSL}_2(\mathbb{R}) = \pi(\mathrm{SO}_2)\pi(P)$. This is a Cartan decomposition of $\mathrm{PSL}_2(\mathbb{R})$. Since $\pi(\mathrm{SO}_2) = \mathrm{SO}_2/\{\pm E\} \simeq \mathrm{SO}_2$, then $\pi_1(\mathrm{PSL}_2(\mathbb{R})) \simeq \mathbb{Z}$, implying $Z(\widetilde{\mathrm{SL}}_2(\mathbb{R})) \simeq \mathbb{Z}$.

Suppose $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{R}$ admitting no complex structure, i.e. a real form of a complex simple Lie algebra. Then the automorphism θ extended by linearity onto $\mathfrak{g}(\mathbb{C})$ is the involutive automorphism of $\mathfrak{g}(\mathbb{C})$ that corresponds to the real form $\mathfrak{g}$ by Theorem 1.4 (see Problem 1.37) and $\mathfrak{k}(\mathbb{C})$ coincides with $\mathfrak{g}(\mathbb{C})^\theta$. According to the classification of Problem 1.38 the case of a semisimple subalgebra $\mathfrak{k}$ corresponds to types I, II, and that of a non-semisimple subalgebra to the type III; in the latter case $\mathfrak{k}$ has a one-dimensional center.

3°. Conjugacy of Maximal Compact Subgroups. In this subsection we will describe maximal compact subgroups of semisimple Lie groups with a finite number of connected components. In particular, we will prove that all maximal compact subgroups are conjugate. First, we consider the general case and formulate a conjugacy theorem for subgroups more general than compact ones.

A subgroup M of a semisimple Lie group G is called *pseudocompact* if the linear group $\mathrm{Ad}\,M \subset \mathrm{GL}(\mathfrak{g})$ is compact. Any compact group is pseudocompact.

Problem 12. The subgroup K considered in Theorem 2 is a maximal pseudocompact subgroup of G.

Theorem 3. *Let $G = KP$ be a Cartan decomposition of a semisimple Lie group G. For any pseudocompact subgroup $M \subset G$ there exists $g \in P$ such that $gMg^{-1} \subset K$.*

Before we prove this theorem let us deduce from it several corollaries. If G has a finite number of connected components then so has K by Corollary 4 of Theorem 2. Since $\mathfrak{k}$ is compact, Theorems 2.5 and 2.6 imply that $K = A \rtimes L$, where $A \simeq \mathbb{R}^S$ and L is a maximal compact subgroup of K.

Problem 13 (Corollary 1). If G has a finite number of connected components then any maximal compact subgroup L of K is a maximal compact subgroup of G. Any maximal compact subgroup of G is conjugate to L by an automorphism of the form $a(g)$, where $g \in G^0$.

Corollary 2. *A semisimple Lie group G with a finite number of connected components is diffeomorphic to $L \times \mathbb{R}^N$, where L is any maximal compact subgroup of G.*

Problem 14 (Corollary 3). Let $\mathfrak{g}$ be a real semisimple Lie algebra and let M be a compact subgroup of $\operatorname{Aut}\mathfrak{g}$. Then $\mathfrak{g}$ admits a Cartan decomposition invariant with respect to M.

The classical proof of Theorem 3 due to E. Cartan (see [6]), as well as its simplified versions (see, e.g., [31]), are based on the study of geometry of the symmetric space G/K. The proof that follows, exploiting an idea presented in [31], does not use Riemannian geometry at all.

Observe that $\mathrm{GL}(\mathbb{E})$ acts on the manifold $P(\mathbb{E})$ of positive definite self-adjoint operators in a Euclidean space $\mathbb{E}$ by the formula

$$\mathrm{Sq}(A)(X) = AXA^* \qquad (X \in P(\mathbb{E}),\ A \in \mathrm{GL}(\mathbb{E})).$$

As it is known from linear algebra, this action is transitive, and the stabilizer of the identity operator $E \in P(\mathbb{E})$ is the orthogonal group $\mathrm{O}(\mathbb{E})$. Consider the differentable function r of two variables on $P(\mathbb{E})$ given by the formula

$$r(X, Y) = \operatorname{tr}(XY^{-1}). \tag{8}$$

Problem 15. $r(\mathrm{Sq}(A)(X), \mathrm{Sq}(A)(Y)) = r(X, Y)$ for any $A \in \mathrm{GL}(\mathbb{E})$.

Let Ω be a compact set in $P(E)$. Let

$$\rho(X) = \max_{Y \in \Omega} r(X, Y), \tag{9}$$

Problem 16. The function ρ is continuous on $P(\mathbb{E})$.

Set $SP(\mathbb{E}) = P(\mathbb{E}) \cap \mathrm{SL}(\mathbb{E})$. Clearly, $SP(\mathbb{E})$ is closed in $\mathrm{SL}(\mathbb{E})$ and therefore is closed in the space $\mathfrak{gl}(\mathbb{E})$.

Lemma 1. *For any compact set $\Omega \subset P(\mathbb{E})$ the function ρ defined by formula (9) assumes its minimum on any closed subset $F \subset SP(\mathbb{E})$.*

Proof. First, prove that

$$\rho(X) \geqslant b\|X\| \qquad (X \in P(\mathbf{E})), \tag{10}$$

where $b > 0$ is a constant and $\|X\|$ is the norm of an operator X in $\mathbf{E}$. Fix $X \in P(\mathbf{E})$ and choose an orthonormal basis of $\mathbf{E}$ in which X is expressed by a diagonal matrix $\operatorname{diag}(x_1, \dots, x_n)$. If (y_{ij}) is the matrix of $Y \in P(\mathbf{E})$ then $y_{ii} > 0$ and

$$r(X, Y) = \sum_{1 \leqslant i \leqslant n} x_i/y_{ii}. \tag{11}$$

Since Ω and the orthogonal group are compact, there exists $b > 0$ such that $1/y_{ii} \geqslant b$ $(i = 1, \dots, n)$ for all $Y \in \Omega$ and all orthonormal bases of $\mathbf{E}$. Then

$$r(X, Y) \geqslant (\operatorname{tr} X)/b \geqslant \left(\max_{1 \leqslant i \leqslant n} x_i\right)/b = \|X\|/b \qquad \text{for any } Y \in \Omega$$

implying (10).

It follows from (10) that for any $N > 0$ the set $\{X \in SP(E)\colon \rho(X) \leqslant N\}$ is compact. In fact, $\rho(X) \leqslant N$ implies $\|X\| \leqslant N/b$ and the intersection of the compact ball $\{X \in S(E)\colon \|X\| \leqslant N/b\}$ with the closed set $SP(\mathbf{E})$ is compact.

Now it is easy to prove the existence of a minimum point. Let $X_0 \in F$. Consider the set $B = \{X \in F\colon \rho(X) \leqslant \rho(X_0)\}$ containing X_0 and compact by the above considerations. Problem 16 implies the existence of $X_i \in B$ such that $\rho(X_1) \leqslant \rho(X)$ for all $X \in B$. The point X_1 is a minimum point of ρ on the whole F since $\rho(X) > \rho(X_0) \geqslant \rho(X_1)$ for $X \in F \backslash B$. □

Now we want to show that under appropriate conditions the minimum point of ρ is unique. We want to prove that the functions r and ρ possess some convexity property.

Problem 17. For any fixed X, $Y \in P(\mathbf{E})$, $X \neq E$ the functions

$$f_{X,Y}(t) = r(X^t, Y),\ \varphi_X(t) = \rho(X^t)$$

are strictly convex on the whole real axis.

Return to the situation of Theorem 3. Consider the tangent algebra $\mathfrak{g}$ of G as a Euclidean space with the scalar product (2) corresponding to our Cartan decomposition. Set $\hat{P} = \exp \operatorname{ad} \mathfrak{p}$.

Problem 18. P is a closed submanifold of $SP(\mathfrak{g})$, coinciding with the orbit of the point E under the action $(\mathrm{Sq})(\mathrm{Ad})$ of G on $P(\mathfrak{g})$. The subgroup $K \subset G$ is the stabilizer of E with respect to this action.

Lemma 2. *For any compact set Ω in $P(\mathfrak{g})$ the function ρ defined by* (9) *has a unique minimum point in $\hat{P}$.*

Proof. Let A, $B \in \hat{P}$ be two different minimum points of ρ. Apply to A, B and Ω the map $\mathrm{Sq}(B^{-1/2})$ which transforms $\hat{P}$ into itself, B into E and Ω into a new compact set. Making use of Problem 15, we shall reduce our problem to the case $B = E$. Clearly, $A^t \in \hat{P}$ for all $t \in \mathbb{R}$. By Problem 17 the function $\varphi_A(t) = \rho(A^t)$ is strictly convex on the segment $[0, 1]$. Therefore, it can not assume its minimum on both ends of this segment. □

Proof of Theorem 3. Let M be a pseudocompact subgroup of G. Consider the action of the subgroup $B \subset G$ on $P(\mathfrak{g})$ defined in Problem 18. Since $\operatorname{Ad} M$ is compact, the orbit $\Omega = \operatorname{Sq}(\operatorname{Ad} M)(E)$ is also compact. By Problem 15 the function ρ on $P(\mathfrak{g})$ given by (9) is invariant with respect to M. Thus, its unique minimum point $A_0 \in \hat{P}$ (see Lemma 2) is fixed under M. Since G acts transitively on $\hat{P}$, it follows that $gMg^{-1} \subset K$ for some $g \in G$. It is easy to see that we may set $g = p^{-1/2}$, where $p \in P = \exp \mathfrak{p}$ is such that $A_0 = \operatorname{Ad} p$.

4°. Canonically Embedded Subalgebras. Given a Cartan decomposition (1) of a real semisimple Lie algebra $\mathfrak{g}$ we call a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ *canonically embedded in* $\mathfrak{g}$ *with respect to the decomposition* (1) if $\theta(\mathfrak{h}) = \mathfrak{h}$, where θ is the automorphism corresponding to the Cartan decomposition, or, equivalently, if

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}). \tag{12}$$

As it is known, any semisimple Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) can be identified with the linear Lie algebra $\operatorname{ad} \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ over the same field. Therefore we may introduce the notion of an algebraic subalgebra of a semisimple Lie algebra. A subalgebra $\mathfrak{h}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is called a (*reductive*) *algebraic subalgebra* if $\operatorname{ad} \mathfrak{h}$ is a (reductive) algebraic linear Lie algebra in the sense of 4.1.1°. A subalgebra $\mathfrak{h}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called *reductive algebraic* if $\mathfrak{h}(\mathbb{C})$ is a reductive algebraic subalgebra of a complex Lie algebra $\mathfrak{g}(\mathbb{C})$. For instance, any semisimple subalgebra of a semisimple Lie algebra (over $\mathbb{C}$ or $\mathbb{R}$) is reductive algebraic.

Problem 19. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Any canonically embedded algebraic subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is reductive algebraic. If $\mathfrak{h}$ is semisimple then the decomposition (12) is its Cartan decomposition.

Our aim is to prove the following statement inverse to the first statement of Problem 19.

Theorem 4. *Any reductive algebraic subalgebra of a real semisimple Lie algebra* $\mathfrak{g}$ *is canonically embedded in* $\mathfrak{g}$ *with respect to a Cartan decomposition.*

Proof is based on the following refinement of one of the statements of Theorem 1.3.

Lemma 3. *Let* $\mathfrak{h}$ *be a reductive algebraic subalgebra of a complex semisimple Lie algebra* $\mathfrak{g}$ *and let* σ *be a real structure on* $\mathfrak{g}$ *such that* $\sigma(\mathfrak{h}) = \mathfrak{h}$. *Then on* $\mathfrak{g}$, *there exists a real structure* τ *such that* $\mathfrak{g}^\tau$ *is compact,* $\sigma\tau = \tau\sigma$ *and* $\tau(\mathfrak{h}) = \mathfrak{h}$.

Proof. Represent $\mathfrak{h}$ in the form $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}'$, where $\mathfrak{z}$ is the center of $\mathfrak{h}$. Clearly, $\sigma(\mathfrak{h}') = \mathfrak{h}'$, $\sigma(\mathfrak{z}) = \mathfrak{z}$. By Theorem 1.3 there exists a real structure τ_1 on the semisimple Lie algebra $\mathfrak{h}'$ such that $(\mathfrak{h}')^{\tau_1}$ is compact and $\tau_1\sigma = \sigma\tau_1$ on $\mathfrak{h}'$. The corresponding compact real form L of the group $\operatorname{Int}\mathfrak{h}' \subset \operatorname{Int}\mathfrak{g}$ satisfies $\sigma L\sigma = L$. The algebraic torus $Z = \exp\operatorname{ad}\mathfrak{z} \subset \operatorname{Int}\mathfrak{g}$ determines a real form $\mathfrak{z}(\mathbb{R})$ of $\mathfrak{z}$. The subgroup $B = \exp\operatorname{ad}(i\mathfrak{z}(\mathbb{R}))$ is the compact part of Z so that $\sigma B\sigma = B$. Then $M = BL$ is a compact Lie subgroup of $\operatorname{Int}\mathfrak{g}$, its tangent algebra $\mathfrak{m} = i\mathfrak{z}(\mathbb{R}) \oplus (\mathfrak{h}')^{\tau_1}$ is a real form of $\mathfrak{h}$ and $\sigma M\sigma = M$. Now consider $\mathfrak{g}$ as a real semisimple Lie algebra $\mathfrak{g}^{\mathbb{R}}$ and denote by M_1 the subgroup of $\operatorname{Aut}\mathfrak{g}^{\mathbb{R}}$ generated by M and $\langle\sigma\rangle$. Clearly, $M_1 = \langle\sigma\rangle M$, so M_1 is compact. By Corollary 3 of Theorem 3 there is an M_1-invariant Cartan decomposition of $\mathfrak{g}^{\mathbb{R}}$. This means (see Example of 1°) that there exists a compact M_1-invariant real form of $\mathfrak{g}$. The corresponding real structure τ satisfies, as is easy to verify, the requirements of Lemma. □

Problem 20. Prove Theorem 4.

5°. Classification of Connected Semisimple Lie Groups. This section is devoted to the global classification of connected real semisimple Lie groups. It turns out that as in the complex case this classification can be given in terms of the tangent algebras and lattices in some commutative subalgebras of these algebras. By a "torus" we always mean a compact torus.

Let G be a connected semisimple Lie group. A connected subgroup $A \subset G$ will be called a *pseudotorus* if $\operatorname{Ad} A$ is a torus. Fix a Cartan decomposition $G = KP$.

Problem 21. The maximal connected commutative subgroups of K are the maximal pseudotori of G belonging to K. All maximal pseudotori of G are conjugate.

A commutative subalgebra $\mathfrak{a}$ of a semisimple Lie algebra $\mathfrak{g}$ will be called *pseudotoral* if $\exp\operatorname{ad}\mathfrak{a} \subset \operatorname{Int}\mathfrak{g}$ is compact, i.e. is a torus.

Problem 22. Let $\mathfrak{g}$ be the tangent algebra of a semisimple Lie group G. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is (maximal) pseudotoral if and only if it is the tangent algebra of a (maximal) pseudotorus in G. Any maximal commutative subalgebra of $\mathfrak{k}$ is pseudotoral. All maximal pseudotoral subalgebras of a semisimple Lie algebra $\mathfrak{g}$ are conjugate.

Let A be a maximal pseudotorus of a connected semisimple Lie group G and let $\mathfrak{a}$ be the corresponding maximal pseudotoral subalgebra of $\mathfrak{g}$. The kernel of the homomorphism $\exp = \exp_G\colon \mathfrak{a} \to A$ is a lattice in $\mathfrak{a}$ which, as we will see, determines together with the Lie algebra $\mathfrak{g}$, the group G uniquely up to an isomorphism. But it is more convenient to consider the lattice $L(G) = \operatorname{Ker}\mathscr{E} \subset \mathfrak{a}(\mathbb{C})$, where $\mathscr{E} = \mathscr{E}_G\colon i\mathfrak{a} \to G$ is the homomorphism defined by $\mathscr{E}(x) = \exp 2\pi i x$. The lattice $L(G)$ is called the *characteristic lattice* of G.

Problem 23. Let G_1, G_2 be two connected semisimple Lie groups with the same tangent algebra $\mathfrak{g}$, $\mathfrak{a} \subset \mathfrak{g}$ a maximal pseudotoral subalgebra. The characteristic lattices of G_1 and G_2 satisfy $L(G_1) \subset L(G_2)$ if and only if there exists a homomorphism $\pi\colon G_1 \to G_2$ such that $d\pi = \mathrm{id}$. In this case $\mathscr{E}_{G_1}^{-1}(\operatorname{Ker}\pi) = L(G_2)$, whence $\operatorname{Ker}\pi \simeq L(G_2)/L(G_1)$.

Theorem 5. *Let* G_j $(j = 1, 2)$ *be two connected semisimple Lie groups,* $\mathfrak{a}_j \subset \mathfrak{g}_j$ *maximal pseudotoral subalgebras of their tangent algebras,* $L(G_j) \subset i\mathfrak{a}_j$ *their characteristic lattices.* G_1 *and* G_2 *are isomorphic if and only if there exists an isomorphism* $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ *such that* $\varphi(\mathfrak{a}_1) = \mathfrak{a}_2$ *and* $\varphi(\mathbb{C})(L(G_1)) = L(G_2)$.

Problem 24. Prove this theorem.

To complete the classification we need to find out which lattices in $i\mathfrak{a}$ might be characteristic ones.

Let G be again a connected semisimple Lie group and $\mathfrak{a}$ a maximal pseudotoral subalgebra of $\mathfrak{g}$. The lattice $L_0 = L(\tilde{G}) \subset i\mathfrak{a}$ corresponds to the simply connected covering $\tilde{G}$ of G. On the other hand, the lattice $L_1 = L(\operatorname{Int}\mathfrak{g}) \subset i \operatorname{ad}\mathfrak{a} \subset i\operatorname{ad}\mathfrak{g}$ corresponds to $\operatorname{Int}\mathfrak{g}$. Identifying $\mathfrak{g}$ and $\operatorname{ad}\mathfrak{g}$ with the help of the isomorphism ad we get $L_1 \subset i\mathfrak{a}$. Problem 24 implies that $L_0 \subset L(G) \subset L_1$.

Problem 25. $\mathscr{E}^{-1}(Z(G)) = L_1$, $Z(G) = \mathscr{E}(L_1) \simeq L_1/L(G)$, $\pi_1(G) \simeq L(G)/L_0$.

Problem 26. Any lattice L such that $L_0 \subset L \subset L_1$ is the characteristic lattice of a connected Lie group with the tangent algebra $\mathfrak{g}$.

Now describe the lattices L_0 and L_1. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Denote by θ the involutive automorphism of $\mathfrak{g}$ associated to this decomposition and the extension of this automorphism onto the complex semisimple Lie algebra. Then $\mathfrak{k}(\mathbb{C}) = \mathfrak{g}(\mathbb{C})^\theta$. By Problem 21 we may assume that $\mathfrak{a}$ is a maximal commutative subalgebra of $\mathfrak{k}$. We have $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}(\mathfrak{k})$. By Theorem 2.15 $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{a}_0$ is a maximal commutative subalgebra of $\mathfrak{k}'$.

Problem 27. The subalgebras $\mathfrak{t} = \mathfrak{a}(\mathbb{C})$ and $\mathfrak{t}_0 = \mathfrak{a}_0(\mathbb{C})$ are maximal diagonalizable subalgebras of the reductive algebraic subalgebra $\mathfrak{k}(\mathbb{C}) \subset \mathfrak{g}(\mathbb{C})$ and the semisimple Lie algebra $\mathfrak{k}'(\mathbb{C}) = \mathfrak{k}(\mathbb{C})'$ respectively.

Problem 28. The lattice L_0 coincides with $Q^\vee(\mathfrak{k}'(\mathbb{C})) \subset \mathfrak{a}_0$, where $Q^\vee(\mathfrak{k}'(\mathbb{C}))$ is the dual root lattice of $\mathfrak{k}'(\mathbb{C}) = \mathfrak{k}(\mathbb{C})'$ with respect to $\mathfrak{t}_0$.

By Problem 4.4.11 the centralizer $\mathfrak{h}$ of $\mathfrak{t}$ in $\mathfrak{g}(\mathbb{C})$ is the only maximal diagonalizable subalgebra of $\mathfrak{g}(\mathbb{C})$ containing $\mathfrak{t}$ and $\theta(\mathfrak{h}) = \mathfrak{h}$.

Problem 29. We have $L_1 = P^\vee \cap \mathfrak{t}$, where $P^\vee$ is the weight lattice of the dual root system $\Delta^\vee_{\mathfrak{g}(\mathbb{C})}$ of $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{h}$.

For a lattice L_1 we may find another expression with the help of θ. By Problem 4.4.12 there is a base Π of $\Delta_{\mathfrak{g}(\mathbb{C})}$ invariant with respect to ${}^t\theta$. Let $\tau = {}^t\theta^{-1} \in \operatorname{Aut}\Pi$ and let $\hat{\tau}$ be the automorphism of $\mathfrak{g}(\mathbb{C})$ defined by (4.4.1). By Problems 4.4.17 and 4.4.29, $\mathfrak{t}$ is a maximal diagonalizable subalgebra of the semisimple Lie algebra $\mathfrak{g}(\mathbb{C})^{\hat{\tau}}$.

Problem 30. The lattice L_1 coincides with $P^\vee(\mathfrak{g}(\mathbb{C})^{\hat{\tau}})$, the weight lattice of the dual root system $\Delta^\vee_{\mathfrak{g}(\mathbb{C})^{\hat{\tau}}}$ of the Lie algebra $\mathfrak{g}(\mathbb{C})^{\hat{\tau}}$ with respect to $\mathfrak{t}$.

Problems 25, 26, 28–30 imply the following statements:

Theorem 6. *Let* $\mathfrak{a}$ *be a maximal commutative subalgebra of* $\mathfrak{k}$. *The lattice* $L \subset i\mathfrak{a}$ *is characteristic for a connected Lie group with the tangent algebra* $\mathfrak{g}$ *if and only if*

$$Q^\vee(\mathfrak{g}(\mathbb{C})^{\theta'}) \subset L \subset P^\vee(\mathfrak{g}(\mathbb{C})^{\hat{\tau}}),$$

where $\tau = \eta(\theta)$ *and* η: $\operatorname{Aut}\mathfrak{g}(\mathbb{C}) \to \operatorname{Aut}\Pi$ *is the homomorphism defined in* 4.4.1°.

Theorem 7. *For any connected Lie group* G *with the tangent algebra* $\mathfrak{g}$ *we have* $\mathscr{E}^{-1}(Z(G)) = P^\vee(\mathfrak{g}(\mathbb{C})^{\hat{\tau}})$, *implying*

$$Z(G) \simeq P^\vee(\mathfrak{g}(\mathbb{C})^{\hat{\tau}})/L(G).$$

We have also

$$\pi_1(G) \simeq L(G)/Q^\vee((\mathfrak{g}(\mathbb{C}))^\theta).$$

In particular, for a simply connected group $\tilde{G}$ *we have*

$$Z(\tilde{G}) \simeq P^\vee(\mathfrak{g}(\mathbb{C})^{\hat{\tau}})/Q^\vee((\mathfrak{g}(\mathbb{C})^\theta)')$$

and

$$\pi_1(G) \simeq L(G)/Q^\vee((\mathfrak{g}(\mathbb{C})^G)').$$

6°. Linearizer. Let G be a Lie group. Denote by $\Lambda(G)$ the intersection of the kernels of all linear representations of G. As follows from Theorem 1.4.2 $\Lambda(G)$ is a normal Lie subgroup of G. Call it the *linearizer* of G and set $G_{\text{lin}} = G/\Lambda(G)$.

Problem 31. Let R: $G \to \mathrm{GL}(V)$ be a linear representation. Then there exists a unique linear representation R_0: $G_{\text{lin}} \to \mathrm{GL}(V)$ such that $R = R_0\pi$, where π: $G \to G_{\text{lin}}$ is the natural homomorphism.

Our aim is to prove the following theorem which justifies the term "linearizer" in case when G is connected and semisimple.

Theorem 8. *Let* G *be a connected semisimple Lie group. The linearizer* $\Lambda(G)$ *is discrete, belongs to* $Z(G)$ *and* G_{lin} *admits a faithful linear representation.*

Proof. It suffices to prove the existence of a locally faithful linear representation R_0 of G such that $\Lambda(G) = \operatorname{Ker} R_0$. Let π: $\tilde{G} \to G$ be a simply connected covering, $\Gamma = \operatorname{Ker}\pi$, and let H be a simply connected complex Lie group with tangent algebra $\mathfrak{g}(\mathbb{C})$. By Theorem 1.2.6 there exists a homomorphism j: $\tilde{G} \to H$ such that dj is the identity embedding $\mathfrak{g} \to \mathfrak{g}(\mathbb{C})$. Then $j(\tilde{G})$ is a real form of H with the tangent algebra $\mathfrak{g}$. Problem 1.3 implies that $j(\Gamma) \subset Z(H)$. Clearly, there exists a homomorphism Φ: $G \to H/j(\Gamma)$ such that the diagram

$$\begin{tikzcd} \tilde{G} \arrow[r, "j"] \arrow[d, "\pi"'] & H \arrow[d, "\tilde{\pi}"] \\ G \arrow[r, "\Phi"] & H/j(\Gamma) \end{tikzcd} \tag{13}$$

where $\tilde{\pi}$ is the natural homomorphism, commutes. By Theorem 4.3.6 $H/j(\Gamma)$ admits a faithful linear representation. Therefore there exists a representation R_0 of G such that $\operatorname{Ker} R_0 = \operatorname{Ker} \Phi$. Let us prove that this representation is the desired one, i.e. the kernel of any linear representation of G contains $\operatorname{Ker} \Phi$.

Let $R\colon G \to \mathrm{GL}(W)$ be an arbitrary linear representation of G. The tangent representation $dR\colon \mathfrak{g} \to \mathfrak{gl}(W)$ extends to a complex representation $(dR)(\mathbb{C})\colon \mathfrak{g}(\mathbb{C}) \to \mathfrak{gl}(W(\mathbb{C}))$. By Theorem 1.2.6 there exists a representation $\tilde{R}\colon H \to \mathrm{GL}(W(\mathbb{C}))$ such that $d\tilde{R} = (dR)(\mathbb{C})$. Since $\tilde{G}$ is connected, Theorem 1.2.4 implies that $R\pi = \tilde{R}j$. Hence, $R(j(\Gamma)) = \{e\}$ so that $\tilde{R} = \tilde{\tilde{R}}\tilde{\pi}$, where $\tilde{\tilde{R}}$ is a representation of $H/j(\Gamma)$. Therefore $R\pi = \tilde{\tilde{R}}\tilde{\pi}j = \tilde{\tilde{R}}\Phi\pi$ and $R = \tilde{\tilde{R}}\Phi$. It follows that $\operatorname{Ker} \Phi \subset \operatorname{Ker} R$. □

Notice that the proof of Theorem 7 gives a method of finding linearizer $\Lambda(G)$: it coincides with $\operatorname{Ker} \Phi$ from (13). Therefore, $G_{\mathrm{lin}} \simeq \Phi(G)$.

Example. Let $G = \mathrm{SL}_2(\mathbb{R})$ (see Example 5 of 1.1°). Then $H = \mathrm{SL}_2(\mathbb{C})$ and $\Lambda(G) = \operatorname{Ker} j$. Clearly, j is the covering $G \to \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$. Since $Z(\mathrm{SL}_2(\mathbb{R})) \simeq \mathbb{Z}_2$ and $Z(G) \simeq \mathbb{Z}$ (see Remark 4 of 2°), we have $\Lambda(G) = 2Z(G) \simeq \mathbb{Z}$. Furthermore, $G_{\mathrm{lin}} \simeq \mathrm{SL}_2(\mathbb{R})$.

Now, we will express the linearizer $\Lambda(G)$ in terms of the characteristic lattice of G. Suppose, as in 5°, that we are given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathfrak{k}$, $\mathfrak{t} = \mathfrak{a}(\mathbb{C}) \subset \mathfrak{k}(\mathbb{C})$, $\mathfrak{h}$ a maximal diagonalizable subalgebra of $\mathfrak{g}(\mathbb{C})$ containing $\mathfrak{t}$.

Theorem 9. *For any connected Lie group G with tangent algebra $\mathfrak{g}$ we have*

$$\mathscr{E}^{-1}(\Lambda(G)) = L(G) + (Q^\vee \cap \mathfrak{t})$$

where $Q^\vee$ is the dual root latice of the Lie algebra $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{h}$. Therefore

$$\Lambda(G) \simeq (Q^\vee \cap \mathfrak{t})/(Q^\vee \cap L(G)).$$

In particular, for a simply connected group $G = \tilde{G}$ we have

$$\mathscr{E}^{-1}(\Lambda(\tilde{G})) = Q^\vee \cap \mathfrak{t}, \qquad \Lambda(\tilde{G}) \simeq (G^\vee \cap \mathfrak{t})/Q^\vee(\mathfrak{k}(\mathbb{C})').$$

Problem 32. Prove this theorem.

Exercises

In exercises 1–4 some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a real semisimple Lie algebra $\mathfrak{g}$ is fixed.

1) If $\mathfrak{g}$ is simple then the adjoint linear representation of $\mathfrak{k}$ in $\mathfrak{p}$ is irreducible and $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$.
2) If $\mathfrak{g}$ contains no non-zero compact ideals, then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and the adjoint representation of $\mathfrak{k}$ in $\mathfrak{p}$ is faithful.

3) In $\mathfrak{p}$, no one-dimensional $\operatorname{ad}\mathfrak{k}$-invariant subspaces exist. In particular, $\dim \mathfrak{p} \geqslant 2$ if $\mathfrak{g}$ is non-compact.
4) $\mathfrak{k}$ coincides with its normalizer in $\mathfrak{g}$.

In exercises 5–7 a Cartan decomposition $G = KP$ of a semisimple Lie group G is fixed.

5) The formula $T_g(x) = gx\Theta(g)^{-1}$ $(g, x \in G)$ defines a G-action on G. The orbit of e under this action is P and the stabilizer of e is K. Therefore P is a homogeneous space of G isomorphic to G/K.
6) P is the connected component of unit in each of the sets $\{g \in G\colon \Theta(g) = g^{-1}\}$, $\{g \in G\colon \operatorname{Ad} g \in P(\mathfrak{g})\}$.
7) If $g \in G$, $a \in K$ are such that $gag^{-1} \in K$ then in the decomposition $g = kp$, where $k \in K$, $p \in P$, the factor p satisfies $pa = ap$.
8) The polar decomposition $G = KP$ of a real semisimple algebraic linear group (see Exercise 2.2) is a Cartan one. If H is an open subgroup of G then its Cartan decomposition is of the form $H = (K \cap H)(P \cap H)$.
9) The maximal compact subgroups of an irreducible reductive algebraic real linear group G are conjugate with respect to automorphisms of the form $a(g)$, where $g \in G^0$.
10) Let G be a semisimple Lie group, H its semisimple Lie subgroup with a finite number of connected components. Then there exists a Cartan decompsotion $G = KP$ such that $H = (H \cap K)(H \cap P)$. This decomposition of H is a Cartan one.
11) Let G be a connected Lie group, H its connected normal subgroup and $\dim G/H = 1$. Then there exists a Lie subgroup $C \subset G$ such that $G = H \rtimes C$. (Hint: reduce the general case to the cases of a solvable and of a semisimple group H. In the solvable case see Exercise 1.4.15. In the semisimple case make use of the fact that $Z(H)$ is contained in a pseudotorus (see Problem 25).)

Hints to Problems

3. To prove the converse statement make use of Problem 1.3.17.
8. Make use of Problem 4.
9. If $k \in N(K^0)$ then the automorphism $\operatorname{Ad} k$ preserves the decomposition (1) and therefore commutes with θ. Next, make use of Problem 7.
10. The decomposition $G^0 = (G^0 \cap K)P$ implies that $G^0 \cap K$ is connected and therefore coincides with K^0.
11. Notice that the group $\operatorname{Ad} K^0 = \operatorname{Ad}(G^0 \cap K) = (\operatorname{Int} \mathfrak{g}) \cap \mathrm{O}(\mathfrak{g})$ is compact and make use of Corollaries 4 and 5 of Theorem 2.
12. First prove that $\operatorname{Ad} K = (\operatorname{Ad} G) \cap \mathrm{O}(\mathfrak{g})$ is a maximal compact subgroup of $\operatorname{Ad} G$, making use of Corollary 3 of Theorem 2.1 and Problem 7.
13. Theorems 3 and 2.7 imply that for any compact subgroup $M \subset G$ there exists $g \in G^0$ such that $gMg^{-1} \subset L$. If $L \subset L_1$, where L_1 is a compact subgroup of G, then applying this statement to L_1 we get $gL_1g^{-1} \subset L$ for some $g \in G^0$. Therefore $gLg^{-1} \subset gL_1g^{-1} \subset L$ implying $gLg^{-1} = L$, since L is a compact

Lie group. Therefore $L = L_1$. If M is a maximal compact subgroup then obviously $gMg^{-1} = L$.

14. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ and consider the corresponding Cartan decomposition $\operatorname{Aut}\mathfrak{g} = KP$ of the group $\operatorname{Aut}\mathfrak{g}$. If $a \in \operatorname{Aut}\mathfrak{g}$ is an element such that $aMa^{-1} \subset K$ then the Cartan decomposition $\mathfrak{g} = a^{-1}(\mathfrak{k}) \oplus a^{-1}(\mathfrak{p})$ is M-invariant.
17. In E, choose an orthonormal basis, such that $\log X = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ for $\lambda_i \in \mathbb{R}$. Then by (11)

$$f_{X,Y}(t) = \sum_{1 \leqslant i \leqslant n} e^{t\lambda_i}/y_{ii} \quad \text{where} \quad y_{ii} > 0.$$

Therefore $f_{X,Y}$ is strictly convex. The strict convexity of φ_x follows from the equality $\varphi_x(t) = \max_{Y \in \Omega} f_{X,Y}(t)$.

18. By Problem 4.1.8 $\operatorname{Ad} G \subset \mathrm{SL}(\mathfrak{g})$, whence $\hat{P} \subset SP(\mathfrak{g})$. Lemma 2.1 implies that $\hat{P}$ is closed in $P(\mathfrak{g})$. By Problem 7 and Corollary 7 of Theorem 2 the action (Sp)(Ad) transforms $\hat{P}$ into itself. Since any $Y = \exp \operatorname{ad} y$, where $y \in \mathfrak{p}$, presents in the form $Y = (\operatorname{Ad}\exp(y))^2 = \operatorname{Sq}(\operatorname{Ad}\exp(y))(E)$, then $\hat{P}$ coincides with the orbit of E.
19. Verify that the Cartan scalar product in $\mathfrak{g}(\mathbb{C})$ is non-degenerate on $\mathfrak{h}(\mathbb{C})$ if $\mathfrak{h}$ is canonically embedded and make use of Theorem 4.1.2.
20. Apply Lemma 3 to $\mathfrak{h}(\mathbb{C})$ and the real structure $\sigma\colon z \mapsto \bar{z}$ on $\mathfrak{g}(\mathbb{C})$. The subalgebra $\mathfrak{h}$ is canonically embedded in $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{u}) \oplus (\mathfrak{g} \cap i\mathfrak{u})$, where $\mathfrak{u} = \mathfrak{g}(\mathbb{C})^{\tau}$.
21. Theorem 2.16 implies that if A is a maximal connected commutative subgroup of K then $\operatorname{Ad} A$ is a maximal torus in the compact Lie group $\operatorname{Ad} K$, whence A is a pseudotorus in G. This makes it obvious that a maximal pseudotorus belonging to K is a maximal connected commutative subgroup of K. The conjugacy follows from Theorems 3 and 2.16.
23. Let $A_j = \exp_{G_j}(\mathfrak{a})$. If there exists a covering $\pi\colon G \to G_2$ such that $d\pi = \mathrm{id}$ then we have the commuting diagram

$$\begin{tikzcd}
 & A_1 \subset G_1 \arrow[dd, "\pi"] \\
i\mathfrak{a} \arrow[ur, "\mathscr{E}_{G_1}"] \arrow[dr, "\mathscr{E}_{G_2}"'] & \\
 & A_2 \subset G_2.
\end{tikzcd} \tag{14}$$

Corollary 5 of Theorem 2 and Theorem 2.16 imply that $\operatorname{Ker}\pi \subset A_1$. Therefore $L(G_2) = \mathscr{E}_{G_1}^{-1}(\operatorname{Ker}\pi) \supset L(G_1)$. To prove the existence of π provided $L(G_1) \subset L(G_2)$, consider a simply connected group $\tilde{G}$ covering G_1 and G_2 and prove that the kernel of the covering $\tilde{G} \to G_1$ is contained in the kernel of the covering $\tilde{G} \to G_2$.

25. Make use of Problem 23.
26. Let $\tilde{G}$ be a simply connected Lie group with the tangent algebra $\mathfrak{g}$. Problem 25 implies that $N = \mathscr{E}_{\tilde{G}}(L)$ is a subgroup of $Z(\tilde{G})$ and $L = \mathscr{E}_{\tilde{G}}^{-1}(N)$. Verify that $L = L(G)$ for $G = \tilde{G}/N$.

28. If G is simply connected then so is K (Corollary 1 of Theorem 2). Making use of Theorem 4.3.5 we deduce that $L(G) = Q^\vee(\mathfrak{k}(\mathbb{C})')$.
29. Use Theorem 4.3.7.
30. Apply Problems 29 and 4.4.30.
32. Let $A = \exp_G \mathfrak{a}$, $\tilde{A} = \exp_{\tilde{G}} \mathfrak{a}$. Consider the commutative diagram which follows from (13) and (14):

$$\begin{tikzcd}
& \tilde{A} \subset \tilde{G} \arrow[r, "j"] \arrow[dd, "\pi"'] & H \arrow[dd, "\tilde{\pi}"] \\
i\mathfrak{a} \arrow[ur, "\mathscr{E}_{\tilde{G}}"] \arrow[urr, "\exp_H"'] \arrow[dr, "\mathscr{E}_G"'] & & \\
& A \subset G \arrow[r, "\Phi"] & H/j(\Gamma)
\end{tikzcd}$$

It implies that

$$\begin{aligned}
\exp_G^{-1}(\operatorname{Ker}\Phi) &= \exp_{\tilde{G}}^{-1}(j^{-1}(j(\Gamma))) \\
&= \exp_{\tilde{G}}^{-1}(\Gamma \operatorname{Ker} j) \\
&= \exp_{\tilde{G}}^{-1}(\Gamma) + \exp_{\tilde{G}}^{-1}(\operatorname{Ker} j) \\
&= \operatorname{Ker}\exp_G + \operatorname{Ker}\exp_H \\
&= L(G) + \operatorname{Ker}\exp_H.
\end{aligned}$$

Theorem 4.3.5 implies that $\operatorname{Ker}\exp_H = 2\pi i(Q^\vee \cap \mathfrak{t})$.

§4. Real Root Decomposition

In this section we consider the root decomposition of a real semisimple Lie algebra with respect to a maximal subalgebra expressed in the adjoint representation by diagonal matrices. The study of the corresponding root system enables us to assign to a real semisimple Lie algebra the so-called Satake diagram which can be considered as a generalization of the Dynkin diagram. Satake diagrams can be used in the classification of real semisimple Lie algebras which we carried out in on § 1 by another method (cf. [33]). Another application of a real root decomposition is Iwasawa's theorem generalizing the classical Gram-Schmidt orthogonalization method.

1°. Maximal $\mathbb{R}$-Diagonalizable Subalgebras. Let $\mathfrak{g}$ be a real Lie algebra. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called $\mathbb{R}$-*diagonalizable* if there is a basis in $\mathfrak{g}$ with respect to which all operators $\operatorname{ad} x$ ($x \in \mathfrak{a}$) are expressed by diagonal matrices. In this case we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda, \tag{1}$$

where Δ is a finite set of non-zero elements of $\mathfrak{a}^*$ and $\mathfrak{g}_\lambda (\lambda \in \Delta \cup \{0\})$ denotes the non-zero subspace $\{x \in \mathfrak{g}: [a, x] = \lambda(a)x (a \in \mathfrak{a})\}$. The set Δ is called the *root system of* $\mathfrak{g}$ *with respect to* $\mathfrak{a}$ and the decomposition (1) is called the *root decomposition.* As in the complex case, for any λ, $\mu \in \Delta \cup \{0\}$ we have

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \begin{cases} \subset \mathfrak{g}_{\lambda+\mu} & \text{if } \lambda + \mu \in \Delta \cup \{0\}, \\ = 0 & \text{otherwise} \end{cases}$$

In particular, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ (the centralizer of $\mathfrak{a}$).

Now suppose that $\mathfrak{g}$ is semisimple. Clearly, any $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is commutative. If $x \in \mathfrak{a}$ and $\alpha(x) = 0$ for all $\alpha \in \Delta$ then $x \in \mathfrak{z}(\mathfrak{g})$ and therefore $x = 0$. This makes it obvious that Δ generates the space $\mathfrak{a}^*$.

Problem 1. Any $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of a real semisimple Lie algebra $\mathfrak{g}$ is contained in some Cartan subspace $\mathfrak{p}$. Conversely, if $\mathfrak{p}$ is a Cartan subspace of $\mathfrak{g}$ then any subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ is $\mathbb{R}$-diagonalizable.

Let $\mathfrak{a}$ be a maximal diagonalizable subalgebra of a semisimple Lie algebra $\mathfrak{g}$. By Problem 1 there exists a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{2}$$

such that $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{a}$ is maximal among the subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}$.

Problem 2. Any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and maximal among such subalgebras is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$. The centralizer $\mathfrak{g}_0$ of such a subalgebra is of the form

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}, \tag{3}$$

where $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$.

Let $\Sigma \subset \mathfrak{a}^*$ be the root system associated to a maximal diagonalizable subalgebra $\mathfrak{a}$. Notice that $\Sigma \neq \varnothing$ if and only if $\mathfrak{a} \neq 0$. Any $\alpha \in \Sigma$ determines the hyperplane $P_\alpha = \operatorname{Ker} \alpha$ in $\mathfrak{a}$. The elements of the non-empty open set

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} P_\alpha$$

are called *regular.*

Problem 3. The centralizer of any regular element of $\mathfrak{a}$ coincides with $\mathfrak{g}_0$.

Theorem 1. *Let K be the maximal compact subgroup of* $\operatorname{Int} \mathfrak{g}$ *corresponding to the subalgebra* $\mathfrak{k}$ *of the decomposition* (2). *Any two maximal subalgebras of* $\mathfrak{p}$ *are transformed into each other by an element of K. Any two maximal $\mathbb{R}$-diagonalizable subalgebras of* $\mathfrak{g}$ *are conjugate.*

The second statement of Theorem 1 reduces to the first one with the help of Problem 1 and Theorem 3.1. It suffices to prove the first statement.

Problem 4. Deduce the first statement of Theorem 1 from the following lemma.

Lemma 1. *Under the assumptions of Theorem* 1, *for any* $x, y \in \mathfrak{p}$ *there exists* $k \in K$ *such that* $[k(x), y] = 0$.

Proof. On K, consider the smooth function $\varphi(k) = (x, k(y))$. Since K is compact, φ possesses a minimum point, k_0. Then for any $z \in \mathfrak{k}$ the function

$$\tilde{\varphi}(t) = \varphi(k_0 \exp(t \operatorname{ad} \mathfrak{z}))$$

assumes its minimum at $t = 0$. Therefore

$$\begin{aligned} 0 = \tilde{\varphi}'(0) &= (x, k_0([z, y])) = (k_0^{-1}(x), [z, y]) \\ &= -([k_0^{-1}(x), y], z), \end{aligned}$$

implying $[k_0^{-1}(x), y] = 0$.

The dimension of a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of a real semisimple Lie algebra $\mathfrak{g}$ (independent by Theorem 1 of the choice of $\mathfrak{a}$) is called the *real rank* of $\mathfrak{g}$ and is denoted by $\operatorname{rk}_{\mathbb{R}} \mathfrak{g}$.

Problem 5. $\operatorname{rk}_{\mathbb{R}} \mathfrak{g} = 0$ if and only if $\mathfrak{g}$ is compact.

Problem 6. If a real semisimple Lie algebra $\mathfrak{g}$ splits into the direct sum of ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ then the maximal $\mathbb{R}$-diagonalizable subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ are of the form $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_i$ $(i = 1, 2)$ is an arbitrary maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}_i$. In particular,

$$\operatorname{rk}_{\mathbb{R}} \mathfrak{g} = \operatorname{rk}_{\mathbb{R}} \mathfrak{g}_1 + \operatorname{rk}_{\mathbb{R}} \mathfrak{g}_2.$$

Under the natural identification of $\mathfrak{a}^*$ with $\mathfrak{a}_1^* \oplus \mathfrak{a}_2^*$ the root system Σ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is identified with $\Sigma_1 \cup \Sigma_2$, where $\Sigma_i \subset \mathfrak{a}_i^*$ is the root system of $\mathfrak{g}_i$ with respect to $\mathfrak{a}_i (i = 1, 2)$.

2°. Real Root Systems. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed decomposition (2), $\mathfrak{a} \subset \mathfrak{g}$ a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, Σ the corresponding root system. Problem 5 implies that $\Sigma \neq \varnothing$ if and only if $\mathfrak{g}$ is non-compact. By Problem 3.3 $\mathfrak{a}$ is a Euclidean space with respect to the Cartan scalar product in $\mathfrak{g}$. Let us naturally transport the scalar product from $\mathfrak{a}$ to $\mathfrak{a}^*$. Our next aim is to prove the following theorem.

Theorem 2. *The root system* $\Sigma \subset \mathfrak{a}^*$ *of a semisimple Lie algebra* $\mathfrak{g}$ *with respect to a maximal* $\mathbb{R}$*-diagonalizable subalgebra* $\mathfrak{a}$ *is a root system in the sense of* 4.2° (*not necessarily reduced*).

Proof is close to the proof of the similar fact for complex Lie algebras (see 4.1.6°). For any $\alpha \in \Sigma$ denote by h_α the element of $\mathfrak{a}$ uniquely determined by the following property:

$$\gamma(h_\alpha) = \langle \gamma | \alpha \rangle \qquad \text{for any} \quad \gamma \in \mathfrak{a}^*.$$

Problem 7. Let θ be an automorphism of $\mathfrak{g}$ transforming $\mathfrak{a}$ into itself. Then ${}^t\theta(\Sigma) = \Sigma$, $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{{}^t\theta^{-1}(\alpha)}(\alpha \in \Sigma \cup \{0\})$, $\theta(h_\alpha) = h_{{}^t\theta^{-1}(\alpha)}\ (\alpha \in \Sigma)$.

Apply Problem 7 to the involutive automorphism θ of $\mathfrak{g}$ defined by the formula

$$\theta(x + y) = x - y \qquad (x \in \mathfrak{k}, y \in \mathfrak{p}).$$

Since $\theta|\mathfrak{a} = -\mathrm{id}$, we see that $-\Sigma = \Sigma$ and $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}\ (\alpha \in \Sigma \cup \{0\})$.

Problem 8. For any $x \in \mathfrak{g}_\alpha$, where $\alpha \in \Sigma$, we have

$$[x, \theta(x)] = (\alpha, \alpha)/2\, (x, \theta(x)) h_\alpha$$

and $(x, \theta(x)) < 0$ if $x \neq 0$.

Fix $\alpha \in \Sigma$ and a non-zero $x \in \mathfrak{g}_\alpha$. Problem 8 easily implies the existence of a $c \in \mathbb{R}$, $c \neq 0$, such that $x_\alpha = cx \in \mathfrak{g}_\alpha$ and $y_\alpha = -c\theta(x) \in \mathfrak{g}_{-\alpha}$ satisfy $[x_\alpha, y_\alpha] = h_\alpha$.

As follows from Problem 2, the maximal commutative subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$ are of the form $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{a}$, where $\mathfrak{h}^+$ is any maximal commutative subalgebra of $\mathfrak{m}$. Now pass to the complexification $\mathfrak{g}(\mathbb{C})$ of $\mathfrak{g}$ and consider its commutative subalgebra

$$\mathfrak{t} = \mathfrak{h}(\mathbb{C}) = \mathfrak{h}^+(\mathbb{C}) \oplus \mathfrak{a}(\mathbb{C}).$$

Let us extend θ to $\mathfrak{g}(\mathbb{C})$ by linearity. Denote by σ the complex conjugation in $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{g}$.

Problem 9. The subalgebra $\mathfrak{t}$ is maximal diagonalizable in $\mathfrak{g}(\mathbb{C})$ and invariant with respect to σ and θ. The subalgebras $\mathfrak{t}^- = \mathfrak{a}(\mathbb{C})$ and $\mathfrak{t}^+ = \mathfrak{h}^+(\mathbb{C})$ are algebraic and diagonalizable in $\mathfrak{g}(\mathbb{C})$ and $\mathfrak{t}^+$ is a maximal diagonalizable subalgebra of the reductive algebraic subalgebra $\mathfrak{m}(\mathbb{C})$. We have

$$\mathfrak{t}(\mathbb{R}) = (i\mathfrak{h}^+) \oplus \mathfrak{a}. \tag{4}$$

Under the natural identification $\mathfrak{a}^* = \mathfrak{t}^-(\mathbb{R})^*$ the root system Σ is identified with the root system $\Delta(\mathfrak{t}^-)$ of $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{t}^-$.

Consider the homomorphism $\varphi_\alpha\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}(\mathbb{C})$ defined by the formulas

$$\varphi_\alpha(\mathbf{e}) = x_\alpha, \qquad \varphi_\alpha(\mathbf{f}) = y_\alpha, \qquad \varphi_\alpha(\mathbf{h}) = h_\alpha.$$

Problem 10. φ_α is an injective Lie algebra homomorphism over $\mathbb{C}$ such that $\varphi_\alpha(\mathfrak{sl}_2(\mathbb{R})) \subset \mathfrak{g}$, $\varphi_\alpha(\mathfrak{so}_2) \subset \mathfrak{k}$.

Denote by F_α a Lie group homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{Int}(\mathfrak{g}(\mathbb{C}))$ such that $d\varphi_\alpha = (\mathrm{ad})\varphi_\alpha$. Problem 10 implies that $F_\alpha(\mathrm{SL}_2(\mathbb{R})) \subset \mathrm{Int}\,\mathfrak{g}$ ($\mathrm{Int}\,\mathfrak{g}$ is naturally embedded into $\mathrm{Int}\,\mathfrak{g}(\mathbb{C})$, see Example 4 of 1.1°). If K is the maximal compact subgroup of $\mathrm{Int}\,\mathfrak{g}$ corresponding to $\mathfrak{k}$ then $F_\alpha(\mathrm{SO}_2) \subset K$. In particular, $h_\alpha = F_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in K$.

Problem 11. The automorphism n_α transforms $\mathfrak{a}$ into itself and induces in $\mathfrak{a}$ the orthogonal reflection r_α with respect to P_α.

Proof of Theorem 2. Let $\alpha \in \Sigma$. Denote also by r_α the orthogonal reflection in $\mathfrak{a}^*$ with respect to the hyperplane $L_\alpha = \{\gamma \in \mathfrak{a}^*\colon (\alpha, \gamma) = 0\}$ (this reflection coincides with ${}^t r_\alpha$). Problems 11 and 7 imply that $r_\alpha(\Sigma) = \Sigma$ (cf. Theorem 4.1). Further, $h_\alpha \in \mathfrak{t}^-(\mathbb{Z})$ implying $\langle \beta | \alpha \rangle = \beta(h_\alpha) \in \mathbb{Z}$ for all $\beta \in \Sigma$ (cf. Problem 4.1.34).

Now consider the relation between $\Delta = \Delta(\mathfrak{t}^-)$ and the root system $\Delta(\mathfrak{t}) = \Delta$ of the Lie algebra $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{t}$. Clearly, the restriction map $\rho\colon \mathfrak{t}(\mathbb{R})^* \to \mathfrak{t}^-(\mathbb{R})^* = \mathfrak{a}^*$ transforms Δ into $\Sigma \cup \{0\}$. Set

$$\Delta_0 = \{\alpha \in \Delta\colon \rho(\alpha) = 0\}, \qquad \Delta_1 = \Delta \backslash \Delta_0.$$

Problem 12. The map $\rho\colon \Delta_{\mathfrak{g}(\mathbb{C})} \cup \{0\} \to \Sigma \cup \{0\}$ is surjective. We have

$$\mathfrak{m}(\mathbb{C}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}(\mathbb{C})_\alpha, \qquad \mathfrak{g}_\lambda(\mathbb{C}) = \bigoplus_{\rho(\alpha) = \lambda} \mathfrak{g}(\mathbb{C})_\alpha \quad (\lambda \in \Sigma).$$

In particular, Δ_0 is the root system of the semisimple Lie algebra $\mathfrak{m}(\mathbb{C})'$ with respect to $\mathfrak{t} \cap \mathfrak{m}(\mathbb{C})'$.

Since $\theta(\mathfrak{t}) = \mathfrak{t}$, Problem 4.1.10 implies that ${}^t\theta(\Delta) = \Delta$.

Problem 13. $\mathrm{Ker}\,\rho = \{\gamma \in \mathfrak{t}^*\colon {}^t\theta(\gamma) = \gamma\}$. In particular, $\Delta_0 = \{\alpha \in \Delta\colon {}^t\theta(\alpha) = \alpha\}$.

Set

$${}^t\sigma(\gamma)(x) = \overline{\gamma(\sigma(x))} \qquad (\gamma \in \mathfrak{t}^*, x \in \mathfrak{t}).$$

Then ${}^t\sigma(\gamma) \in \mathfrak{t}^*$. Therefore an antilinear transformation ${}^t\sigma\colon \mathfrak{t}^* \to \mathfrak{t}^*$ is defined.

Problem 14. The transformations σ and ${}^t\sigma$ send $\mathfrak{t}(\mathbb{R})$ and $\mathfrak{t}(\mathbb{R})^*$ into themselves and coincide on these subspaces with $-\theta$ and $-({}^t\theta)$ respectively. We have $\sigma(\mathfrak{g}(\mathbb{C})_\alpha) = \mathfrak{g}(\mathbb{C})_{{}^t\sigma(\alpha)} = \mathfrak{g}(\mathbb{C})_{({}^t\theta(\alpha))}$ for all $\alpha \in \Delta$.

3°. Satake Diagram. We retain the notation of 2°. In $\mathfrak{t}(\mathbb{R})$, choose a basis $v_1, \ldots, v_l$ such that $v_1, \ldots, v_r$ is a basis of $\mathfrak{a}$ and consider the lexicographic orderings with respect to these bases in $\mathfrak{t}(\mathbb{R})^*$ and $\mathfrak{a}^*$ (see 4.2.2°). Then $\rho(\lambda) > 0$ implies $\lambda > 0$ for $\lambda \in \mathfrak{t}(\mathbb{R})^*$. Denote by Δ^+, Σ^+ (resp. Δ^-, Σ^-) the sets of positive (negative) roots with respect to these orderings. Set $\Delta_i^\pm = \Delta_i \cap \Delta^\pm$ $(i = 0, 1)$.

Problem 15. $\rho(\Delta_1^\pm) = \Sigma^\pm$, ${}^t\theta(\Delta_1^\pm) = \Delta_1^\mp$, ${}^t\sigma(\Delta_1^\pm) = \Delta_1^\pm$. Let $\Pi \subset \Delta^+$ and $\Theta \subset \Sigma^+$ be bases. Set $\Pi_i = \Delta_i \cap \Pi$ $(i = 0, 1)$.

Problem 16. Π_0 is a base of Δ_0 and $\rho(\Pi_1) \supset \Theta$.
Actually, as we will show, $\rho(\Pi_1) = \Theta$.
Let us prove the following important statement.

Lemma 2. *There exists an involutive transformation* $\omega\colon \Pi_1 \to \Pi_1$ *such that for any* $\alpha \in \Pi_1$ *we have*

$$'\theta(\alpha) = -\omega(\alpha) - \sum_{\gamma \in \Pi_0} c_{\alpha\gamma}\gamma,$$

where $c_{\alpha\gamma}$ *are non-negative integers.*

Problem 17. Let C be a square matrix with non-negative integer entries such that $C^2 = E$. Then C is the matrix corresponding to an involutive permutation of elements of the basis.

Problem 18. Prove Lemma 2.

Problem 19. For $\alpha, \beta \in \Pi_1$ we have $\rho(\alpha) = \rho(\beta)$ if and only if $\alpha = \beta$ or $\alpha = \omega(\beta)$. The system $\rho(\Pi_1)$ is linearly independent and therefore coincides with Θ.

Lemma 2 enables us to assign to any real semisimple Lie algebra $\mathfrak{g}$ the *Satake diagram* obtained from the Dynkin diagram of the complex Lie algebra $\mathfrak{g}(\mathbb{C})$ as follows: the vertices corresponding to the roots from Π_0 are blackened and the pairs of different roots from Π_1 transformed into each other by an involution ω are joined by arrows.

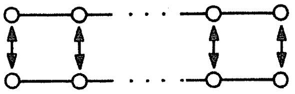
Problem 20. $\operatorname{rk} \mathfrak{g}(\mathbb{C}) = \operatorname{rk}_{\mathbb{R}} \mathfrak{g} + |\Pi_0| + s$, where s is the number of arrows on the Satake diagram.

Problem 21. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be real semisimple Lie algebras. Then the Satake diagram of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the disjoint union of the Satake diagrams of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Problem 22. A real semisimple Lie algebra is simple if and only if its Satake diagram is connected.

Example 1. The Satake diagram of a semisimple compact Lie algebra $\mathfrak{g}$ is obtained from the Dynkin diagram of $\mathfrak{g}(\mathbb{C})$ by blackening all vertices. Any semisimple Lie algebra over $\mathbb{R}$, all verices of whose Satake diagram are black, is compact.

Example 2. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then the Satake diagram of $\mathfrak{g}^{\mathbb{R}}$ is obtained from the Dynkin diagram of $\mathfrak{g}$ by doubling and joining the corresponding vertices of the two diagrams by arrows. For instance, the Satake diagram of $\mathfrak{sl}_{l+1}(\mathbb{C})^{\mathbb{R}}$ contains $2l$ vertices and is of the form



In fact, consider a compact real form $\mathfrak{u} \subset \mathfrak{g}$. If $\mathfrak{h}^+$ is a maximal commutative subalgebra of $\mathfrak{u}$ then $\mathfrak{h} = \mathfrak{h}^+(\mathbb{C})$ is a maximal diagonalizable subalgebra of $\mathfrak{g}$ and $\mathfrak{a} = I\mathfrak{h}^+$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}^{\mathbb{R}}$. Furthermore, $\mathfrak{g}^{\mathbb{R}}(\mathbb{C})$ is identified with $\mathfrak{g} \oplus \mathfrak{g}$ and the maximal diagonalizable subalgebra $\mathfrak{t} = \mathfrak{h}(\mathbb{C})$ of this algebra with $\mathfrak{h} \oplus \mathfrak{h}$. Moreover, $\sigma(x, y) = (\bar{y}, \bar{x})$ $(x, y \in \mathfrak{g})$, where $z \mapsto \bar{z}$ $(z \in \mathfrak{g})$ is the complex conjugation with respect to $\mathfrak{u}$ (see Problem 1.8). The root system Δ of $\mathfrak{g}^{\mathbb{R}}(\mathbb{C})$ with respect to $\mathfrak{t}$ is of the form $\Delta = \Delta_{\mathfrak{g}} \cup {}^t\sigma(\Delta_{\mathfrak{g}})$, where $\Delta_{\mathfrak{g}}$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Similarly, $\Pi = \Pi_{\mathfrak{g}} \cup {}^t\sigma(\Pi_{\mathfrak{g}})$, where $\Pi_{\mathfrak{g}} \subset \Delta_{\mathfrak{g}}$, $\Pi \subset \Delta$ are bases, and $\omega = {}^t\sigma$.

As is clear from Problem 22, Examples 1 and 2, to list the Satake diagrams of semisimple Lie algebras $\mathfrak{g}$ over $\mathbb{R}$ we may confine ourselves to the case when $\mathfrak{g}$ is a non-compact real form of a simple Lie algebra $\mathfrak{g}(\mathbb{C})$. The Satake diagrams of all such Lie algebras $\mathfrak{g}$ are listed in Table 9, which also contains the Dynkin diagrams of the corresponding root systems Σ, the types of these systems and dimensions of root subspaces $m_\lambda = \dim \mathfrak{g}_\lambda$ $(\lambda \in \Sigma)$. This Table quite easily implies

Theorem 3. *Two semisimple Lie algebras over $\mathbb{R}$ are isomorphic if and only if so are (in the natural sense) their Satake diagrams.*

4°. Split Semisimple Lie Algebras. A real semisimple Lie algebra is called *split* if any of its maximal $\mathbb{R}$-diagonalizable subalgebras is a maximal commutative subalgebra.

Problem 23. The following conditions are equivalent: $\mathfrak{g}$ is split; $\mathfrak{a}(\mathbb{C})$ is a maximal diagonalizable subalgebra of $\mathfrak{g}(\mathbb{C})$ for any maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$; $\operatorname{rk}_{\mathbb{R}} \mathfrak{g} = \operatorname{rk} \mathfrak{g}(\mathbb{C})$; the Satake diagram of $\mathfrak{g}$ has neither black vertices nor arrows.

If $\mathfrak{g}$ is split then under the notation of 2° we have $\mathfrak{m} = \mathfrak{a}$, $\Delta = \Sigma$, $\mathfrak{g}(\mathbb{C})_\alpha = \mathfrak{g}_\alpha(\mathbb{C})$ for all $\alpha \in \Delta$. Therefore, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

Problem 24. Any ideal of a split semisimple Lie algebra is split. The direct sum of two split Lie algebras is split.

Theorem 4. *Any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a unique up to an isomorphism split real form $\mathfrak{s}$ which is simple if and only if so is $\mathfrak{g}$.*

Problem 25. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. The normal real form of $\mathfrak{g}$ associated with an arbitrary canonical system of generators (see Problem 1.6) is split. Conversely, any split real form of $\mathfrak{g}$ is normal with respect to a canonical system of generators.

The first statement of Theorem 4 follows from Problem 25 and Theorem 4.3.1. If $\mathfrak{s}$ is simple then by Theorem 1.1 so is $\mathfrak{g}$ since a complex Lie algebra considered as a real one is not split (see Example 2 of 3°).

Example. Simple split Lie algebras over $\mathbb{R}$ are $\mathfrak{sl}_n(\mathbb{R})$ $(n \geqslant 2)$, $\mathfrak{so}_{k,k+1}$ $(k \geqslant 1)$, $\mathfrak{so}_{k,k}$ $(k \geqslant 3)$, $\mathfrak{sp}_n(\mathbb{R})$ $(n \geqslant 2)$, *EI*, *EV*, *EVIII*, *FI*, *G*. This is clear: look at the values of the real rank listed in Table 9.

5°. Iwasawa Decomposition. Let again $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ a maximal $\mathbb{R}$-diagonalizable subalgebra, Σ the root system with respect to $\mathfrak{a}$. In Δ, choose a system of simple roots Θ and denote by $\Sigma^+ \subset \Sigma$ the corresponding subsystem of positive roots. Set

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda .$$

Problem 26. The subspace $\mathfrak{n}$ is a unipotent algebraic subalgebra of $\mathfrak{g}$. We have $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$ so that $\mathfrak{d} = \mathfrak{a} \oplus \mathfrak{n}$ is a solvable algebraic subalgebra of $\mathfrak{g}$.

Theorem 5. *The following decompositions into direct sums of subalgebras take place*: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{d}$.

Problem 27. Prove this theorem.

We want to construct decompositions of a connected semisimple Lie group into products of its Lie subgroups corresponding to the decompositions of Theorem 5. Let G be a connected semisimple Lie group with the tangent algebra $\mathfrak{g}$. As is shown in § 3, there exists a connected Lie subgroup $K \subset G$ with the tangent algebra $\mathfrak{k}$. If G has a finite center then K is a maximal compact subgroup of G.

Problem 28. In G, there exist simply connected Lie subgroups A, N, D with the tangent algebras $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{d}$ respectively and $D = A \rtimes N$.

Problem 29. In $\mathfrak{g}$, there exists a basis by means of which all elements $\operatorname{ad} x$ $(x \in \mathfrak{d})$ and $\operatorname{Ad} g (g \in D)$ are expressed by upper triangular matrices (for $\operatorname{Ad} g$, $g \in D$, with positive diagonal entries) and $D \cap K = \{e\}$.

Problem 30. Prove the following theorem:

Theorem 6. *Let G be a connected semisimple Lie group and K, A, N, D its connected Lie subgroups defined above. Then the maps*

$$K \times A \times N \to G, \qquad (k, a, n) \mapsto kan$$

and

$$K \times D \to G, \qquad (k, d) \mapsto kd$$

are diffeomorphisms. In particular, $G = KAN = KD$.

The decompositions of $\mathfrak{g}$ and G described in Theorems 5 and 6 are called the *Iwasawa decompositions.*

Now we will characterize the subalgebra $\mathfrak{d} \subset \mathfrak{g}$ and the subgroup $D \subset G$ without incorporating the root decomposition.

Let $\mathfrak{g}$ be a real Lie algebra. A subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is called *triangular* if in a basis of $\mathfrak{g}$ all operators $\operatorname{ad} x$ $(x \in \mathfrak{c})$ are expressed by upper triangular matrices. Let G be a Lie group with the tangent algebra $\mathfrak{g}$. A subgroup $C \subset G$ is called *triangular* if there is a basis in $\mathfrak{g}$ with respect to which all operators $\operatorname{Ad} g\ (g \in C)$ are expressed by upper triangular matrices.

Problem 31. A connected virtual Lie subgroup of G is triangular if and only if its tangent subalgebra of $\mathfrak{g}$ is triangular. A maximal connected triangular subgroup is a Lie subgroup of G; its tangent algebra is a maximal triangular subalgebra of $\mathfrak{g}$. Any maximal triangular subalgebra of $\mathfrak{g}$ is tangent to a maximal connected triangular subgroup of G.

Problem 32. Let G be a connected semisimple Lie group, $\mathfrak{g}$ its tangent algebra. The subgroup $D \subset G$ and the subalgebra $\mathfrak{d} \subset \mathfrak{g}$ defined in problems 26 and 28 are a maximal connected triangular subgroup and a maximal triangular subalgebra, respectively.

Example. Let $G = \mathrm{SL}_n(\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. Under an appropriate choice of a base in $\Sigma = \Delta_{\mathfrak{sl}_n(\mathbb{C})}$ the subalgebra $\mathfrak{d}$ defined in Problem 26 is the subalgebra of all upper triangular traceless matrices, D is the subgroup of all upper triangular matrices with determinant 1 and positive diagonal entries. The group K coincides with SO_n. Theorem 6 easily follows in this case from the classical theorem on the reducing of a positive definite quadratic form to the normal form with the help of a triangular change of basis.

Concluding this section we prove the following theorem which is a real analogue of Theorem 3.2.12 on conjugacy of Borel subgroups.

Theorem 7. *The maximal connected triangular subgroups (maximal triangular subalgebras) of a connected semisimple real Lie group (semisimple Lie algebra over $\mathbb{R}$) are conjugate.*

Proof is based on the following fixed point lemma.

Lemma 3. *Let V be a finite-dimensional vector space, X its linear transformation whose characteristic roots are all real. For any point $p \in \mathrm{P}(V)$ there exists the limit*

$$p_0 = \lim_{t \to \infty} (\exp tX)(p) \in \mathrm{P}(V).$$

The point p_0 is stable with respect to the group $\{\exp tX : t \in \mathbb{R}\}$.

Proof. Express X by a triangular matrix in a basis of V. The diagonal entries of this matrix are the eigenvalues $\lambda_1, \ldots, \lambda_r$ of X (multiplicities counted). The entries of the matrix $\exp tX$ are functions in t of the form

$$\sum_{1 \leqslant i \leqslant r} Q_i(t) e^{\lambda_i t},$$

where Q_i are polynomials. The coordinates of the vector $(\exp tX)v$, where $v \in V$ is a non-zero vector such that $\langle v \rangle = p$, are of the same form. Let Λ be the maximal of the numbers λ_i among the coordinates of this vector and M the highest of the degrees of the corresponding polynomials Q_i. Then $(\exp tX)v = t^M e^{\Lambda t}(v_0 + \varepsilon(t))$, where $v_0 \neq 0$ and $\varepsilon(t) \to 0$ as $t \to \infty$. Clearly, $\langle v_0 \rangle = \lim_{t\to\infty}(\exp tX)(p)$ and $p_0 = \langle v_0 \rangle$ is fixed under $\exp tX (t \in \mathbb{R})$. □

Using Lemma 3 we will prove that the connected triangular linear group in V over $\mathbb{R}$ has a fixed point in any invariant closed subset of the flag variety $\mathrm{F}(V)$. For this we need the embedding j of $\mathrm{F}(V)$ into the projective space constructed in 2.2.7°. Recall that this embedding is of the form

$$\begin{aligned}\mathrm{F}(V) &\to \mathrm{Gr}_1(V) \times \cdots \times \mathrm{Gr}_n(V) \to \mathrm{P}(V) \times \mathrm{P}(\Lambda^2 V) \times \cdots \times \mathrm{P}(\Lambda^n V)\\ &\to \mathrm{P}(V \otimes \Lambda^2 V \otimes \cdots \otimes \Lambda^n V),\end{aligned}$$

where the last arrow is described in 2.2.6° (here $n = \dim V$).

Problem 33. The embedding $j\colon \mathrm{F}(V) \to \mathrm{P}(W)$, where $W = V \otimes \Lambda^2 V \otimes \cdots \Lambda^n V$, constructed in 2.2.7° has the following property: $j(gf) = R(g)j(f)$ $(g \in \mathrm{GL}(V), f \in \mathrm{F}(V))$, where $R\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$ is the natural representation.

Problem 34. Let F be the flag variety of a finite-dimensional vector space V over $\mathbb{R}$ and $C \subset \mathrm{GL}(V)$ a connected virtual Lie subgroup with a fixed point in F. Then any non-empty closed C-invariant subset $\Omega \subset \mathrm{F}$ contains a point fixed under C.

Problem 35. Prove Theorem 7.

Exercises

Let G be an irreducible semisimple real algebraic group, $\mathfrak{g}$ its tangent algebra. An algebraic torus $T \subset G(\mathbb{C})$ is called *split* if in a basis of $\mathfrak{g}(\mathbb{C})$ contained in $\mathfrak{g}$ all elements of the torus $\operatorname{Ad} T$ are expressed by diagonal matrices.

1) An algebraic torus $T \subset G(\mathbb{C})$ is split if and only if $\mathfrak{t} = \mathfrak{a}(\mathbb{C})$, where $\mathfrak{a}$ is an $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$.
2) The maximal split tori in $G(\mathbb{C})$ are conjugate with respect to the inner automorphisms generated by the elements of G^0.
3) $\mathfrak{g}$ is split if and only if $G(\mathbb{C})$ has a split maximal torus.

Let $\mathfrak{a}$ be a subalgebra of the real Lie algebra $\mathfrak{g}$ and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a real linear representation. The subalgebra $\mathfrak{a}$ is called *ρ-diagonalizable* (or *ρ-triangular*) if all $\rho(x)$ $(x \in \mathfrak{g})$ are expressed by diagonal (triangular) matrices in a basis of V.

4) Let $\mathfrak{g}$ be a semisimple real Lie algebra. Any $\mathbb{R}$-diagonalizable (i.e. ad-diagonalizable) subalgebra of $\mathfrak{g}$ is ρ-diagonalizable for any linear representation ρ. Conversely, if $\mathfrak{a} \subset \mathfrak{g}$ is a ρ-diagonalizable subalgebra for some faithful representation ρ then $\mathfrak{a}$ is $\mathbb{R}$-diagonalizable.
5) Any triangular subalgebra of a semisimple real Lie algebra $\mathfrak{g}$ is ρ-triangular for any linear representation of $\mathfrak{g}$. Conversely, if the subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is ρ-triangular for some faithful representation ρ of $\mathfrak{g}$ then $\mathfrak{c}$ is triangular.
6) Under the notation of 2° denote by $W \subset \mathrm{GL}(\mathfrak{a})$ the Weyl group of the root system Σ (see 4.2.4°). Set

$$N_K(\mathfrak{a}) = \{k \in K \colon k(\mathfrak{a}) = \mathfrak{a}\},$$

$$Z_K(\mathfrak{a}) = \{k \in K \colon k(x) = x \quad \text{for any} \quad x \in \mathfrak{a}\}.$$

Then $N_K(\mathfrak{a})$ and $Z_K(\mathfrak{a})$ are Lie subgroups of K with the tangent algebras isomorphic to $\mathfrak{m}$. The correspondence $k \mapsto k|\mathfrak{a}$ is the surjective homomorphism of $N_K(\mathfrak{a})$ onto W with the kernel $Z_K(\mathfrak{a})$, whence

$$W \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

7) Let, under the same notation, $\dim \mathfrak{g}_\lambda = 1$ for all $\lambda \in \Sigma$ and let $\mathfrak{g}$ have no compact ideals. Then $\mathfrak{g}$ is split.
8) In a complex semisimple Lie algebra $\mathfrak{g}$ with a maximal diagonalizable subalgebra $\mathfrak{h}$ there exists a unique up to a conjugacy in $\operatorname{Aut}\mathfrak{g}$ involutive automorphism θ such that $\theta(x) = -x$ for all $x \in \mathfrak{h}$. The corresponding automorphism $\eta(\theta) \in \operatorname{Aut}\Pi$ coincides with the automorphism θ of Exercise 4.3.6. The correspondence established in Theorem 1.4 assignes to θ the class of the normal real form of $\mathfrak{g}$.
9) For the classical Lie algebras $\mathfrak{g}$ the automorphism θ of Exercise 8 is conjugate to the following automorphism (under notation of 1.2°):

$$\theta \colon X \to -X^T \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}),\ n \geqslant 2;$$

$$\theta = \operatorname{Ad} I_{n,n+1} \qquad \text{for} \quad \mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}),\ n \geqslant 1;$$

$$\theta = \operatorname{Ad} I_{n,n} \qquad \text{for} \quad \mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}),\ n \geqslant 2;$$

$$\theta = \operatorname{Ad} S_n \qquad \text{for} \quad \mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}),\ n \geqslant 2.$$

A subalgebra $\mathfrak{p}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called *parabolic* if $\mathfrak{p}(\mathbb{C})$ is a parabolic subalgebra of $\mathfrak{g}(\mathbb{C})$ (see Exercises to 4.2°). Let, under the notation of 3°, M be a subset of a base $\Theta \subset \Sigma^+$. Denote by $\Sigma^{(M)}$ the subset of Σ consisting of all positive roots and those negative roots which can be linearly expressed in terms of M.

10) For any $M \subset \Theta$ the system $\Sigma^{(M)}$ is closed.
11) The subalgebra $\mathfrak{p}^{(M)} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^{(M)}} \mathfrak{g}_\alpha$ of $\mathfrak{g}$ is parabolic.
12) Any parabolic subalgebra of $\mathfrak{g}$ is conjugate to exactly one of the $\mathfrak{p}^{(M)}$.
13) Prove Theorem 2.15 by the method used in the proof of Theorem 1 of this section.
14) Let $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional irreducible linear representation of a split real semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Then the complex representation $\rho(\mathbb{C}) \colon \mathfrak{g}(\mathbb{C}) \to \mathfrak{gl}(V(\mathbb{C}))$ is irreducible and $\rho \mapsto \rho(\mathbb{C})$ is a one-to-one correspondence between the classes of equivalent real irreducible representations of $\mathfrak{g}$ and the classes of complex irreducible representations of $\mathfrak{g}(\mathbb{C})$. Similar statement holds for arbitrary finite-dimensional representations.

Hints to Problems

1. Clearly, the algebraic closure $\mathfrak{a}^a \subset \mathfrak{g}$ is also an $\mathbb{R}$-diagonalizable subalgebra. Therefore we may assume that $\mathfrak{a}$ is an algebraic subalgebra. Obviously, $\mathfrak{a}(\mathbb{C})$ is a diagonalizable subalgebra of $\mathfrak{g}(\mathbb{C})$, whence $\mathfrak{a}$ is a reductive algebraic subalgebra. The inclusion $\mathfrak{a} \subset \mathfrak{p}$ follows now from Theorem 3.4. Conversely, any subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is commutative and $\operatorname{ad} x$ is diagonalizable for any $x \in \mathfrak{a}$ (see Problems 3.1 and 3.4) implying that $\mathfrak{a}$ is an $\mathbb{R}$-diagonalizable subalgebra.
2. First prove that $\mathfrak{g}_0$ is of the form (3).
4. Apply Lemma 1 to the regular elements of two maximal subalgebras of $\mathfrak{p}$ and use Problems 3 and 2.
9. The subalgebra $\mathfrak{t}$ is a maximal commutative subalgebra of $\mathfrak{g}(\mathbb{C})$ and consists of semisimple elements. Therefore $\mathfrak{t}$ is a maximal diagonalizable subalgebra. Let T be the corresponding maximal torus of $H = \operatorname{Int} \mathfrak{g}(\mathbb{C})$, Θ the automorphism of H defined by the formula $\Theta(g) = \theta g \theta^{-1}$ $(g \in H)$. Then $\Theta(T) = T$. The subalgebras $\mathfrak{t}^-$ and $\mathfrak{t}^+$ are tangent to the algebraic subgroups $T^- = \{g \in T\colon \Theta(g)^{-1} = g\}$ and $T^+ = \{g \in T\colon \Theta(g) = g\}$ respectively. Formula (4) follows from the fact that $\mathfrak{h}^+ \oplus (i\mathfrak{a})$ belongs to the compact real form $\mathfrak{k} \oplus (i\mathfrak{p})$ of $\mathfrak{g}(\mathbb{C})$ and therefore the differential $d\chi$ of any character $\chi \in \mathscr{X}(T)$ has only purely imaginary values of $\mathfrak{h}^+ \oplus (i\mathfrak{a})$.
11. Is similar to Problem 4.1.37.
18. Problem 15 implies that for any $\alpha \in \Pi_1$ we have

$${}^t\theta(\alpha) = -\sum_{\beta \in \Pi_1} c_{\alpha\beta}\beta - \sum_{\gamma \in \Pi_0} c_{\alpha\gamma}\gamma,$$

    where $c_{\alpha\beta}, c_{\alpha\gamma}$ are non-negative integers. Verify that $(c_{\alpha\beta})^2_{\alpha, \beta \in \Pi_1} = E$ and apply Problem 17 to the matrix $C = (c_{\alpha\beta})$.
19. Make use of Lemma 2 and Problem 13.
22. Let the Satake diagram of $\mathfrak{g}$ be not connected and $\Delta = \Delta' \cup \Delta''$ the corresponding decomposition of the root system of $\mathfrak{g}(\mathbb{C})$ into the union of nonempty disjoint subsystems. Then $\Delta' \cap \Pi_1$ and $\Delta'' \cap \Pi_1$ are ω-invariant. With the help of Problem 14 we deduce from here that ${}^t\sigma(\Delta') = \Delta'$, ${}^t\sigma(\Delta'') = \Delta''$. Therefore, the ideals $\mathfrak{h}'$, $\mathfrak{h}''$ of $\mathfrak{g}(\mathbb{C})$ corresponding to Δ' and Δ'' (see Problem 4.1.32) are σ-invariant implying $\mathfrak{g} = \mathfrak{h}'^{\sigma} \oplus \mathfrak{h}''^{\sigma}$.
25. Let $\mathfrak{s}$ be a split real form of $\mathfrak{g}$, $\mathfrak{a}$ a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{s}$. By Problem 23 $\mathfrak{t} = \mathfrak{a}(\mathbb{C})$ is a maximal diagonalizable subalgebra of $\mathfrak{g}$ and $\mathfrak{a} = \mathfrak{t}(\mathbb{R})$ by Problem 9. Let Π be a system of simple roots of the root system $\Sigma = \Delta_{\mathfrak{g}}$. Then the elements $h_\alpha, x_\alpha, y_\alpha (\alpha \in \Pi)$ of $\mathfrak{s}$ constructed in 2° form a canonical system of generators of $\mathfrak{g}$. Clearly, $\mathfrak{s}$ coincides with the subalgebra generated by these elements over $\mathbb{R}$.
27. Make use of (1), (3) and the inclusion $\mathfrak{g}_{-\lambda} \subset \mathfrak{k} + \mathfrak{g}_\lambda$.
28. First, let $G = \operatorname{Int} \mathfrak{g} = (\operatorname{Aut} \mathfrak{g})^0$. The unipotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$ determines a connected unipotent algebraic subgroup $N \subset G$ and $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism. The algebraic subalgebra $\mathfrak{a}$ determines the commutative algebraic

subgroup $\tilde{A} \subset \operatorname{Aut} \mathfrak{g}$ and $A = \tilde{A}^0 = \exp \mathfrak{a} \subset G$. Since $\mathfrak{a}$ is an $\mathbb{R}$-diagonalizable subalgebra, $A \simeq \mathbb{R}^l$, where $l = \operatorname{rk}_{\mathbb{R}} \mathfrak{g}$. In an arbitrary connected semisimple Lie group G with tangent algebra $\mathfrak{g}$, consider the Lie subgroups $\hat{A} = (\operatorname{Ad}^{-1} A)^0$ and $\hat{N} = (\operatorname{Ad}^{-1} N)^0$. The simple connectedness of A and N implies that $\hat{A}$ and $\hat{N}$ are simply connected and $\hat{A} \cap Z(G) = \hat{N} \cap Z(G) = \{e\}$. If $g \in \hat{A} \cap \hat{N}$ then $\operatorname{Ad} g \in A \cap N$ implying $g \in Z(G)$ and $g = e$. Clearly, $\hat{A}$ normalizes $\hat{N}$ so that $\hat{A}\hat{N} = \hat{A} \ltimes \hat{N}$ is a Lie subgroup of G.

29. Consider the ascending filtration of $\mathfrak{g}$ by the subspaces $\mathfrak{g}(\lambda) = \sum_{\mu \geqslant \lambda} \mathfrak{g}_\mu$ $(\lambda \in \Sigma \cup \{0\})$, where $\geqslant$ is the partial ordering determined by Θ. Complementing this filtration by the subspaces of missing dimensions we get a flag in $\mathfrak{g}$ invariant with respect to all $\operatorname{ad} x$ $(x \in \mathfrak{d})$ and $\operatorname{Ad} g$ $(g \in D)$. If $g \in D \cap K$ then $\operatorname{Ad} g$ is a diagonalizable operator with all eigenvalues equal to 1 so that $\operatorname{Ad} g = E$ and $g \in Z(G)$. Since the group $\operatorname{Ad} G = (\operatorname{Ad} A) \ltimes (\operatorname{Ad} N)$ is simply connected, $Z(G) \cap D = \{e\}$ and $g = e$.

30. Let $\mu\colon K \times D \to G$ be the map defined by the formula $\mu(k, d) = k$. Since $K \cap D = \{e\}$, then μ is injective. Theorem 5 implies that the map $d_{(e,e)}\mu$: $\mathfrak{k} \times \mathfrak{d} \to \mathfrak{g}$ sending (x, y) into $x + y$ is injective. Therefore so is $d_{(a,b)}\mu$ for any $a \in K$, $b \in D$. In fact, $\mu(l(a)u, r(b^{-1})\sigma) = l(a)r(b^{-1})\mu(u, v)$ $(u \in K, v \in D)$, implying $(d_{(a,b)}\mu)(d_e l(a) \times d_e r(b^{-1})) = (d_e l(a))(d_e r(b^{-1}))d_{(e,e)}\mu$. Therefore μ is a diffeomorphism of $K \times D$ on an open set $KD \subset G$. In particular, $(\operatorname{Ad} K)(\operatorname{Ad} D)$ is open in $\operatorname{Int} \mathfrak{g} = \operatorname{Ad} G$. Since $\operatorname{Ad} K$ is compact, the set $(\operatorname{Ad} K)(\operatorname{Ad} D)$ is closed in $\operatorname{Int} \mathfrak{g}$, implying $\operatorname{Int} \mathfrak{g} = (\operatorname{Ad} K)(\operatorname{Ad} D) = \operatorname{Ad}(KD)$. Taking into account that $Z(D) \subset K$ (by Corollary 2 of Theorem 3.2) we deduce that $G = KD$.

31. Let F be the flag variety of the vector space $\mathfrak{g}$. Consider the G-action on F defined by the adjoint representation Ad. A subgroup $C \subset G$ (a subalgebra $\mathfrak{c} \subset \mathfrak{g}$) is triangular if and only if $C \subset G_f$ (resp. $\mathfrak{c} \subset \mathfrak{g}_f$) for some $f \in$ F. By Theorem 1.1.1 G_f is a Lie subgroup of G with the tangent algebra $\mathfrak{g}_f$. This implies the first statement.

Any maximal connected triangular subgroup coincides with G_f^0 for some $f \in$ F, hence is a Lie subgroup; similarly, any maximal triangular subalgebra coincides with $\mathfrak{g}_f$ for some $f \in$ F. This easily implies the other statements of the problem.

32. If $\mathfrak{c}$ is a triangular subalgebra containing $\mathfrak{d}$ then by Theorem 5 $\mathfrak{c} = (\mathfrak{c} \cap \mathfrak{k}) + \mathfrak{d}$. If $x \in \mathfrak{c} \cap \mathfrak{k}$ then $\operatorname{ad} x$ is a semisimple (in $\mathfrak{g}(\mathbb{C})$) operator with zero eigenvalues implying $\operatorname{ad} x = 0$ and $x = 0$. Thus $\mathfrak{c} = \mathfrak{d}$.

34. Let us carry out the induction in $\dim C$. The existence of a C-invariant flag implies that C is solvable. Therefore $C = C_1 C_0$, where C_1, C_0 are connected virtual Lie subgroups of $\mathrm{GL}(V)$, C_0 is normal in C and $\dim C_1 = 1$, $\dim C_0 = \dim C - 1$ (Problem 1.4.7). By the inductive hypothesis we may assume that the closed set $\Omega_0 = \{f \in \Omega\colon gf = f \text{ for all } g \in C_0\}$ is non-empty. The subgroup C_1 transforms Ω_0 into itself. It is clear from Problem 33 that under the embedding $j\colon \mathrm{F}(V) \to \mathrm{P}(W)$ the group $C_1 = \{\exp tX\colon t \in \mathbb{R}\}$ where $X \in \mathfrak{gl}(V)$, is identified with the group of projective transformations $\{\exp tY\colon t \in \mathbb{R}\}$, where $Y = (dR)X$. By hypothesis all characteristic roots of X are real. Since R is equivalent to a subrepresentation of a power $(\mathrm{Id})^s$ of the identity

representation, so is Y. Lemma 3 implies that there exists a flag $f_0 \in \Omega_0$ invariant with respect to C_1 and therefore with respect to C.

35. Consider the G-action on $\mathrm{F}(\mathfrak{g})$ defined by the adjoint representation. Let D be the maximal triangular subgroup of G described in Problem 38 and let $f_0 \in \mathrm{F}(\mathfrak{g})$ be a D-invariant flag. It follows from Theorem 6 that the orbit $\Omega = Gf_0 \subset \mathrm{F}(\mathfrak{g})$ is compact. Now let C be any maximal triangular subgroup of G. Applying Problem 34 to the linear group $\operatorname{Ad} C$ we get the flag $f_1 \in \Omega$ invariant with respect to C. If $f_1 = gf_0$, where $g \in G$, then $C = gDg^{-1}$.

Chapter 6
Levi Decomposition

In this chapter, which owing to its brevity is not divided into sections, we prove Levi's theorem on the decomposition of an arbitrary Lie algebra into a semidirect sum of a solvable ideal (radical) and a semisimple subalgebra and the theorem on the uniqueness of this decomposition due to A.I. Malcev. Levi's theorem implies the result which concludes the classical Lie group theory—the existence of a Lie group with an arbitrary given tangent algebra. Next we will consider an analogue of Levi decomposition for algebraic groups.

1°. Levi's Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $K = \mathbb{C}$ or $\mathbb{R}$. A subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is called a *Levi subalgebra* if $\mathfrak{g}$ splits into the semidirect sum

$$\mathfrak{g} = \mathfrak{rad}\,\mathfrak{g} \oplus \mathfrak{l}. \tag{1}$$

Decomposition (1) is called the *Levi decomposition* of $\mathfrak{g}$.

Problem 1. The natural homomorphism $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$ isomorphically maps any Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ onto the semisimple Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$. Any Levi subalgebra is a maximal semisimple subalgebra of $\mathfrak{g}$.

Problem 2. An automorphism of a Lie algebra transforms any of its Levi subalgebras into a Levi subalgebra.

In this section we will prove the following.

Theorem 1 (Levi). *Any finite-dimensional Lie algebra $\mathfrak{g}$ over $K = \mathbb{C}$ or $\mathbb{R}$ contains a Levi subalgebra.*

First, prove Theorem 1 when $\mathfrak{g}$ has a commutative radical and the center of $\mathfrak{g}$ is trivial.

Problem 3. The kernel of any derivation of a Lie algebra is a subalgebra.

It follows from Problem 3 that it suffices to construct a derivation $\delta \in \operatorname{Der}\mathfrak{g}$ which is the projection of $\mathfrak{g}$ onto $\mathfrak{rad}\,\mathfrak{g}$, i.e. such that $\delta(\mathfrak{g}) \subset \mathfrak{rad}\,\mathfrak{g}$ and $\delta(x) = x$ ($x \in \mathfrak{rad}\,\mathfrak{g}$).

Problem 4. Suppose there exists a projection h of $\mathfrak{g}$ onto $\mathfrak{rad}\,\mathfrak{g}$ belonging to the normalizer of the subalgebra $\operatorname{ad}\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$. If $\mathfrak{z}(\mathfrak{g}) = 0$ then $\mathfrak{g}$ contains a Levi subalgebra.

Now let us construct a projection $h\colon \mathfrak{g} \to \mathfrak{rad}\,\mathfrak{g}$ satisfying the conditions of Problem 4. Let $P = \{v \in \mathfrak{gl}(\mathfrak{g})\colon v(\mathfrak{g}) = \mathfrak{rad}\,\mathfrak{g}$ and $v|\mathfrak{rad}\,\mathfrak{g}$ is a scalar operator$\}$ and $Q = \{v \in P\colon v|\mathfrak{rad}\,\mathfrak{g} = 0\}$. Set $R = \operatorname{ad}(\mathfrak{rad}\,\mathfrak{g}) = \{\operatorname{ad} x\colon x \in \mathfrak{rad}\,\mathfrak{g}\}$.

Problem 5. The sets P, Q, R are subspaces of $\mathfrak{gl}(\mathfrak{g})$ such that $R \subset Q \subset P$ and $\dim P - \dim Q = 1$.

Consider the linear representation ρ of $\mathfrak{g}$ in the space $\mathfrak{gl}(\mathfrak{g})$ defined by the formula

$$\rho(x) = \operatorname{ad}(\operatorname{ad} x) \qquad (x \in \mathfrak{g}).$$

Problem 6. The subspaces P, Q, R are $\rho(\mathfrak{g})$-invariant and $\rho(x)P \subset Q$ for all $x \in \mathfrak{rad}\,\mathfrak{g}$. If $\mathfrak{rad}\,\mathfrak{g}$ is commutative then $\rho(x)P \subset R$ for all $x \in \mathfrak{rad}\,\mathfrak{g}$.

Now suppose $\mathfrak{rad}\,\mathfrak{g}$ is commutative and $\mathfrak{z}(\mathfrak{g}) = 0$. Problem 6 implies that ρ induces a representation $\hat{\rho}$ of $\mathfrak{s} = \mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$ in P/R such that $\hat{\rho}(\xi)(P/R) \subset Q/R$ for all $\xi \in \mathfrak{s}$. By Problem 5 $\dim P/R - \dim Q/R = 1$. Since $\mathfrak{s}$ is semisimple, $\hat{\rho}$ is completely reducible (Corollary 3 of Theorem 5.2.13). Therefore there exists $v_0 \in P \backslash Q$, such that $\hat{\rho}(\xi)(v_0 + R) = 0$ for all $\xi \in \mathfrak{s}$. This means that $[\operatorname{ad} x, v_0] \in R \subset \operatorname{ad} \mathfrak{g}$ for all $x \in \mathfrak{g}$, i.e. v_0 normalizes $\operatorname{ad} \mathfrak{g}$. Furthermore, $v_0|\mathfrak{rad}\,\mathfrak{g} = \lambda E$, where $\lambda \neq 0$, and the operator $h = v_0/\lambda$ satisfies the conditions of Problem 4. Therefore Theorem 1 is proved under the above assumptions.

Notice that Problem 5.2.30 implies that Theorem 1 holds in another particular case: when $\mathfrak{rad}\,\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$.

To prove Levi's theorem in the general case we will need two properties of the radical of a Lie algebra.

Problem 7. An ideal $\mathfrak{h} \subset \mathfrak{g}$ contains $\mathfrak{rad}\,\mathfrak{g}$ if and only if $\mathfrak{g}/\mathfrak{h}$ is semisimple.

Problem 8. Let $\mathfrak{r}$ be a solvable ideal of $\mathfrak{g}$. Then $\mathfrak{rad}(\mathfrak{g}/\mathfrak{r}) = (\mathfrak{rad}\,\mathfrak{g})/\mathfrak{r}$. The image of any Levi subalgebra of $\mathfrak{g}$ under the natural homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$ is a Levi subalgebra of $\mathfrak{g}/\mathfrak{r}$.

Now we prove Theorem 1 by induction in $\dim(\mathfrak{rad}\,\mathfrak{g})$. Suppose it holds for Lie algebras with radicals of dimensions $< \dim(\mathfrak{rad}\,\mathfrak{g})$. Consider, separately, the cases of non-commutative and commutative radical.

Let $(\mathfrak{rad}\,\mathfrak{g})' \neq 0$. Then $0 < \dim \mathfrak{rad}\,\mathfrak{g}/\mathfrak{rad}\,\mathfrak{g})') < \dim(\mathfrak{rad}\,\mathfrak{g})$ and $(\mathfrak{rad}\,\mathfrak{g})'$ is an ideal of $\mathfrak{g}$. By Problem 8 $\mathfrak{rad}\,\mathfrak{g}/(\mathfrak{rad}\,\mathfrak{g})'$ is the radical of $\mathfrak{g}_1 = \mathfrak{g}/(\mathfrak{rad}\,\mathfrak{g})'$. Therefore $\mathfrak{g}_1$ contains a Levi subalgebra $\mathfrak{l}_1$. Let $\mathfrak{g}_2 = \pi^{-1}(\mathfrak{l}_1) \subset \mathfrak{g}$, where $\pi\colon \mathfrak{g} \to \mathfrak{g}_1$ is the natural homomorphism. Then $\mathfrak{g}_2/(\mathfrak{rad}\,\mathfrak{g})' = \mathfrak{l}_1$ so that $(\mathfrak{rad}\,\mathfrak{g})'$ is the radical of $\mathfrak{g}_2$ by Problem 7. Applying the inductive hypothesis to $\mathfrak{g}_2$ we see that $\mathfrak{g}_2$ contains a Levi subalgebra $\mathfrak{l}$. Clearly, $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$.

Let $\mathfrak{rad}\,\mathfrak{g}$ be commutative. By what we have already proved we may assume that $\dim \mathfrak{z}(\mathfrak{g}) > 0$. Then $\dim(\mathfrak{rad}\,\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) < \dim(\mathfrak{rad}\,\mathfrak{g})$. By Problem 8 $\mathfrak{rad}\,\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is the radical of $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. By the inductive hypothesis $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ contains a Levi subalgebra $\mathfrak{l}_1$. If $\mathfrak{g}_1$ is the preimage of $\mathfrak{l}_1$ with respect to the natural homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ then $\mathfrak{z}(\mathfrak{g}) = \mathfrak{rad}\,\mathfrak{g}_1$. By Problem 5.2.30 $\mathfrak{g}_1$ contains a Levi subalgebra which is clearly a Levi subalgebra of $\mathfrak{g}$.

2°. Existence of a Lie Group with the Given Tangent Algebra. In this section we will make use of Theorem 1 to prove the following theorem which is one of the fundamental facts of the Lie group theory.

Theorem 2. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra* (*over* $\mathbb{C}$ *or* $\mathbb{R}$), $\mathfrak{l}$ *its Levi subalgebra. Then there exists a simply connected Lie group* G (*either complex or real respectively*) *whose tangent algebra is isomorphic to* $\mathfrak{g}$. *Moreover,*

$$G = A \rtimes L, \tag{2}$$

where $A = \operatorname{Rad} G$, L *is a simply connected Lie subgroup with the tangent algebra* $\mathfrak{l}$.

Proof. As it was shown in 1.4.4 there exists a simply connected Lie group A whose tangent algebra is isomorphic to $\mathfrak{rad}\,\mathfrak{g}$. On the other hand, it is clear that there exists a simply connected Lie group L with the tangent algebra isomorphic to $\mathfrak{l}$ (e.g. the simply connected covering group for $\operatorname{Int}\mathfrak{l}$, see Problem 5.1.4). Applying Problem 1.2.39 to the adjoint representation $\operatorname{ad}: \mathfrak{l} \to \mathfrak{der}(\mathfrak{rad}\,\mathfrak{g})$ we get the simply connected Lie group $G = A \rtimes L$ with the tangent algebra $(\mathfrak{rad}\,\mathfrak{g}) \oplus \mathfrak{l} = \mathfrak{g}$. □

3°. Malcev's Theorem. Our goal is the proof of the following statement.

Theorem 3 (A.I. Malcev [43]). *Let* $\mathfrak{l}$ *be a Levi subalgebra of* $\mathfrak{g}$. *For any semisimple subalgebra* $\mathfrak{s} \subset \mathfrak{g}$ *there exists* $\varphi \in \operatorname{Int}\mathfrak{g}$ *such that* $\varphi(\mathfrak{s}) \subset \mathfrak{l}$. *The automorphism* φ *can be chosen from the connected virtual Lie subgroup of* $\operatorname{Int}\mathfrak{g}$ *with the tangent algebra* $\operatorname{ad}(\mathfrak{rad}\,\mathfrak{g})$.

To prove it we will need an embedding of the group of affine transformations of an affine space into the group of linear transformations of a vector space of dimension greater by 1. Let V be a vector space over $K = \mathbb{C}$ or $\mathbb{R}$. Consider the vector space $W = V \oplus K$. The affine hyperplane $\mathbb{A} = (V, 1) \subset W$ is an affine space with the associated vector space V. Consider the subgroup $G(W; W, V) \subset \mathrm{GL}(W)$ consisting of transformations preserving V and inducing on W/V the identity transformation (see Example 3 of 3.1.1°).

Problem 9. The subgroup $G(W; W, V)$ coincides with the subgroup of all invertible linear transformations of W preserving $\mathbb{A}$. If $X \in G(W; W, V)$ then X induces an affine transformation of $\mathbb{A}$. Conversely, any affine transformation of $\mathbb{A}$ is obtained in this way from a uniquely determined element of $G(W; W, V)$.

Therefore the group $\mathrm{GA}(\mathbb{A})$ is naturally identified with the subgroup $G(W; W, V) \subset \mathrm{GL}(W)$.

Lemma 1. *If all finite-dimensional linear representations of a Lie group* H *are completely reducible then any affine action of* H *has a fixed point.*

Proof. Let $R: H \to \mathrm{GA}(\mathbb{A})$ be an affine H-action. By Problem 9 R may be considered as a linear representation of H in the space W so that V is an invariant subspace. The complete reducibility implies that there exists a vector $v_0 \in \mathbb{A}$, such that $R(h)v_0 = cv_0$, where $c \in k$, for any $h \in H$. Since $R(h)v_0 \in \mathbb{A}$, then $c = 1$, hence v_0 is a fixed point for R. □

Proof of Theorem 3. First suppose that $\mathfrak{rad}\,\mathfrak{g}$ is commutative. Consider a simply connected Lie group G with the tangent algebra $\mathfrak{g}$ constructed in 2°. Its radical

$A = \operatorname{Rad} \mathfrak{g}$ is a vector group. A connected semisimple virtual Lie subgroup $S \subset G$ corresponds to the subalgebra $\mathfrak{s}$ by Theorem 1.2.8. Consider the affine action $\tilde{R}$ of G in A defined in Problem 5.2.16. Since all linear representations of S are completely reducible (Corollary 2 of Theorem 5.2.13), Lemma 1 implies that S has a fixed point in A. As in 5.2.2° we derive from here that $aSa^{-1} \subset L$ for some $a \in A$. Therefore $(\operatorname{Ad} a)\mathfrak{s} \subset \mathfrak{l}$. It remains to notice that $\operatorname{Ad} a = \exp(\operatorname{ad})$, where $z \in \mathfrak{rad}\, \mathfrak{g}$ is an element such that $\exp z = a$.

Now consider the general case and apply the induction in $\dim(\mathfrak{rad}\, \mathfrak{g})$. Suppose the theorem is proved for all Lie algebras whose radical is of dimension $< \dim(\mathfrak{rad}\, \mathfrak{g})$. Set $\mathfrak{g}_1 = \mathfrak{g}_1/(\mathfrak{rad}\, \mathfrak{g})'$ and let $\mathfrak{l}_1$, $\mathfrak{s}_1$ be the projections of $\mathfrak{l}$, $\mathfrak{s}$ into $\mathfrak{g}_1$. By Problem 8 $\mathfrak{l}_1$ is a Levi subalgebra of $\mathfrak{g}_1$ having the commutative radical $\mathfrak{rad}\, \mathfrak{g}/(\mathfrak{rad}\, \mathfrak{g})'$. Therefore there exists $z_1 \in \mathfrak{rad}\, \mathfrak{g}$ such that $\exp \operatorname{ad}(z_1 + (\mathfrak{rad}\, \mathfrak{g})')\mathfrak{s}_1 \subset \mathfrak{l}_1$ implying $\exp(\operatorname{ad} z_1)\mathfrak{s} \subset (\mathfrak{rad}\, \mathfrak{g})' + \mathfrak{l}$. Since $\dim(\mathfrak{rad}\, \mathfrak{g})' < \dim(\mathfrak{rad}\, \mathfrak{g})$, we may apply the inductive hypothesis to $\mathfrak{g}_2 = (\mathfrak{rad}\, \mathfrak{g})' + \mathfrak{l} \subset \mathfrak{g}$. Therefore there exist $z_2, \ldots, z_r \in (\mathfrak{rad}\, \mathfrak{g})'$, such that $(\exp \operatorname{ad} z_r) \cdots (\exp \operatorname{ad} z_2)(\exp \operatorname{ad} z_1)\mathfrak{s} \subset \mathfrak{l}$. □

Corollary 1. *Any two Levi subalgebras of* $\mathfrak{g}$ *are transformed into each other by a product of automorphisms of the form* $\exp(\operatorname{ad} z)$, *where* $z \in \mathfrak{rad}\, \mathfrak{g}$.

Corollary 2. *Any maximal semisimple subalgebra of a Lie algebra is its Levi subalgebra.*

4°. Algebraic Levi Decomposition. In this section we consider algebraic groups over $\mathbb{C}$.

Let G be an algebraic group. By Problem 3.3.10 the radical $\operatorname{Rad} G$ of G is an irreducible solvable algebraic subgroup. Consider the unipotent radical of $\operatorname{Rad} G$, i.e. the set of all unipotent elements of this group (see 3.2.7°). We will call it the *unipotent radical* of G and denote by $\operatorname{Rad}_u G$.

Problem 10. $\operatorname{Rad}_u G$ is the largest unipotent normal subgroup of G.

Problem 11. An algebraic group is reductive if and only if its unipotent radical is trivial.

Problem 12. Let N be an algebraic normal subgroup of an algebraic group G. The algebraic group G/N is reductive if and only if $N \supset \operatorname{Rad}_u G$.

The *reductive Levi subgroup* of an algebraic group G is an algebraic subgroup $H \subset G$, such that

$$G = \operatorname{Rad}_u G \rtimes H. \tag{3}$$

Problem 13. Any reductive Levi subgroup H of an algebraic group G is a maximal reductive algebraic subgroup of this group and is isomorphic to $G/\operatorname{Rad}_u G$.

Problem 14. If a reductive algebraic subgroup $H \subset G$ satisfies $G = (\operatorname{Rad}_u G)\, H$, then H is a reductive Levi subgroup of G.

Problem 15. Let U be a unipotent algebraic normal subgroup of G. Then $\operatorname{Rad}_u(G/V) = (\operatorname{Rad}_u G)/V$. The image of a reductive Levi subgroup of G under the natural homomorphism $G \to G/H$ is a reductive Levi subgroup of G/U.

The decomposition (3) is called the *algebraic Levi decomposition* of G. Our goal is to prove the existence and the uniqueness (up to inner automorphisms) of an algebraic Levi decomposition.

Theorem 4. *In any algebraic group G there exists a reductive Levi subgroup.*

Proof of this theorem will be divided into two parts. First, we consider the case when the radical of G consists of unipotent elements and then the general case.

Suppose that $\operatorname{Rad}_u G = \operatorname{Rad} G$. In this case the proof will be carried out along the same lines as for Theorem 1, i.e. first we consider the subcases a) $\operatorname{Rad} G$ is commutative and $\mathfrak{z}(\mathfrak{g}) = 0$; b) $\operatorname{rad}\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$ and then reduce the general case to these two ones.

a) Let $\operatorname{Rad} G = \operatorname{Rad}_u G$ be commutative and $\mathfrak{z}(\mathfrak{g}) = 0$. Let $\mathfrak{h}$ be a Levi subalgebra of the tangent algebra $\mathfrak{g}$ of G existing by Theorem 1. Set $H = N(\mathfrak{h}) = \{g \in G : (\operatorname{Ad} g)\mathfrak{h} = \mathfrak{h}\}$. Clearly, H is an algebraic subgroup of G. Its tangent algebra is $\mathfrak{n}(\mathfrak{h}) = (\mathfrak{n}(\mathfrak{h}) \cap \operatorname{rad}\mathfrak{g}) \dotplus \mathfrak{h}$. Clearly, $\mathfrak{n}(\mathfrak{h}) \cap \operatorname{rad}\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) = 0$, so that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and H is semisimple. By Problem 14 it remains to prove that $G = (\operatorname{Rad} G)\cdot H$. To do this consider the action of G on the set of all Levi subalgebras of $\mathfrak{g}$ by inner automorphisms $a(g)$ $(g \in G)$. The stabilizer of $\mathfrak{h}$ is H and (by Theorem 3) the subgroup $\operatorname{Rad} G$ acts transitively on the set of all Levi subalgebras. This implies the required decomposition.

b) Let $\mathfrak{rad}\,\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$. Then $\mathfrak{g}$ is a reductive Lie algebra, i.e. $G^0 = (\operatorname{Rad} G)(G^0)'$ (Problem 5.2.3). In this case we apply the same arguments as in the proof of Theorem 5.2.5. Consider the algebraic group $G_1 = G/(G^0)'$. Clearly, G_1^0 is a unipotent commutative group. By Theorem 3.2.2 $G_1^0 \simeq \mathbb{C}^p$. By Lemma 5.2.1 $G_1 = G_1^0 \rtimes H_1$, where H_1 is a finite subgroup. The preimage H of H_1 with respect to the natural homomorphism $G \to G_1$ is a reductive Levi subgroup of G.

Problem 16. Prove Theorem 4 when $\operatorname{Rad}_u G = \operatorname{Rad} G$.

Now prove Theorem 4 in the general case. For this fix a maximal torus T in $\operatorname{Rad} G$. By Theorem 3.2.10 $\operatorname{Rad} G = \operatorname{Rad}_u G \rtimes T$. Set $G_1 = N(T)$.

Problem 17. We have $G = (\operatorname{Rad}_u G)G_1$.

Problem 18. $\operatorname{Rad}_u G_1$ coincides with $(\operatorname{Rad}_u G) \cap G_1$.

Now let us carry out the induction in $\dim(\operatorname{Rad}_u G)$. Suppose that Theorem 4 is proved for all algebraic groups whose unipotent radical is of dimension $< \dim(\operatorname{Rad}_u G)$. By Problem 18 $\operatorname{Rad}_u G_1 \subset \operatorname{Rad}_u G$. If $\dim(\operatorname{Rad}_u G_1) < \dim(\operatorname{Rad}_u G)$ then by the inductive hypothesis $G_1 = (\operatorname{Rad}_u G_1) \rtimes H$, where H is a reductive algebraic subgroup. Then problems 17, 18 and 14 imply that H is a reductive Levi subgroup of G. If $\dim \operatorname{Rad}_u G_1 = \dim \operatorname{Rad}_u G$, then by Problem 17 $G = G_1$ so that T is a normal subgroup of G. Problem 8 implies that the radical of the algebraic group $G_2 = G/T$ coincides with $(\operatorname{Rad} G)/T \simeq \operatorname{Rad}_u G$ and therefore consists of unipotent elements. By what we have proved above, G_2 possesses a reductive Levi subgroup H_2 which is actually semisimple. Let $p\colon G \to G_2$ be the natural homomorphism and $H = p^{-1}(H_2)$. Then $T = \operatorname{Rad} H$ (see Problem 7) whence H is a reductive algebraic subgroup by Problem 5.2.31. Clearly, H is a reductive Levi subgroup of G. Proof of Theorem 4 is completed. □

Theorem 5. *Let* $G = \operatorname{Rad}_u G \rtimes H$ *be an algebraic Levi decomposition of* G. *Then for any reductive algebraic subgroup* $Q \subset G$ *there exists* $u \in \operatorname{Rad}_u G$ *such that* $uQu^{-1} \subset H$.

Proof will be carried out along the same lines as that of Theorem 3. First prove Theorem 5 when the unipotent radical of G is commutative. By Theorem 3.2.2 $\operatorname{Rad}_u G$ is a vector group in this case. Therefore the argument used in 3° in the proof of Theorem 3 for the case of a commutative radical is applicable (Lemma 1 is applicable to Q thanks to Corollary 1 of Theorem 5.2.13).

Problem 19. Prove Theorem 5 in the general case. □

Corollary 1. *If* H_1 *and* H_2 *are two reductive Levi subgroups of an algebraic group* G *then there exists* $u \in \operatorname{Rad}_u G$, *such that* $uH_1u^{-1} = H_2$.

Corollary 2. *Any maximal reductive algebraic subgroup of an algebraic group is its reductive Levi subgroup.*

Exercises

Let G be a Lie group. A *Levi subgroup* of G is a virtual Lie subgroup $L \subset G$, such that $G = (\operatorname{Rad} G)L$, $\dim((\operatorname{Rad} G) \cap L) = 0$.

1) If L is a Levi subgroup of G then its tangent algebra $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$.
2) If G is connected then any of its virtual Lie subgroups whose tangent algebra is a Levi subgroup of $\mathfrak{g}$ is a Levi subgroup.
3) In a connected Lie group there always exists a connected Levi subgroup.
4) If L is a Levi subgroup of a Lie group G then for any connected semisimple virtual Lie subgroup $S \subset G$ there exists $g \in \operatorname{Rad} G$ such that $gSg^{-1} \subset L$.
5) In a connected Lie group all connected Levi subgroups are conjugate.
6) A connected virtual Lie subgroup L of the connected Lie group G is a Levi subgroup if and only if L is a maximal connected semisimple virtual Lie subgroup of G.
7) Let a (not necessarily connected) Lie group G is such that $\operatorname{Rad} G$ is commutative and $Z(G^0)$ is discrete. Then there exists a Levi subgroup L of G such that $G = \operatorname{Rad} G \rtimes L$ and $\operatorname{Rad} G$ is a vector group. (Hint: for L take $N(\mathfrak{l})$, where $\mathfrak{l}$ is a Levi subalgebra of the tangent algebra $\mathfrak{g}$ and make use of Theorem 3.)
8) In a simply connected Lie group G the radical is simply connected, any connected Levi subgroup L is a simply connected Lie subgroup and $G = \operatorname{Rad} G \rtimes L$.
9) Let G be a simply connected Lie group, $\mathfrak{h}$ an ideal of its Lie algebra $\mathfrak{g}$. Then G contains a connected normal Lie subgroup H with the tangent algebra $\mathfrak{h}$.

(Hint: consider a connected Lie group Q with the tangent algebra $\mathfrak{g}/\mathfrak{h}$ and the homomorphism $G \to Q$ whose differential is the natural homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.)

10) Let G be a unipotent (i.e. consisting of unipotent elements) real algebraic linear group. Then $\exp\colon \mathfrak{g} \to G$ is an isomorphism of real algebraic varieties. If G is commutative then $G \simeq \mathbb{R}^p$.

11) A real algebraic linear group G is unipotent if and only if so is $G(\mathbb{C})$.

Therefore we may speak about *unipotent real algebraic groups.*

12) Let G be a real algebraic group (which may be considered linear). The set $\operatorname{Rad}_u G$ of all unipotent elements contained in $\operatorname{Rad} G$ is a normal algebraic subgroup of G and $\operatorname{Rad}_u G(\mathbb{C}) = (\operatorname{Rad}_u G)(\mathbb{C})$.

$\operatorname{Rad}_u G$ is called the *unipotent radical* of G.

13) $\operatorname{Rad}_u G$ is the largest unipotent normal subgroup of a real algebraic group G.

14) Let N be a normal algebraic subgroup of a real algebraic group G. The algebraic group G/N is reductive if and only if $N \supset \operatorname{Rad}_u G$.

15) A real algebraic group has a finite number of connected components (in the usual topology). (Hint: make use of Exercises 14 and 5.2.5.)

A *reductive Levi subgroup* of a real algebraic group G is an algebraic subgroup $H \subset G$, such that $G = \operatorname{Rad}_u G \rtimes H$.

16) Any real algebraic group G has a reductive Levi subgroup. (Hint: reduce to the case when $\operatorname{Rad}_u G$ is commutative. In the latter case consider the group $G(\mathbb{C})$ and making use of Theorem 4 and Corollary of Theorem 3.4.1 prove the existence of a reductive Levi subgroup H of $G(\mathbb{C})$ such that $\sigma(H) = H$, where σ is the complex conjugation in $G(\mathbb{C})$ with respect to G.)

17) Prove the analogue of Theorem 5 for real algebraic groups.

Hints to Problems

4. Since $\operatorname{ad} h\colon \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g})$ induces a derivation of the algebra $\operatorname{ad}\mathfrak{g}$ and since $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{ad}\mathfrak{g}$ is an isomorphism, there exists $\delta \in \mathfrak{der}\,\mathfrak{g}$ such that

$$[h, \operatorname{ad} x] = \operatorname{ad} \delta(x) \qquad (x \in \mathfrak{g}).$$

Clearly δ is a projection of $\mathfrak{g}$ onto $\mathfrak{rad}\,\mathfrak{g}$.

10. Follows from the fact that any unipotent normal subgroup is connected and solvable (Theorem 3.3.7) and therefore is contained in $\operatorname{Rad} G$.

11. Make use of Problem 5.2.31.

14. Problem 11 implies that $(\operatorname{Rad}_u G) \cap H = \{e\}$.

16. Carry out the induction in $\dim(\operatorname{Rad} G)$ as in the proof of Theorem 1.

17. Consider the G-action on the set of maximal tori of $\operatorname{Rad} G$ via inner automorphhisms and take into account the fact that the subgroup $\operatorname{Rad} G \subset G$ acts transitively on this set (Problem 3.2.23).

18. Problem 17 implies that the algebraic group $G_1/(\operatorname{Rad}_u G) \cap G_1 \simeq G/\operatorname{Rad}_u G$ is reductive so that $(\operatorname{Rad}_u G) \cap G_1 \supset \operatorname{Rad}_u G_1$ by Problem 12. The converse inclusion follows from Problem 10.

19. Carry out the induction in $\dim(\operatorname{Rad}_u G)$ as in the proof of Theorem 3.

Reference Chapter

§ 1. Useful Formulae

1°. Weyl Groups and Exponents. Let G be a simply connected non-commutative simple complex Lie group, $\mathfrak{g}$ its tangent algebra, W the Weyl group, $(\alpha_0, \alpha_1, \dots, \alpha_l)$ the extended system of simple roots. Denote by n_0, $n_1, \dots, n_l$ the coefficients of the linear relation among $\alpha_0, \alpha_1, \dots, \alpha_l$ normed so that $n_0 = 1$ (see Table 6).

Let us arrange the positive roots of $\mathfrak{g}$ in a table in such a way that the k-th row consist of the roots of height k (see Exercise 4.3.25) with aligned last elements of all rows. The lengths of the rows of this table form a non-increasing sequence with the first row of length l. Let m_i be the number of elements in the i-th column. The numbers $m_1, \dots, m_l$ are called the *exponents* of G (or $\mathfrak{g}$). (See Table 4).

Define the *Killing—Coxeter* element $c \in W$:

$$c = r_1 \dots r_l,$$

where $r_1, \dots, r_l$ are the reflections associated with the simple roots. The element c does not depend on the numbering of simple roots up to conjugacy in W.

In this notation we have the following formulas.

(F1) The number of roots of $\mathfrak{g}$ equals $l \sum n_i = 2 \sum m_i$.

(F2) The order z of $Z(G)$ equals the number of 1's among n_i's.

(F3) The order of W equals

$$z l! \prod n_i = \prod (m_i + 1).$$

(F4) If g_k is the number of elements of W, whose space of fixed elements is of dimension $l - k$, then

$$\sum g_k t^k = \prod (1 + m_i t)$$

(F5) The order h of c (the *Coxeter number*) equals

$$\sum n_i = \max m_i + 1.$$

(F6) The eigenvalues of c are $\varepsilon^{m_1}, \dots, \varepsilon^{m_l}$, where ε is a primitive root of degree h of 1.

(F7) The algebra of W-invariant polynomials on a maximal diagonalizable subalgebra is freely generated by homogeneous polynomials of degrees $m_1 + 1, \ldots, m_l + 1$.

(F8) The Poincaré polynomial of G is $\prod (1 + t^{2m_i+1})$.

2°. Linear Representations of Complex Semisimple Lie Algebras. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We will use the following notation:

$R(\Lambda)$ is the irreducible linear representation of $\mathfrak{g}$ with the highest weight Λ;

$V(\Lambda)$ is the space of this representation;

$V_\lambda(\Lambda)$ is the weight subspace of $V(\Lambda)$ corresponding to λ;

$m_\lambda(\Lambda) = \dim V_\lambda(\Lambda)$ is the multiplicity of the weight λ in $R(\Lambda)$;

$\Lambda' = \nu(\Lambda)$ is the highest weight of $R(\Lambda)^*$;

$\Lambda_i = \Lambda(h_i)$ $(i = 1, \ldots, l)$ are the "numerical labels" of the weight Λ;

ρ is the half sum of positive roots (see Exercise 4.2.5 and Tables 1 and 2).

The following formulas are valid:

(F9) *H. Weyl's formula*

$$\dim R(\Lambda) = \prod_{\alpha>0} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

(F10) *Freudenthal's formula* (see [37] and Exercise 5 to § 9 of Chapter VIII in [3]):

$$[(\Lambda + \rho, \Lambda + \rho) - (\lambda + \rho, \lambda + \rho)]m_\lambda(\Lambda) = 2 \sum_{\alpha>0, k>0} (\lambda + k\alpha, \alpha) m_{\lambda+k\alpha}(\Lambda).$$

(F11) The multiplicity of $R(N)$ in $R(\Lambda) \otimes R(M)$ equals

$$\begin{aligned} &\dim\{v \in V_{N-\Lambda}(M) \colon dR(M)(e_i)^{\Lambda_i+1} v = 0 \quad \text{for} \quad i = 1, \ldots, l\} \\ &\quad = \dim\{v \in V_{\Lambda-M'}(N) \colon dR(N)(e_i)^{M'_i+1} v = 0 \quad \text{for} \quad i = 1, \ldots, l\} \end{aligned}$$

(see [47] and Exercise 14 to § 9 of Ch. VIII in [3]).

3°. Linear Representations of Real Semisimple Lie Algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra. We will use the following notation.

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a real linear representation, then $\rho(\mathbb{C})\colon \mathfrak{g} \to \mathfrak{gl}(V(\mathbb{C}))$ is the complex extension of ρ.

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex linear representation then $\bar{\rho}$ is the representation ρ considered in the space $\bar{V}$ obtained from V by the change of the sign of the complex structure and $\rho^{\mathbb{R}}$ is the representation ρ considered in the real space $V^{\mathbb{R}}$.

We will say that a complex representation ρ *admits a real* (*quaternionic*) *structure* if there is an antilinear operator J in V such that $J^2 = E$ (resp. $-E$) commuting with any $\rho(x)$ $(x \in \mathfrak{g})$. A real structure exists if and only if $\rho =$

$\rho_0(\mathbb{C})$, where $\rho_0\colon \mathfrak{g} \to \mathfrak{gl}(V^J)$ is a real representation, and a quaternionic structure is the same as a quaternionic vector space structure on V compatible with ρ.

The irreducible real representations of $\mathfrak{g}$ are divided into two classes (see [40]): a) irreducible representations ρ, for which $\rho(\mathbb{C})$ is irreducible (over $\mathbb{C}$); b) representations $\rho^{\mathbb{R}}$, where ρ is a complex irreducible representation that admits no real structure. In the class a) $\rho_1 \sim \rho_2 \Leftrightarrow \rho_1(\mathbb{C}) \sim \rho_2(\mathbb{C})$ and in the class b) $\rho_1^{\mathbb{R}} \sim \rho_2^{\mathbb{R}} \Leftrightarrow$ either $(\rho_1 \sim \rho_2)$ or $(\rho_1 \sim \bar{\rho}_2)$ (see Exercises 5.1.16 and 5.1.17). Therefore the description of the irreducible real representations reduces to the following two questions on irreducible complex representations: When $\rho_1 \sim \bar{\rho}_2$ (in particular, when $\rho \sim \bar{\rho}$)? Which irreducible representations such that $\rho \sim \bar{\rho}$ admit a real structure? These questions are answered in terms of highest weights.

The *highest weight of a real irreducible representation* ρ of $\mathfrak{g}$ is the highest weight Λ of the extension of this representation to $\mathfrak{g}(\mathbb{C})$; we will write $\rho = \rho(\Lambda)$, since Λ defines ρ up to an isomorphism (theorem 4.3.2). Let θ be the canonical involutive automorphism of $\mathfrak{g}(\mathbb{C})$ corresponding to the real form $\mathfrak{g}$ and $\tau = \eta(\theta)$ the corresponding automorphism of the system of simple roots. Then

$$\overline{\rho(\Lambda)} = \rho(\nu\tau(\Lambda)). \tag{F12}$$

In particular,

$$\rho(\Lambda) \sim \overline{\rho(\Lambda)} \Leftrightarrow \nu\tau(\Lambda) = \Lambda. \tag{F13}$$

Now let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible complex representation such that $\rho \sim \bar{\rho}$. Then there exists an invertible antilinear operator J in V, commuting with $\rho(x)$ $(x \in \mathfrak{g})$, such that $J^2 = cE$, where $c \in \mathbb{R}^*$. The number $\varepsilon(\rho) = \operatorname{sign} c = \pm 1$ does not depend on the choice of J and is called the *index of* ρ (see [41]).

Suppose $\tau = \mathrm{id}$, i.e. $\theta \in \operatorname{Int} \mathfrak{g}(\mathbb{C})$. Then $\rho(\Lambda) \sim \overline{\rho(\Lambda)}$ is expressed as $\nu\Lambda = \Lambda$ and the index is calculated via the formula

$$\varepsilon(\rho(\Lambda)) = (-1)^{2\Lambda(2u+\rho^\vee)}, \tag{F14}$$

where $\exp(2\pi i u) = \theta$ and $\rho^\vee = 1/2\sum_{\alpha>0} h_\alpha = \sum_{1 \leqslant i \leqslant l} \pi_{\alpha_i^\vee}$ (see [42]). In particular, for compact Lie algebras $\mathfrak{g}$ we have $\varepsilon(\rho(\Lambda)) = (-1)^{2\Lambda(\rho^\vee)} = 1$ or -1 depending on whether the nondegenerate bilinear form invariant with respect to ρ is symmetric or skew-symmetric, respectively (Exercise 4.3.12). If $\tau \neq \mathrm{id}$ and $\mathfrak{g}(\mathbb{C})$ is simple then $\rho(\Lambda) \sim \overline{\rho(\Lambda)}$ if and only if $\Lambda_{2p-1} = \Lambda_{2p}$ for $\mathfrak{g}(\mathbb{C}) = \mathfrak{so}_{4p}(\mathbb{C})$ and it is always so if $\mathfrak{g}(\mathbb{C})$ is of the type A_l $(l > 1)$, D_{2p+1}, E_6. The corresponding indices were calculated in [41].

In the following table listed are the indices of irreducible complex representations of non-compact real forms $\mathfrak{g}$ of simple complex Lie algebras; for $\mathfrak{g}$ not mentioned in the table $\varepsilon(\rho) = 1$:*

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* We are thankful to B.P. Komrakov for a correction of this table.

$\mathfrak{g}$	$\mathfrak{su}_{k,2p-k}$	$\mathfrak{u}_l^*(\mathbb{H})$	$\mathfrak{sl}_p(\mathbb{H})$	$\mathfrak{so}_{2k-1,2(l-k)+1}$
$\varepsilon(\rho(\Lambda))$	$(-1)^{(k+1)p\Lambda_{p-1}}$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2[l/2]-1}}$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2p-1}}$	$(-1)^{(k+1)+(l-1)(l-2)/2)(\Lambda_{l-1}+\Lambda_l)}$

$\mathfrak{g}$	$\mathfrak{so}_{2k,2(l-k)+1}$	$\mathfrak{so}_{2k,2(2p-k)}$	$\mathfrak{sp}_{k,l-k}$	EVI
$\varepsilon(\rho(\Lambda))$	$(-1)^{(k+l(l-1)/2)\Lambda_l}$	$(-1)^{(k+p)(\Lambda_{2p-1}+\Lambda_{2p})}$	$(-1)^{\Lambda_1+\Lambda_3+\cdots+\Lambda_{2[(l+1)/2]-1}}$	$(-1)^{\Lambda_1+\Lambda_3+\Lambda_7}$

Let $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{R}}$, where $\mathfrak{g}_0$ is a simple Lie algebra over $\mathbb{C}$. Making use of the normal real form of $\mathfrak{g}_0$ we may identify $\mathfrak{g}(\mathbb{C}) \simeq \mathfrak{g}_0 \oplus \bar{\mathfrak{g}}_0$ with $\mathfrak{g}_0 \oplus \mathfrak{g}_0$. A dominant weight of $\mathfrak{g}(\mathbb{C})$ is expressed in the form $\Lambda = (\Lambda_1, \Lambda^1)$, where Λ_1, Λ^1 are dominant weights of $\mathfrak{g}_0$. The condition $\rho(\Lambda) \sim \rho(\Lambda)$ is expressed by the identity $\Lambda_1 = \Lambda^1$ with $\varepsilon(\rho(\Lambda)) = 1$.

Finally, let us describe how to calculate the index of a representation of any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Let $\mathfrak{g} = \bigoplus_{1 \leqslant i \leqslant s} \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple, and $\Lambda = (\Lambda_1, \ldots, \Lambda_s)$, where Λ_i is a dominant weight of $\mathfrak{g}_i(\mathbb{C})$. Then $\rho(\Lambda) \sim \rho(\Lambda)$ if and only if $\rho(\Lambda_i) = \rho(\Lambda_i)$ for all $i = 1, \ldots, s$ and $\varepsilon(\rho(\Lambda)) = \prod_{1 \leqslant i \leqslant s} \varepsilon(\rho(\Lambda_i))$.

§2. Tables

Table 1. Weights and Roots. The weights of the groups B_l, C_l, D_l and F_4 are expressed in the table in terms of an orthonormal basis $(\varepsilon_1, \ldots, \varepsilon_l)$ of $\mathfrak{t}(\mathbb{Q})$. The weights of the groups A_l, E_7, E_8 and G_2 are expressed in terms of vectors $\varepsilon_1, \ldots, \varepsilon_{l+1} \in \mathfrak{t}(\mathbb{Q})^*$, such that $\sum \varepsilon_i = 0$. For these vectors

$$(\varepsilon_i, \varepsilon_i) = l/(l+1), \qquad (\varepsilon_i, \varepsilon_j) = -1/(l+1) \qquad \text{for} \quad i \neq j.$$

It is convenient to remember, however, that if $\sum a_i = 0$, then $(\sum a_i\varepsilon_i, \sum b_j\varepsilon_j) = \sum a_i b_i$. The weights of E_6 are expressed in terms of vectors $\varepsilon_1, \ldots, \varepsilon_6 \in \mathfrak{t}(\mathbb{C})^*$ constructed as for A_5 and of an auxiliary vector $\varepsilon \in \mathfrak{t}(\mathbb{Q})^*$, which is orthogonal to all ε_i and satisfies $(\varepsilon, \varepsilon) = 1/2$.

The indices $i, j, \ldots$ in the expression of any weight are assumed to be different.

In all cases the Weyl group contains all permutations of the vectors ε_i. For B_l, C_l and F_4 the Weyl group contains also all transformations of the form $\varepsilon_i \mapsto \pm\varepsilon_i$ and for D_l all such transformations with an even number of minus signs. The Weyl group of E_6 contains the transformation $\varepsilon_i \mapsto \varepsilon_i$, $\varepsilon \mapsto -\varepsilon$. The Weyl groups of E_7, E_8 and G_2 contain $-\mathrm{id}$.

In the column "Dynkin diagrams" the numbering of simple roots accepted in all tables is given.

In the column "Simple roots" given is also the highest root δ and in the column "Fundamental weights" there is also indicated their sum (equal to the half sum of positive roots).

Table 1

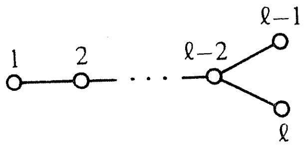
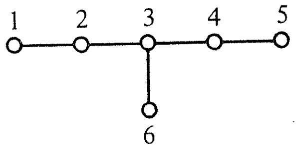
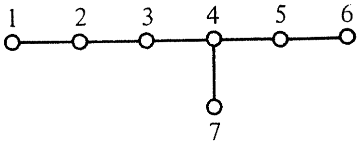
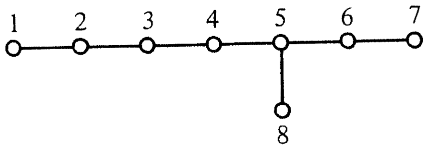
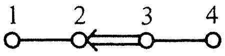
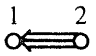
type of G	Dynkin diagrams	$\dim G$	Roots and simple roots
A_l $(l \geqslant 1)$		$l^2 + 2l$	$\varepsilon_i - \varepsilon_j$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\delta = \varepsilon_1 - \varepsilon_{l+1} = \pi_1 + \pi_l$
B_l $(l \geqslant 2)$		$2l^2 + l$	$\pm\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = \varepsilon_l$, $\delta = \varepsilon_1 + \varepsilon_2 = \pi_2$
C_l $(l \geqslant 2)$		$2l^2 + l$	$\pm\varepsilon_i \pm \varepsilon_j$, $\pm 2\varepsilon_i$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = 2\varepsilon_l$, $\delta = 2\varepsilon_1 = 2\pi_1$
D_l $(l \geqslant 3)$		$2l^2 - l$	$\pm\varepsilon_i \pm \varepsilon_j$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$, $\delta = \varepsilon_1 + \varepsilon_2 = \begin{cases} \pi_2 \text{ for } l \geqslant 4, \\ \pi_2 + \pi_3 \text{ for } l = 3 \end{cases}$
E_6		78	$\varepsilon_i - \varepsilon_j$, $\pm 2\varepsilon$, $\varepsilon_i + \varepsilon_j + \varepsilon_k \pm \varepsilon$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < 6)$, $\alpha_6 = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon$, $\delta = 2\varepsilon = \pi_6$
E_7		133	$\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < 7)$, $\alpha_7 = \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8$, $\delta = -\varepsilon_7 + \varepsilon_8 = \pi_6$
E_8		248	$\varepsilon_i - \varepsilon_j$, $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < 8)$, $\alpha_8 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8$, $\delta = \varepsilon_1 - \varepsilon_9 = \pi_1$

Table 1 (cont.)

type of G	Dynkin diagrams	dim G	Roots and simple roots
F_4		52	$\pm\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$ $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$ $\alpha_1 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2,$ $\alpha_2 = \varepsilon_4,$ $\alpha_3 = \varepsilon_3 - \varepsilon_4,$ $\alpha_4 = \varepsilon_2 - \varepsilon_3,$ $\delta = \varepsilon_1 + \varepsilon_2 = \pi_4$
G_2		14	$\varepsilon_i - \varepsilon_j$, $\pm\varepsilon_i$ $\alpha_1 = -\varepsilon_2,$ $\alpha_2 = \varepsilon_2 - \varepsilon_3$ $\delta = \varepsilon_1 - \varepsilon_3 = \pi_2$

Type of G	Fundamental weights	dim $R(\pi_1)$	Weights of $R(\pi_1)$
A_l $(l \geqslant 1)$	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i,$ $\rho = l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l$	$l+1$	ε_i
B_l $(l \geqslant 2)$	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i < l),$ $\pi_l = (\varepsilon_1 + \cdots + \varepsilon_l)/2$ $\rho = [(2l-1)\varepsilon_1 + (2l-3)\varepsilon_2 + \cdots + \varepsilon_l]/2$	$2l+1$	$\pm\varepsilon_i$, 0
C_l $(l \geqslant 2)$	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i,$ $\rho = l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l$	$2l$	$\pm\varepsilon_i$
D_l $(l \geqslant 3)$	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i < l-1),$ $\pi_{l-1} = (\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)/2$ $\pi_l = (\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l)/2$ $\rho = (l-1)\varepsilon_1 + (l-2)\varepsilon_2 + \cdots + \varepsilon_{l-1}$	$2l$	$\pm\varepsilon_i$
E_6	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i + \min\{i, 6-i\}\cdot\varepsilon \quad (i < 6),$ $\pi_6 = 2\varepsilon,$ $\rho = 5\varepsilon_1 + 4\varepsilon_2 + \cdots + \varepsilon_5 + 11\varepsilon$	27	$\varepsilon_i \pm \varepsilon,$ $-\varepsilon_i - \varepsilon_j$
E_7	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i + \min\{i, 8-i\}\cdot\varepsilon_8 \quad (i < 7),$ $\pi_7 = 2\varepsilon_8,$ $\rho = 6\varepsilon_1 + 5\varepsilon_2 + \cdots + \varepsilon_6 + 17\varepsilon_8$	56	$\pm(\varepsilon_i + \varepsilon_j)$

Table 1 (cont.)

Type of G	Fundamental weights	dim $R(\pi_1)$	Weights of $R(\pi_1)$
E_8	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i - \min\{i, 15 - 2i\} \cdot \varepsilon_9 \quad (i < 8)$, $\pi_8 = -3\varepsilon_9$, $\rho = 7\varepsilon_1 + 6\varepsilon_2 + \cdots + \varepsilon_7 - 22\varepsilon_9$	248	$\varepsilon_i - \varepsilon_j$, $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$, 0 (of multiplicity 8)
F_4	$\pi_1 = \varepsilon_1$, $\pi_2 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$ $\pi_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\pi_4 = \varepsilon_1 + \varepsilon_2$, $\rho = (11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4)/2$	26	$\pm\varepsilon_i$, $(\pm\varepsilon_1 \pm \varepsilon_2$ $\pm\, \varepsilon_3 \pm \varepsilon_4)/2$ 0 (of multiplicity 2)
G_2	$\pi_1 = \varepsilon_1$, $\pi_2 = \varepsilon_1 - \varepsilon_3$, $\rho = 2\varepsilon_1 - \varepsilon_3$	7	$\pm\varepsilon_i$, 0

Table 2. Matrices Inverse to Cartan Matrices. The matrix $(A^t)^{-1}$ inverse to the transposed Cartan matrix A is the matrix of the passage from a system of simple roots to the system of fundamental weights, i.e. its i-th column contains the coefficients of the expression of π_i via simple roots. In particular, the doubled sum of all of its columns (shown in the last column of the table) contains the coefficient of the expression of the sum 2ρ of positive roots via simple roots. The matrix $\operatorname{diag}\{d_1, \ldots, d_l\}\ (A^T)^{-1}$, where $d_i = (\alpha_i, \alpha_i)/2$ (these numbers are indicated in the column "d") is the Gram matrix of the system of fundamental weights.

Table 2

type of G	$(A^T)^{-1}$	d	2ρ
A_l	$\frac{1}{l+1}\begin{pmatrix} l & l-1 & l-2 & \ldots & 2 & 1 \\ l-1 & 2(l-1) & 2(l-2) & \ldots & 2\cdot 2 & 2 \\ (l-2) & 2(l-2) & 3(l-2) & \ldots & 3\cdot 2 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 2\cdot 2 & 3\cdot 2 & \ldots & (l-1)2 & l-1 \\ 1 & 2 & 3 & \ldots & l-1 & l \end{pmatrix}$	1 1 1 … 1 1	l $2(l-1)$ $3(l-2)$ … $(l-1)2$ l

Table 2 (cont.)

type of G	$(A^T)^{-1}$	d	2ρ
B_l	$\frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \ldots & 2 & 1 \\ 2 & 4 & 4 & \ldots & 4 & 2 \\ 2 & 4 & 6 & \ldots & 6 & 3 \\ \hdotsfor{6} \\ 2 & 4 & 6 & \ldots & 2(l-1) & l-1 \\ 2 & 4 & 6 & \ldots & 2(l-1) & l \end{pmatrix}$	1 1 1 … 1 1/2	$2l-1$ $2(2l-2)$ $3(2l-3)$ … $(l-1)(l+1)$ l^2
C_l	$\frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 4 & 4 & \ldots & 4 & 4 \\ 2 & 4 & 6 & \ldots & 6 & 6 \\ \hdotsfor{6} \\ 2 & 4 & 6 & \ldots & 2(l-1) & 2(l-1) \\ 1 & 2 & 3 & \ldots & l-1 & l \end{pmatrix}$	1 1 1 … 1 2	$2l$ $2(2l-1)$ $3(2l-2)$ … $(l-1)(l+2)$ $l(l+1)/2$
D_l	$\frac{1}{4}\begin{pmatrix} 4 & 4 & 4 & \ldots & 4 & 2 & 2 \\ 4 & 8 & 8 & \ldots & 8 & 4 & 4 \\ 4 & 8 & 12 & \ldots & 12 & 6 & 6 \\ \hdotsfor{7} \\ 4 & 8 & 12 & \ldots & 4(l-2) & 2(l-2) & 2(l-2) \\ 2 & 4 & 6 & \ldots & 2(l-2) & l & l-2 \\ 2 & 4 & 6 & \ldots & 2(l-2) & l-2 & l \end{pmatrix}$	1 1 1 … 1 1 1	$2l-2$ $2(2l-3)$ $3(2l-4)$ … $(l-2)(l+1)$ $(l-1)l/2$ $l(l-1)/2$
E_6	$\frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$	1 1 1 1 1 1	16 30 42 30 16 22
E_7	$\frac{1}{2}\begin{pmatrix} 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 4 & 8 & 10 & 12 & 8 & 4 & 6 \\ 5 & 10 & 15 & 18 & 12 & 6 & 9 \\ 6 & 12 & 18 & 24 & 16 & 8 & 12 \\ 4 & 8 & 12 & 16 & 12 & 6 & 8 \\ 2 & 4 & 6 & 8 & 6 & 4 & 4 \\ 3 & 6 & 9 & 12 & 8 & 4 & 7 \end{pmatrix}$	1 1 1 1 1 1 1	27 52 75 96 66 34 49

Table 2 (cont.)

type of G	$(A^T)^{-1}$	d	2ρ
E_8	$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}$	1 1 1 1 1 1 1 1	58 114 168 220 270 182 92 136
F_4	$\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$	1/2 1/2 1 1	22 42 30 16
G_2	$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$	1/3 1	10 6

Table 3. Centers, Outer Automorphisms and Bilinear Invariants. Here there are listed centers and groups of outer automorphisms of simply connected simple complex Lie groups.

The fifth column contains the order of the automorphism ν of the Dynkin diagram that transforms the numerical labels of the highest weight of an irreducible representation into the numerical labels of the highest weight of the dual representation (see Exercise 4.3.6).

In the space of the representation $R(\Lambda)$, there exists a nondegenerate symmetric or skew-symmetric invariant bilinear form if and only if $R(\Lambda)$ is self-adjoint, i.e. $\Lambda_{\nu(i)} = \Lambda_i$ for $i = 1, \ldots, l$ (see Exercises 4.3.9 and 4.3.7). This form is symmetric if and only if $\operatorname{Ker} R(\Lambda)$ contains the element of the center $Z(G) \simeq P^\vee/Q^\vee$ corresponding to the element $b \in P^\vee$, indicated in the last column, i.e. if $\Lambda(b) \in \mathbb{Z}$ (see Exercises 4.3.12 and 4.3.13).

For the groups E_8, F_4 and G_2 not mentioned in the table the centers and the groups of outer automorphisms are trivial and any their linear representation possesses a nondegenerate symmetric invariant bilinear form.

Table 3

Type of G	$Z(G)$ $\simeq P^\vee/Q^\vee$	The generators of $P^\vee/Q^\vee$	Aut G/Int G	$\lvert v \rvert$	b
A_l $(l > 1)$	$\mathbb{Z}_{l+1}$	$(h_1 + 2h_2 + \cdots + lh_l)/(l+1)$	$\mathbb{Z}_2$	2	$(h_1 + h_3 + \cdots + h_l)/2$ for $l = 2q+1$, 0 otherwise
A_1	$\mathbb{Z}_2$	$h_1/2$	$\{e\}$	1	$h_1/2$
B_l	$\mathbb{Z}_2$	$h_l/2$	$\{e\}$	1	$h_l/2$ for $l = 4q+1, 4q+2$, 0 otherwise
C_l	$\mathbb{Z}_2$	$(h_1 + h_3 + h_5 + \cdots)/2$	$\{e\}$	1	$(h_1 + h_3 + h_5 + \cdots)/2$
D_l $(l$ odd$)$	$\mathbb{Z}_4$	$(h_1 + h_3 + \cdots + h_{l-2})/2 + (h_{l-1} - h_l)/4$	$\mathbb{Z}_2$	2	$(h_{l-1} + h_l)/2$ for $l = 4q+3$, 0 otherwise
D_l $(l$ even$)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(h_1 + h_3 + \cdots + h_{l-1})/2$, $(h_{l-1} + h_l)/2$	$\mathbb{Z}_2$ for $l > 4$ S_3 for $l = 4$	1	$(h_{l-1} + h_l)/2$ for $l = 4q+2$, 0 otherwise
E_6	$\mathbb{Z}_3$	$(h_1 - h_2 + h_4 - h_5)/3$	$\mathbb{Z}_2$	2	0
E_7	$\mathbb{Z}_2$	$(h_1 + h_3 + h_7)/2$	$\{e\}$	1	$(h_1 + h_3 + h_7)/2$

Table 4. Exponents. On exponents $m_1, \ldots, m_l$ see 1.1°. Besides the exponents, the table contains the order $|h|$ of the Killing-Coxeter element and the order $|W|$ of the Weyl group.

Table 4

Type of $\mathfrak{g}$	$m_1, m_2, \ldots, m_l$	$\|h\|$	$\|W\|$
A	$1, 2, 3, \ldots, l$	$l+1$	$(l+1)!$
B_l, C_l	$1, 3, 5, \ldots, 2l-1$	$2l$	$2^l \cdot l!$
D_l	$1, 3, 5, \ldots, 2l-1, l-1$	$2(l-1)$	$2^{l-1} \cdot l!$
E_6	$1, 4, 5, 7, 8, 11$	12	$2^7 \cdot 3^4 \cdot 5$
E_7	$1, 5, 7, 9, 11, 13, 17$	18	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	30	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
F_4	$1, 5, 7, 11$	12	$2^7 \cdot 3^2$
G_2	$1, 5$	6	$2^2 \cdot 3$

Table 5. Decomposition of Tensor Products and Dimensions of Certain Representations. This table contains the decomposition into the irreducible components of tensor products and also of exterior and symmetric powers of certain irreducible linear representations of simple complex Lie groups. Besides, there are listed the dimensions of all the irreducible representations occuring in the formulas of the table. The following notation is used:

$R = R(\pi_1)$ the simplest representation,

$n = \dim R = l+1, 2l+1, 2l, 2l$ for the groups A_l, B_l, C_l, D_l, respectively,

$\mathrm{Ad} = R(\delta)$ the adjoint representation.

$1 = R(0)$ the unit (trivial) representation,

$\Delta(p,q)$, $p \geqslant q \geqslant 0$ the set of pairs $(x,y) \in \mathbb{Z}_+^2$ such that $x+y \leqslant p+q$, $x-y \geqslant p-q$, $x-y \equiv p-q \pmod 2$, see Fig. 2.

If a representation on the right-hand side of a formula is denoted by a meaningless symbol (e.g., $R(-\pi_1+\pi_2)$) it is meant to be zero.

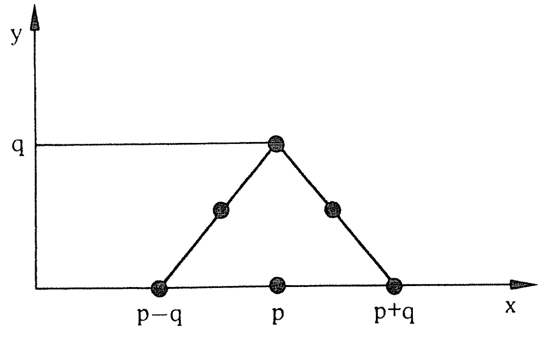


Fig. 2

Table 5

A_1

1. $S^pR = R(p\pi_1)$.
2. $R(p\pi_1)R(q\pi_1) = \sum_{0 \leqslant i \leqslant q} R((p+q-2i)\pi_1), \qquad p \geqslant q;$

$$S^2R(p\pi_1) = \sum_{i \geqslant 0} R((2p-4i)\pi_1).$$

$\dim R(p\pi_1) = p + 1$

$A_l, \quad l \geqslant 2$ $\qquad (n = l + 1)$

In the right hand sides of formulas we assume that $\pi_0 = \pi_n = 0$.

1. $\bigwedge^p R = R(\pi_p)$.
2. $S^pR = R(p\pi_1)$.
3. $R(\pi_p)R(\pi_q) = \sum_{i \geqslant 0} R(\pi_{p+i} + \pi_{q-i}), \qquad p \geqslant q;$

$$S^2R(\pi_p) = \sum_{i \geqslant 0} R(\pi_{p+2i} + \pi_{p-2i}).$$

4. $R(p\pi_1)R(\pi_q) = R(p\pi_1 + \pi_q) + R((p-1)\pi_1 + \pi_{q+1})$.
5. $R(p\pi_1)R(q\pi_1) = \sum_{0 \leqslant i \leqslant q} R((p+q-2i)\pi_1 + i\pi_2), \qquad p \geqslant q;$

$$S^2R(p\pi_1) = \sum_{i \geqslant 0} R((2p-4i)\pi_1 + 2i\pi_2).$$

6. $R(p\pi_1)\,\mathrm{Ad} = R((p+1)\pi_1 + \pi_l) + R(p\pi_1) + R((p-1)\pi_1 + \pi_2 + \pi_l) + R((p-2)\pi_1 + \pi_2)$.
7. $R(\pi_p)\,\mathrm{Ad} = R(\pi_1 + \pi_p + \pi_l) + R(\pi_1 + \pi_{p-1}) + R(\pi_{p+1} + \pi_l) + R(\pi_p), \qquad 2 \leqslant p \leqslant l-1.$
8. $R(p\pi_1)R(q\pi_l) = \sum_{i \geqslant 0} R((p-i)\pi_1 + (q-i)\pi_l)$.
9. $\bigwedge^2 \mathrm{Ad} = R(2\pi_1 + \pi_{l-1}) + R(\pi_2 + 2\pi_l) + \mathrm{Ad};$

$$S^2\,\mathrm{Ad} = \begin{cases} R(2\pi_1 + 2\pi_l) + R(\pi_2 + \pi_{l-1}) + \mathrm{Ad} + 1, & l \geqslant 3, \\ R(2\pi_1 + 2\pi_2) + \mathrm{Ad} + 1, & l = 2. \end{cases}$$

Λ	$\dim R(\Lambda)$
π_p	$\binom{n}{p}$
$p\pi_1$	$\binom{n+p-1}{p}$
$\pi_p + \pi_q$	$\frac{p-q+1}{p+1}\binom{n}{p}\binom{n+1}{q}, \qquad p \geqslant q$
$p\pi_1 + \pi_q$	$\frac{q}{p+q}\binom{n+p}{p}\binom{n}{q}$
$p\pi_1 + q\pi_l$	$\frac{n+p+q-1}{n-1}\binom{n+p-2}{p}\binom{n+q-2}{q}$
$p\pi_1 + q\pi_2$	$\frac{p+1}{p+q+1}\binom{n+p+q-1}{p+q}\binom{n+q-2}{q}$
$p\pi_1 + \pi_q + \pi_l$	$\frac{n(n-q)q}{(p+q)(n+p)}\binom{n+p+1}{p}\binom{n+1}{q}$

Table 5 (cont.)

$\boxed{B_l, \quad l \geqslant 2}$ $\qquad (n = 2l + 1)$

Notation:

$$\hat{\pi}_p = \begin{cases} \pi_p & \text{for } 1 \leqslant p \leqslant l - 1, \\ 2\pi_l & \text{for } p = l, l+1, \\ \pi_{n-p} & \text{for } l + 2 \leqslant p \leqslant 2l, \end{cases} \qquad \hat{\pi}_0 = \hat{\pi}_n = 0.$$

1. $\bigwedge^p R = R(\hat{\pi}_p)$.
2. $S^p R = \sum\limits_{i \geqslant 0} R((p - 2i)\pi_1)$.
3. $R(\hat{\pi}_p)R(\hat{\pi}_q) = \sum\limits_{(x,y) \in \Delta(p,q)} R(\hat{\pi}_x + \hat{\pi}_y), \qquad q \leqslant p \leqslant l;$

   $S^2 R(\hat{\pi}_p) = \sum\limits_{\substack{(x,y) \in \Delta(p,p) \\ x \equiv y (\mathrm{mod}\ 4)}} R(\hat{\pi}_x + \hat{\pi}_y), \qquad p \leqslant l.$
4. $R(p\pi_1)R(\hat{\pi}_q) = R(p\pi_1 + \hat{\pi}_q) + R((p-1)\pi_1 + \hat{\pi}_{q-1})$

   $+ R((p-1)\pi_1 + \hat{\pi}_{q+1}) + R((p-2)\pi_1 + \hat{\pi}_q), \qquad 2 \leqslant q \leqslant n - 2.$
5. $R(p\pi_1)R(q\pi_1) = \sum\limits_{(x,y) \in \Delta(p,q)} R((x - y)\pi_1 + y\pi_2);$

   $S^2 R(p\pi_1) = \sum\limits_{\substack{(x,y) \in \Delta(p,p) \\ x \equiv y \equiv 0 (\mathrm{mod}\ 2)}} R((x - y)\pi_1 + y\pi_2).$
6. $R(\hat{\pi}_p)R(\pi_l) = \sum\limits_{0 \leqslant i \leqslant p} R(\hat{\pi}_{p-i} + \pi_l) \qquad p \leqslant l.$
7. $R(p\pi_1)R(\pi_l) = R(p\pi_1 + \pi_l) + R((p-1)\pi_1 + \pi_l).$
8. $R(\pi_l)^2 = \sum\limits_{0 \leqslant i \leqslant l} R(\hat{\pi}_{l-i});$

   $S^2 R(\pi_l) = \sum\limits_{0 \leqslant i \leqslant l;\, i \equiv 0,\, 3 (\mathrm{mod}\ 4)} R(\hat{\pi}_{l-i}).$

Λ	$\dim R(\Lambda)$
$\hat{\pi}_p$	$\binom{n}{p}$
$p\pi_1$	$\frac{n + 2p - 2}{n + p - 2}\binom{n + p - 2}{p}$
$\hat{\pi}_p + \hat{\pi}_q$	$\frac{(p - q + 1)(n - p - q + 1)}{(p + 1)(n - p + 1)}\binom{n}{p}\binom{n+2}{q}, \qquad q \leqslant p \leqslant l.$
$p\pi_1 + \hat{\pi}_q$	$\frac{(n + 2p)q}{(p + q)(n + p - q)}\binom{n + p - 1}{p}\binom{n - 1}{q}, \qquad 1 \leqslant q \leqslant n - 1$
$(p - q)\pi_1 + q\pi_2$	$\frac{(p - q + 1)(n + 2p - 2)(n + p + q - 3)(n + 2q - 4)}{(p + 1)(n - 2)(n - 3)(n - 4)}\binom{n + p - 4}{p}\binom{n + q - 5}{q}$
π_l	2^l
$\hat{\pi}_p + \pi_l$	$2^l \frac{n - 2p + 1}{n - p + 1}\binom{n}{p}, \qquad p \leqslant l$
$p\pi_1 + \pi_l$	$2^l \binom{n + p - 2}{p}$

Table 5 (cont.)

Λ	$\dim R(\Lambda)$

$\boxed{C_l, \quad l \geqslant 2}$ $\qquad (n = 2l)$

In the right-hand sides of formulas we assume that $\pi_0 = 0$.

1. $\bigwedge^p R = \sum_{i \geqslant 0} R(\pi_{p-2i}), \qquad p \leqslant l.$
2. $S^p R = R(p\pi_1).$
3. $R(\pi_p)R(\pi_q) = \sum_{\substack{(x,y) \in \Delta(p,q) \\ x-y \leqslant n-p-q}} R(\pi_x + \pi_y), \qquad p \geqslant q;$

$$S^2 R(\pi_p) = \sum_{\substack{(x,y) \in \Delta(p,p) \\ x-y \leqslant n-2p \\ x \equiv y \equiv p \pmod 2}} R(\pi_x + \pi_y).$$

4. $R(p\pi_1)R(\pi_q) = R(p\pi_1 + \pi_q) + R((p-1)\pi_1 + \pi_{q+1})$

$$+ R((p-1)\pi_1 + \pi_{q-1}) + R((p-2)\pi_1 + \pi_q).$$

5. $R(p\pi_1)R(q\pi_1) = \sum_{(x,y) \in \Delta(p,q)} R((x-y)\pi_1 + y\pi_2);$

$$S^2 R(p\pi_1) = \sum_{\substack{(x,y) \in \Delta(p,p) \\ x-y \equiv 2p \pmod 4}} R((x-y)\pi_1 + y\pi_2).$$

Λ	$\dim R(\Lambda)$
π_p	$\dfrac{n-2p+2}{n-p+2}\dbinom{n+1}{p}$
$p\pi_1$	$\dbinom{n+p-1}{p}$
$\pi_p + \pi_q$	$\dfrac{(p-q+1)(n-2p+2)(n-p-q+3)(n-2q+4)}{(p+1)(n-p+2)(n-p+3)(n-q+4)}\dbinom{n+1}{p}\dbinom{n+3}{q}, p \geqslant q$
$p\pi_1 + \pi_q$	$\dfrac{(n-2q+2)q}{(p+q)(n+p-q+2)}\dbinom{n+p+1}{p}\dbinom{n+1}{q}$
$(p-q)\pi_1 + q\pi_2$	$\dfrac{(p-q+1)(n+p+q-1)}{(p+1)(n-1)}\dbinom{n+p-2}{p}\dbinom{n+q-3}{q}$

$\boxed{D_l, \quad l \geqslant 3}$ $\qquad (n = 2l)$

Notation:

$$\hat{\pi}_p = \begin{cases} \pi_p & \text{for } 1 \leqslant p \leqslant l-2, \\ \pi_{l-1} + \pi_l & \text{for } p = l-1, l+1, \\ \pi_{n-p} & \text{for } l+2 \leqslant p \leqslant 2l-1, \end{cases} \qquad \hat{\pi}_0 = \hat{\pi}_n = 0$$

$$R(\hat{\pi}_l) = R(2\pi_{l-1}) + R(2\pi_l),$$

$$R(\hat{\pi}_l + \Lambda) = R(2\pi_{l-1} + \Lambda) + R(2\pi_l + \Lambda),$$

$$R(2\hat{\pi}_l) = R(4\pi_{l-1}) + R(4\pi_l).$$

Formulas 1–5 are the same as for B_l.

Table 5 (cont.)

3a. $R(2\pi_l)R(\hat{\pi}_p) = \sum_{i \geqslant 0} R(2\pi_l + \hat{\pi}_{p-2i}) + \sum_{\substack{(x,y) \in \Delta(l,q) \\ x<l}} R(\hat{\pi}_x + \hat{\pi}_y), \qquad p \leqslant l-1.$

3b. $R(2\pi_l)R(2\pi_{l-1}) = \sum_{\substack{y \leqslant x < l \\ x \equiv y \equiv l-1 \pmod 2}} R(\hat{\pi}_x + \hat{\pi}_y).$

3c. $R(2\pi_l)^2 = R(4\pi_l) + \sum_{i \geqslant 1} R(2\pi_l + \hat{\pi}_{l-2i}) + \sum_{\substack{y \leqslant x < l \\ x \equiv y \equiv l \pmod 2}} R(\hat{\pi}_x + \hat{\pi}_y);$

$S^2R(2\pi_l) = R(4\pi_l) + \sum_{i \geqslant 1} R(2\pi_l + \hat{\pi}_{l-4i}) + \sum_{\substack{y \leqslant x < l \\ x \equiv y \equiv l \pmod 2 \\ x \equiv y \pmod 4}} R(\hat{\pi}_x + \hat{\pi}_y).$

4a. $R(p\pi_1)R(2\pi_l) = R(p\pi_1 + 2\pi_l) + R((p-1)\pi_1 + \hat{\pi}_{l-1}) + R((p-2)\pi_1 + 2\pi_l).$

6. $R(\hat{\pi}_p)R(\pi_l) = \sum_{i \geqslant 0} R(\hat{\pi}_{p-2i} + \pi_l) + \sum_{i \geqslant 0} R(\hat{\pi}_{p-2i-1} + \pi_{l-1}), \qquad p \leqslant l.$

6a. $R(2\pi_l)R(\pi_l) = R(3\pi_l) + \sum_{i \geqslant 1} R(\hat{\pi}_{l-2i} + \pi_l).$

7. $R(p\pi_1)R(\pi_l) = R(p\pi_1 + \pi_l) + R((p-1)\pi_1 + \pi_{l-1}).$

8. $R(\pi_l)R(\pi_{l-1}) = \sum_{i \geqslant 0} R(\hat{\pi}_{l-2i-1}).$

9. $R(\pi_l)^2 = R(2\pi_l) + \sum_{i \geqslant 1} R(\hat{\pi}_{l-2i});$

$S^2R(\pi_l) = R(2\pi_l) + \sum_{i \geqslant 1} R(\hat{\pi}_{l-4i}).$

Λ	$\dim R(\Lambda)$
$\hat{\pi}_p$	$\binom{n}{p}$
$p\pi_1$	$\frac{n+2p-2}{n+p-2}\binom{n+p-2}{p}$
$\hat{\pi}_p + \hat{\pi}_q$	$\frac{(p-q+1)(n-p-q+1)}{(p+1)(n-p+1)}\binom{n}{p}\binom{n+2}{q}, \qquad q \leqslant p \leqslant l$
$p\pi_1 + \hat{\pi}_q$	$\frac{(n+2p)q}{(p+q)(n+p-q)}\binom{n+p-1}{p}\binom{n-1}{q}, \qquad 1 \leqslant q \leqslant n-1.$
$(p-q)\pi_1 + q\pi_2$	$\frac{(p-q+1)(n+2p-2)(n+p+q-3)(n+2q-4)}{(p+1)(n-2)(n-3)(n-4)}\binom{n+p-4}{p}\binom{n+q-5}{q}$
$2\pi_l$	$\binom{n-1}{l-1}$
$2\pi_l + \hat{\pi}_p$	$\frac{2(l-p+1)^2}{(l+1)(n-p+2)}\binom{n-1}{l-1}\binom{n+1}{p}, \qquad p \leqslant l-1$
$4\pi_l$	$\frac{2}{(l+1)(l+2)}\binom{n-1}{l-1}\binom{n+1}{l}$
$p\pi_1 + 2\pi_l$	$\frac{l}{l+p}\binom{n-1}{l-1}\binom{n+p-1}{p}$

Table 5 (cont.)

Λ	$\dim R(\Lambda)$
π_l	2^{l-1}
$\hat{\pi}_p + \pi_l$	$2^{l-1} \cdot \dfrac{n-2p+1}{n-p+1} \dbinom{n}{p}, \qquad p \leqslant l$
$p\pi_1 + \pi_l$	$2^{l-1} \dbinom{n+p-2}{p}$
$3\pi_l$	$2^{l} \cdot \dfrac{1}{l+1} \dbinom{n-1}{l-1}$

$\boxed{E_6}$

1. $\bigwedge^2 R = R(\pi_2)$;

   $S^2 R = R(2\pi_1) + R^*$.
2. $RR^* = R(\pi_1 + \pi_5) + \mathrm{Ad} + 1$.
3. $R \cdot \mathrm{Ad} = R(\pi_1 + \pi_6) + R(\pi_2)^* + R$.
4. $\bigwedge^2 \mathrm{Ad} = R(\pi_3) + \mathrm{Ad}$;

   $S^2 \mathrm{Ad} = R(2\pi_6) + R(\pi_1 + \pi_5) + 1$.

   $(R = R(\pi_1), \qquad \mathrm{Ad} = R(\pi_6).)$

Λ	π_1	π_6	π_2	$2\pi_1$	$\pi_1 + \pi_5$	$\pi_1 + \pi_6$	$2\pi_6$	π_3
$\dim R(\Lambda)$	27	78	351	351	650	1728	2430	2925

$\boxed{E_7}$

1. $\bigwedge^2 R = R(\pi_2) + 1$;

   $S^2 R = R(2\pi_1) + \mathrm{Ad}$.
2. $R \cdot \mathrm{Ad} = R(\pi_1 + \pi_6) + R(\pi_7) + R$.
3. $\bigwedge^2 \mathrm{Ad} = R(\pi_5) + \mathrm{Ad}$;

   $S^2 \mathrm{Ad} = R(2\pi_6) + R(\pi_2) + 1$.

   $(R = R(\pi_1), \qquad \mathrm{Ad} = R(\pi_6).)$

Λ	π_1	π_6	π_7	$2\pi_1$	π_2	$\pi_1 + \pi_6$	$2\pi_6$	π_5
$\dim R(\Lambda)$	56	133	912	1463	1539	6480	7371	8645

$\boxed{E_8}$

1. $\bigwedge^2 R = R(\pi_2) + R$;

   $S^2 R = R(2\pi_1) + R(\pi_7) + 1$.

   $(R = \mathrm{Ad} = R(\pi_1).)$

Table 5 (cont.)

Λ	π_1	π_7	$2\pi_1$	π_2
$\dim R(\Lambda)$	248	3875	27 000	30 380

F_4

1. $\bigwedge^2 R = R(\pi_2) + \mathrm{Ad}$;

   $S^2 R = R(2\pi_1) + R + 1.$

2. $R \cdot \mathrm{Ad} = R(\pi_1 + \pi_4) + R(\pi_2) + R.$

3. $\bigwedge^2 \mathrm{Ad} = R(\pi_3) + \mathrm{Ad}$;

   $S^2 \mathrm{Ad} = R(2\pi_4) + R(2\pi_1) + 1.$

   ($R = R(\pi_1)$, $\mathrm{Ad} = R(\pi_4)$.)

Λ	π_1	π_4	π_2	$2\pi_1$	$\pi_1 + \pi_4$	$2\pi_4$	π_3
$\dim R(\Lambda)$	26	52	273	324	1053	1053	1274

G_2

1. $\bigwedge^2 R = \mathrm{Ad} + R$;

   $S^2 R = R(2\pi_1) + 1.$

2. $R \cdot \mathrm{Ad} = R(\pi_1 + \pi_2) + R(2\pi_1) + R.$

3. $\bigwedge^2 \mathrm{Ad} = R(3\pi_1) + \mathrm{Ad}$;

   $S^2 \mathrm{Ad} = R(2\pi_2) + R(2\pi_1) + 1.$

4. $R(2\pi_1) \cdot R = R(3\pi_1) + R(\pi_1 + \pi_2) + R(2\pi_1) + \mathrm{Ad} + R.$

   ($R = R(\pi_1)$, $\mathrm{Ad} = R(\pi_2)$.)

Λ	π_1	π_2	$2\pi_1$	$\pi_1 + \pi_2$	$2\pi_2$	$3\pi_1$
$\dim R(\Lambda)$	7	14	27	64	77	77

Table 6. Affine Dynkin Diagrams. The table lists connected affine Dynkin diagrams. On each diagram there are indicated the coefficients of the linear relation among vectors of the corresponding admissible system. They are positive integers normed so as to be relatively prime (see Problem 4.4.47).

Table 6

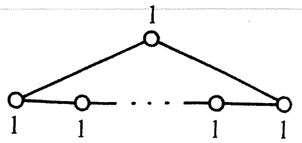
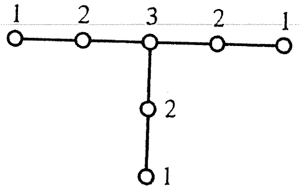
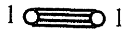
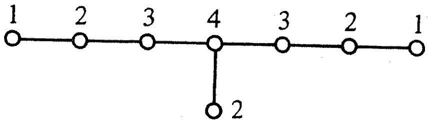
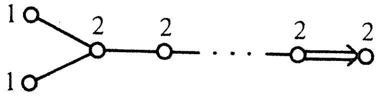
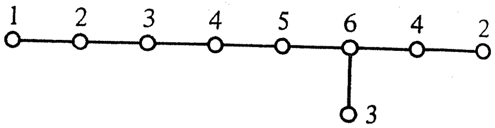
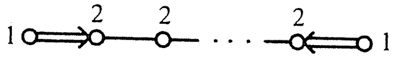
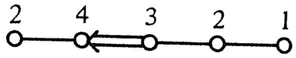
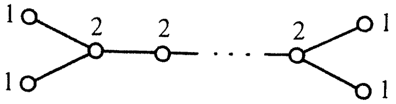
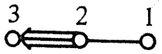
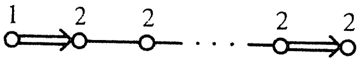
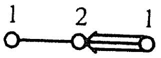
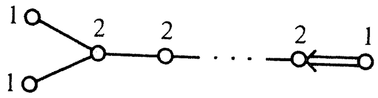
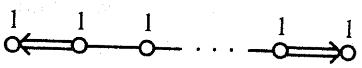
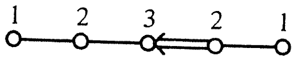
Type	Affine diagram	Type	Affine diagram
$A_l^{(1)}$ ($l \geqslant 2$)		$E_6^{(1)}$	
$A_1^{(1)}$		$E_7^{(1)}$	
$B_l^{(1)}$ ($l \geqslant 3$)		$E_8^{(1)}$	
$C_l^{(1)}$ ($l \geqslant 2$)		$F_4^{(1)}$	
$D_l^{(1)}$ ($l \geqslant 4$)		$G_2^{(1)}$	
$A_{2l}^{(2)}$ ($l \geqslant 2$)		$D_4^{(3)}$	
$A_2^{(2)}$			
$A_{2l-1}^{(2)}$ ($l \geqslant 3$)			
$D_{l+1}^{(2)}$ ($l \geqslant 2$)			
$E_6^{(2)}$			

Table 7. Involutive Automorphisms of Complex Simple Lie Algebras. In the table there are listed all the Kac diagram of all order 2 automorphisms θ of complex simple Lie algebras $\mathfrak{g}$ (up to conjugacy in the group Aut $\mathfrak{g}$). Since all the nonzero numerical labels of Kac diagrams of automorphisms of order 2 equal 1/2, it suffices to distinguish the vertices of the corresponding affine Dynkin diagram endowed with nonzero numerical labels. Therefore the numerical labels are omitted and the vertices with nonzero labels are black, the other being white. The vertices of an affine Dynkin diagram $L_n^{(k)}$ are numbered so that if $\Psi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is the corresponding numbered admissible system of vectors then $\Pi^\tau = \{(\alpha_0, 1/k), (\alpha_1, 0), \ldots, (\alpha_l, 0)\}$ is the system of simple roots of the pair $(\mathfrak{g}, \tau)$, where $\tau = \eta(\theta) \in \operatorname{Aut} \Pi$, and $\Pi_0 = \{\alpha_1, \ldots, \alpha_l\}$ is the system of simple roots of $\mathfrak{g}^\theta$ numbered as in Table 1. There are also indicated: the type of $\mathfrak{g}^\theta$ and the real form of $\mathfrak{g}$ corresponding to θ. The automorphisms θ are divided into the following three types (see Problem 5.1.38): type I—the inner automorphisms with a semisimple $\mathfrak{g}^\theta$, type II—the inner automorphisms with a nonsemisimple $\mathfrak{g}^\theta$, type III—the outer automorphisms.

Table 7

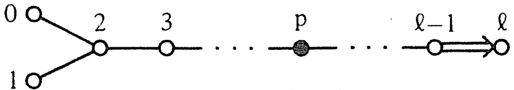
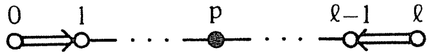
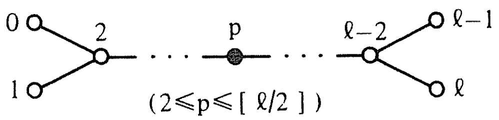
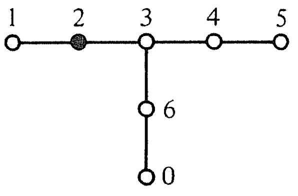
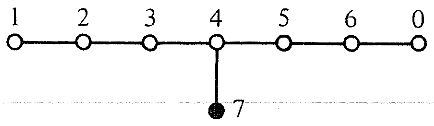
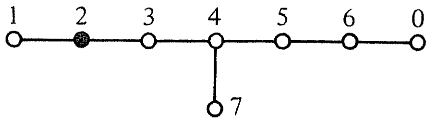
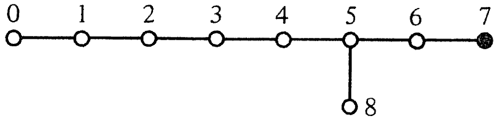
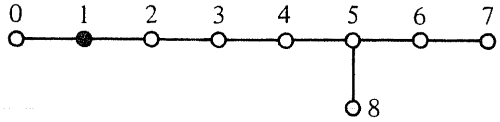
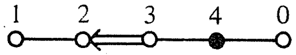
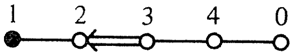
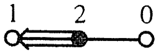
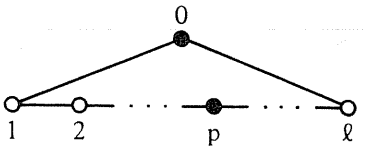
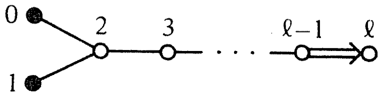
Type I				
$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^\theta$	Real form
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geqslant 3)$	$B_l^{(1)}$	 $(2 \leqslant p \leqslant l)$	$D_p \oplus B_{l-p}$	$\mathfrak{so}_{2p,2(l-p)+1}$
$\mathfrak{sp}_{2l}(\mathbb{C})$ $(l \geqslant 2)$	$C_l^{(1)}$	 $(1 \leqslant p \leqslant [l/2])$	$C_p \oplus C_{l-p}$	$\mathfrak{sp}_{p,l-p}$
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$D_l^{(1)}$	 $(2 \leqslant p \leqslant [l/2])$	$D_p \oplus D_{l-p}$	$\mathfrak{so}_{2p,2(l-p)}$
E_6	$E_6^{(1)}$		$A_1 \oplus A_5$	EII

Table 7 (cont.)

$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^\theta$	Real form
Type I				
E_7	$E_7^{(1)}$		A_7	EV
			$A_1 \oplus D_6$	EVI
E_8	$E_8^{(1)}$		D_8	$EVIII$
			$A_1 \oplus E_7$	EIX
F_4	$F_7^{(1)}$		$C_3 \oplus A_1$	FI
			B_4	FII
G_2	$G_2^{(1)}$		$A_1 \oplus A_1$	G
Type II				
$\mathfrak{sl}_{l+1}(\mathbb{C})$ $(l \geqslant 2)$	$A_l^{(1)}$	 $(1 \leqslant p \leqslant [\,(\ell+1)/2\,])$	$A_{p-1} \oplus$ $A_{l-p} \oplus \mathbb{C}$	$\mathfrak{su}_{p,l+1-p}$
$\mathfrak{sl}_2(\mathbb{C})$	$A_1^{(1)}$		$\mathbb{C}$	$\mathfrak{su}_{1,1}$
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geqslant 3)$	$B_l^{(1)}$		$B_{l-1} \oplus \mathbb{C}$	$\mathfrak{so}_{2,2l-1}$
$\mathfrak{sp}_{2l}(\mathbb{C})$ $(l \geqslant 2)$	$C_l^{(1)}$		$A_{l-1} \oplus \mathbb{C}$	$\mathfrak{sp}_{2l}(\mathbb{R})$

$\mathfrak{g}$	Type of affine diagram
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$D_l^{(1)}$
E_6	$E_6^{(1)}$
E_7	$E_7^{(1)}$
$\mathfrak{sl}_{2l+1}(\mathbb{C})$ $(l \geqslant 2)$	$A_{2l}^{(2)}$
$\mathfrak{sl}_3(\mathbb{C})$	$A_2^{(2)}$
$\mathfrak{sl}_{2l}(\mathbb{C})$ $(l \geqslant 3)$	$A_{2l-1}^{(2)}$
$\mathfrak{so}_{2l+2}(\mathbb{C})$ $(l \geqslant 2)$	$D_{l+1}^{(2)}$
E_6	$E_6^{(2)}$

Table 8. Mat
given matrix re
Cartan decomp
$\mathfrak{a} \subset \mathfrak{g}$. The mat

Table 7 (cont.)

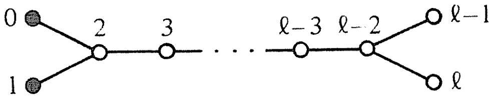
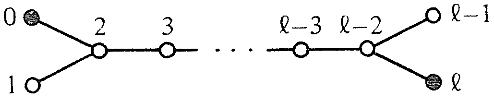
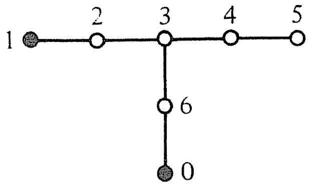
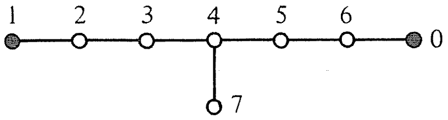
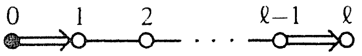
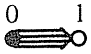
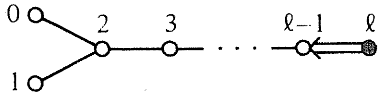
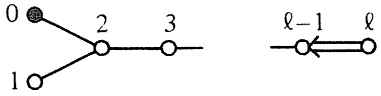
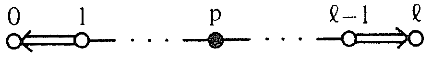
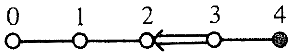
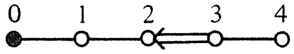
$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^\theta$	Real form
Type II				
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$D_l^{(1)}$		$D_{l-1} \oplus \mathbb{C}$	$\mathfrak{so}_{2,2l-2}$
			$A_{l-1} \oplus \mathbb{C}$	$\mathfrak{u}_l^*(\mathbb{H})$
E_6	$E_6^{(1)}$		$D_5 \oplus \mathbb{C}$	$EIII$
E_7	$E_7^{(1)}$		$E_6 \oplus \mathbb{C}$	$EVII$
Type III				
$\mathfrak{sl}_{2l+1}(\mathbb{C})$ $(l \geqslant 2)$	$A_{2l}^{(2)}$		B_l	$\mathfrak{sl}_{2l+1}(\mathbb{R})$
$\mathfrak{sl}_3(\mathbb{C})$	$A_2^{(2)}$		A_1	$\mathfrak{sl}_3(\mathbb{R})$
$\mathfrak{sl}_{2l}(\mathbb{C})$ $(l \geqslant 3)$	$A_{2l-1}^{(2)}$		D_l	$\mathfrak{sl}_{2l}(\mathbb{R})$
			C_l	$\mathfrak{sl}_l(\mathbb{H})$
$\mathfrak{so}_{2l+2}(\mathbb{C})$ $(l \geqslant 2)$	$D_{l+1}^{(2)}$	 $(0 \leqslant p \leqslant [\ell/2])$	$B_p \oplus B_{l-p}$	$\mathfrak{so}_{2p+1,2(l-p)+1}$
E_6	$E_6^{(2)}$		C_4	EI
			F_4	EIV

Table 8. Matrix Realizations of Classical Real Lie Algebras. In the table are given matrix realizations of real forms $\mathfrak{g}$ of classical complex Lie algebras, their Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the maximal $\mathbb{R}$-diagonalizable subalgebras $\mathfrak{a} \subset \mathfrak{g}$. The matrices are real for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_{p,q}$, $\mathfrak{sp}_n(\mathbb{R})$ and complex otherwise.

Table 8

$\mathfrak{g}$		$\mathfrak{k}$		$\mathfrak{p}$	$\mathfrak{a}$
Type	Matrix description	Type	description		
$\mathfrak{sl}_n(\mathbb{R})$	$X \in \mathfrak{gl}_n(\mathbb{R})$, $\operatorname{tr} X = 0$	$\mathfrak{so}_n$	$X^T = -X$	$X^T = X$, $\operatorname{tr} X = 0$	$\operatorname{diag}(x_1, \ldots, x_n)$, $x_1 + \cdots + x_n = 0$
$\mathfrak{gl}_n(\mathbb{H})$ $(n \geqslant 2)$	$\begin{array}{ccc} & n & n \\ n & X & Y \\ n & -\bar{Y} & \bar{X} \end{array}$ $\operatorname{Re} \operatorname{tr} X = 0$	$\mathfrak{sp}_n$	$\bar{X}^T = -X$, $Y^T = Y$	$\bar{X}^T = X$, $\operatorname{tr} X = 0$, $Y^T = -Y$	$\operatorname{diag}(x_1, \ldots, x_n, x_1, \ldots, x_n)$, $x_i \in \mathbb{R}, x_1 + \cdots + x_n = 0$
$\mathfrak{su}_{p,q}$ $(p \leqslant q)$	$\begin{array}{ccc} & p & q \\ p & X_1 & Y \\ q & \bar{Y}^T & X_2 \end{array}$ $\bar{X}_1^T = -X_1, \bar{X}_2^T = -X_2$, $\operatorname{tr} X_1 + \operatorname{tr} X_2 = 0$	$\mathfrak{su}_p \oplus \mathfrak{u}_q$	$Y = 0$	$X_1 = 0$, $X_2 = 0$	$\bigoplus_{1 \leqslant j \leqslant p} \mathbb{R}(E_{j,p+j} + E_{p+j,j})$
$\mathfrak{so}_{p,q}$ $(p \leqslant q)$	$\begin{array}{ccc} & p & q \\ p & X_1 & Y \\ q & Y^T & X_2 \end{array}$ $X_1^T = -X_1, X_2^T = -X_2$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	$Y = 0$	$X_1 = 0$, $X_2 = 0$	$\bigoplus_{1 \leqslant j \leqslant p} \mathbb{R}(E_{j,p+j} + E_{p+j,j})$

$\mathfrak{sp}_{2n}(\mathbb{R})$ $(n \geqslant 1)$	$\begin{array}{ccc} & n & n \\ n & \left(\begin{array}{c} X \\ Y_2 \end{array}\right. & \left.\begin{array}{c} Y_1 \\ -X^T \end{array}\right) \\ n & & \end{array}$ $Y_1^T = Y_1,\ Y_2^T = Y_2$	$\mathfrak{u}_n$	$X^T = -X,$ $Y_2 = -Y_1$	$X^T = X,$ $Y_2 = Y_1$	$\mathrm{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n)$
$\mathfrak{sp}_{p,q}$ $(p \leqslant q)$	$\begin{array}{c\|cccc} & p & q & p & q \\ \hline p & X_{11} & X_{12} & X_{13} & X_{14} \\ q & \bar{X}_{12}^T & X_{22} & X_{14}^T & X_{24} \\ p & -\bar{X}_{13} & \bar{X}_{14} & \bar{X}_{11} & -\bar{X}_{12} \\ q & \bar{X}_{14}^T & -\bar{X}_{24} & -X_{12}^T & \bar{X}_{22} \end{array}$ $\bar{X}_{11}^T = -X_{11},\ \bar{X}_{22}^T = -X_{22},$ $X_{13}^T = X_{13},\ X_{24}^T = X_{24}$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$	$X_{12} = X_{14} = 0$	$X_{11} = X_{13} = 0$ $X_{22} = X_{24} = 0$	$\bigoplus_{1 \leqslant j \leqslant p} \mathbb{R}(E_{j,p+j} + E_{p+j,j}$ $-E_{p+q+j,2p+q+j} - E_{2p+q+j,p+q+j})$
$\mathfrak{u}_n^*(\mathbb{H})$	$\begin{array}{ccc} & n & n \\ n & \left(\begin{array}{c} X \\ -\bar{Y} \end{array}\right. & \left.\begin{array}{c} Y \\ \bar{X} \end{array}\right) \\ n & & \end{array}$ $X^T = -X,\ Y^T = \bar{Y}$	$\mathfrak{u}_n$	$\bar{X} = X,$ $\bar{Y} = Y$	$\bar{X} = -X,$ $\bar{Y} = -Y$	$i\mathbb{R}(E_{12} - E_{21} - E_{n+1,n+2}$ $+ E_{n+2,n+1}) \oplus i\mathbb{R}(E_{34} - E_{43}$ $-E_{n+3,n+4} + E_{n+4,n+3}) \oplus \cdots$

Table 9. Real Simple Lie Algebras. In the table are listed noncompact real Lie algebras $\mathfrak{g}$ that do not admit a complex structure, i.e. the real forms of complex simple Lie algebras $\mathfrak{g}(\mathbb{C})$. The column "Type of Σ" contains the type of the system Σ of real roots. The column "r" describes the restriction map $r\colon \Pi_1 \to \Theta$, where Θ is the base of Σ. The simple roots from Π are denoted by α_j, those from Θ by λ_j; the numbering in both these systems is the same as in Table 1.

Table 9

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$	$\dim \mathfrak{p}$	$\operatorname{rk}_{\mathbb{R}} \mathfrak{g}$
$\mathfrak{sl}_{l+1}(\mathbb{C})$ $(l \geqslant 1)$	$\mathfrak{sl}_{l+1}(\mathbb{R})$	$\mathfrak{so}_{l+1}$	$l(l+1)/2$	$l(l+3)/2$	l
	$\mathfrak{sl}_{p+1}(\mathbb{H})$ $(l=2p+1,\ p\geqslant 1)$	$\mathfrak{sp}_{p+1}$	$(p+1)(2p+3)$	$p(2p+3)$	p
	$\mathfrak{su}_{p,l+1-p}$ $(1 \leqslant p \leqslant l/2)$	$\mathfrak{su}_p \oplus \mathfrak{u}_{l+1-p}$	$p^2+(l+1-p)^2-1$	$2p(l+1-p)$	p
	$\mathfrak{su}_{p,p}$ $(l=2p-1,\ p\geqslant 2)$	$\mathfrak{su}_p \oplus \mathfrak{u}_p$	$2p^2-1$	$2p^2$	p
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l\geqslant 1)$	$\mathfrak{so}_{p,2l+1-p}$ $(1\leqslant p\leqslant l)$	$\mathfrak{so}_p \oplus \mathfrak{so}_{2l+1-p}$	$p(2p+1)$ $+(2l+1-p)$ $\cdot(4l+3-2p)$	$p(2l+1-p)$	p
$\mathfrak{sp}_{2l}(\mathbb{C})$	$\mathfrak{sp}_{2l}(\mathbb{R})$	$\mathfrak{u}_l$	l^2	$l(l+1)$	l
	$\mathfrak{sp}_{p,l-p}$ $(1\leqslant p<\frac{1}{2}(l-1))$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_{l-p}$	$p(2p+1)+(l-p)$ $(2l-2p+1)$	$4p(l-p)$	p
	$\mathfrak{sp}_{p,p}$ $(l=2p)$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_p$	$2p(2p+1)$	$4p^2$	p

Table 9 (cont.)

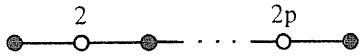
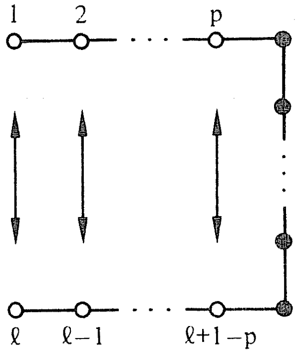
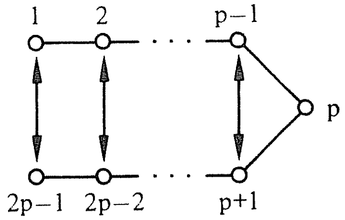
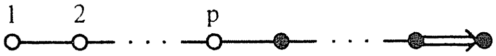
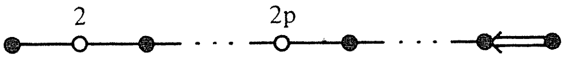
Satake diagram	Type of Σ	r	$\dim \mathfrak{g}_{\lambda_j}$	$\dim \mathfrak{g}_{2\lambda_j}$
	A_l	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant l)$	1	0
	A_p	$r(\alpha_{2j})=\lambda_j$ $(1\leqslant j\leqslant p)$	4	0
	BC_p	$r(\alpha_j) = r(\alpha_{l+1-j})$ $= \lambda_j$ $(1\leqslant j\leqslant p)$	$2\ (j \leqslant p+1)$	0
			$2(l+1-2p)\ (j=p)$	1
	C_p	$r(\alpha_j) = r(\alpha_{2p-j})$ $=\lambda_j$ $(1\leqslant j\leqslant p)$	$2\ (j \leqslant p-1)$ $1\ (j=p)$	0
	B_p	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant p)$	$1\ (j \leqslant p-1)$ $2(l-p)+1\ (j=p)$	0
	C_l	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant l)$	1	0
	BC_p	$r(\alpha_{2j})=\lambda_j$ $(1\leqslant j\leqslant p)$	$4\ (j \leqslant p-1)$	0
			$4(l-2p)\ (j=p)$	3
	C_p	$r(\alpha_{2j})=\lambda_j$ $(1\leqslant j\leqslant p)$	$4\ (j \leqslant p-1)$ $3\ (j=p)$	0

Table 9 (cont.)

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$	$\dim \mathfrak{p}$	$\mathrm{rk}_{\mathbb{R}}\,\mathfrak{g}$
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$\mathfrak{so}_{p,2l-p}$ $(1 \leqslant p \leqslant l-2)$	$\mathfrak{so}_p \times \mathfrak{so}_{2l-p}$	$p(p-1)/2+(2l-p)$ $\cdot(2l-p-1)/2$	$p(2l-p)$	p
	$\mathfrak{so}_{l-1,l+1}$	$\mathfrak{so}_{l-1} \times \mathfrak{so}_{l+1}$	$(l-1)(l-2)/2$ $+l(l+1)/2$	l^2-1	$l-1$
	$\mathfrak{so}_{l,l}$	$\mathfrak{so}_l \times \mathfrak{so}_l$	$l(l-1)$	l^2	l
	$\mathfrak{u}^*_{2p}(\mathbb{H})$ $(l=2p)$	$\mathfrak{u}_{2p}$	$4p^2$	$2p(2p-1)$	p
	$\mathfrak{u}^*_{2p+1}(\mathbb{H})$ $(l=2p+1)$	$\mathfrak{u}_{2p+1}$	$(2p+1)^2$	$2p(2p+1)$	p
E_6	EI	$\mathfrak{sp}_4$	36	42	6
	EII	$\mathfrak{su}_2 \oplus \mathfrak{su}_6$	38	40	4
	$EIII$	$\mathfrak{so}_{10} \oplus \mathbb{R}$	46	32	2
	EIV	F_4	52	26	2

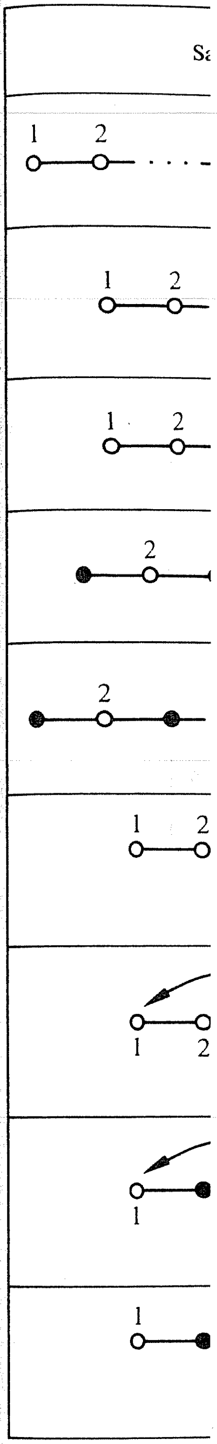


Table 9 (cont.)

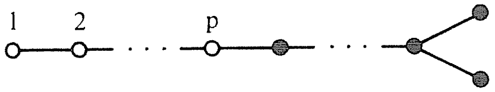
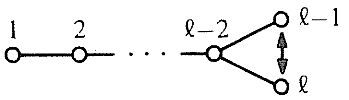
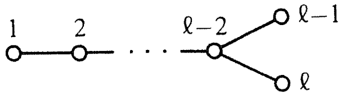
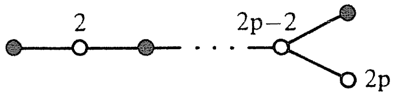
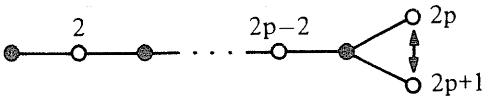
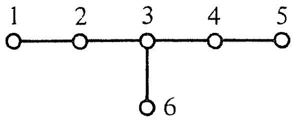
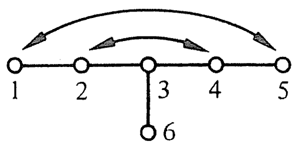
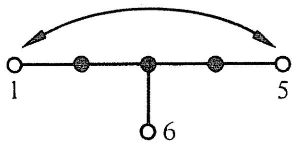
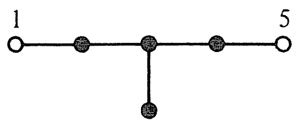
Satake diagram	Type of Σ	r	$\dim \mathfrak{g}_{\lambda_j}$	$\dim \mathfrak{g}_{2\lambda_j}$
	B_p	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant p)$	$1\ (j\leqslant p-1)$ $2(l-p)\ (j=p)$	0
	B_{-1}	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant l-1)$, $r(\alpha_l)=\lambda_{l-1}$	$1\ (j\leqslant l-2)$ $2\ (j=l-1)$	0
	D_l	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant l)$	1	0
	C_p	$r(\alpha_{2j})=\lambda_j$ $(1\leqslant j\leqslant p)$	$4\ (j\leqslant p-1)$ $1\ (j=p)$	0
	BC_p	$r(\alpha_{2j})=\lambda_j$ $(1\leqslant j\leqslant p)$ $r(\alpha_{2p+1})=\lambda_p$	4	$0\ (j\leqslant p-1)$ $1\ (j=p)$
	E_6	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant 6)$	1	0
	F_4	$r(\alpha_1)=r(\alpha_5)=\lambda_1$, $r(\alpha_2)=r(\alpha_4)=\lambda_2$, $r(\alpha_3)=\lambda_3$, $r(\alpha_6)=\lambda_4$	$2\ (j=1,2)$ $1\ (j=3,4)$	0
	BC_2	$r(\alpha_1)=r(\alpha_5)=\lambda_2$, $r(\alpha_6)=\lambda_1$	$6\ (j=1)$	0
			$8\ (j=2)$	1
	A_2	$r(\alpha_1)=\lambda_1$, $r(\alpha_5)=\lambda_2$	8	0

Table 9 (cont.)

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$	$\dim \mathfrak{p}$	$\mathrm{rk}_{\mathbb{R}}\, \mathfrak{g}$
E_7	*EV*	$\mathfrak{su}_8$	63	70	7
	EVI	$\mathfrak{su}_2 \oplus \mathfrak{so}_{12}$	69	64	4
	EVII	$E_6 \oplus \mathbb{R}$	79	54	3
E_8	*EVIII*	$\mathfrak{so}_{16}$	120	128	8
	EIX	$\mathfrak{su}_2 \oplus E_7$	136	112	4
F_4	*FI*	$\mathfrak{su}_2 \oplus \mathfrak{sp}_3$	24	28	4
	FII	$\mathfrak{so}_9$	36	16	1
G_2	*G*	$\mathfrak{so}_3 \oplus \mathfrak{so}_3$	6	8	2

Table 9 (cont.)

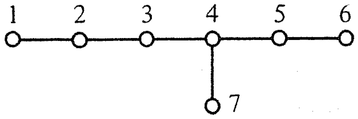
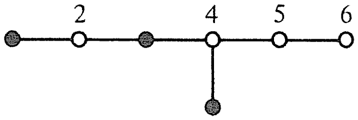
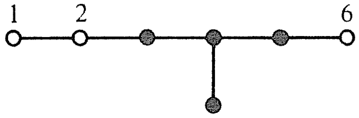
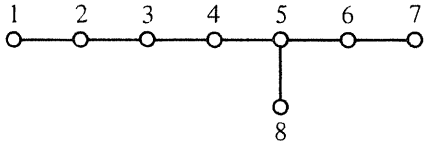
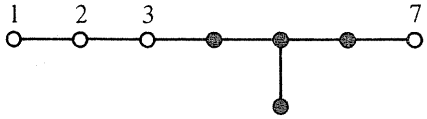
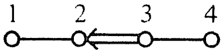
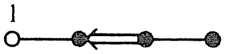
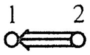
Satake diagram	Type of Σ	r	dim $\mathfrak{g}_{\lambda_j}$	dim $\mathfrak{g}_{2\lambda_j}$
	E_7	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant 7)$	1	0
	F_4	$r(\alpha_2)=\lambda_1,\ r(\alpha_4)=\lambda_2,$ $r(\alpha_5)=\lambda_3,\ r(\alpha_6)=\lambda_4$	4 $(j=1,2)$ 1 $(j=3,4)$	0
	C_3	$r(\alpha_6)=\lambda_1,\ r(\alpha_2)=\lambda_2,$ $r(\alpha_1)=\lambda_3$	8 $(j=1,2)$ 1 $(i=3)$	0
	E_8	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant 8)$	1	0
	F_4	$r(\alpha_7)=\lambda_1,\ r(\alpha_3)=\lambda_2,$ $r(\alpha_2)=\lambda_3,\ r(\alpha_1)=\lambda_4$	8 $(j=1,2)$ 1 $(j=3,4)$	0
	F_4	$r(\alpha_j)=\lambda_j$ $(1\leqslant j\leqslant 4)$	1	0
	BC_1	$r(\alpha_1)=\lambda_1$	8	7
	G_2	$r(\alpha_j)=\lambda_j$ $(j=1,2)$	1	0

Table 10. Centers and Linearizers of Simply Connected Real Simple Lie Groups. Denote by $\mathfrak{g}$ a noncompact real simple Lie algebra that does not admit a complex structure, G the corresponding simply connected Lie group. Denote by $\langle Z \rangle_m$ the cyclic group of order $m = 2, 3, \ldots, \infty$ with generator Z. In the column "Generators" representatives of the generators of $Z(G)$ in the lattice $P^\vee(\Delta_{\mathfrak{g}(\mathbb{C})}) \cap \mathfrak{k}(\mathbb{C})$ (see Theorem 5.3.7) are listed. In the fifth column the group $G_{\text{lin}} = G/\Lambda(G)$ is given (for classical $\mathfrak{g}$); here $\text{Spin}_{p,q}$ denotes the connected real form of the group $\text{Spin}_{p+q}(\mathbb{C})$ (see exercises to § 4.3) corresponding to the real form $\mathfrak{so}_{p,q}$ of $\mathfrak{so}_{p+q}(\mathbb{C})$. In the column "$b_0$" indicated is a representative of an element $b_0 \in Z(G)$ with the property $R(b_0) = \varepsilon(dR)id$, where R is an irreducible complex representation of G such that $\overline{dR} \sim dR$ (see 1.3°).

Table 10

$\mathfrak{g}$	$Z(G)$	Generators of $Z(G)$	$\Lambda(G)$	G_{lin}	b_0

Table 10

$\mathfrak{g}$	$Z(G)$	Generators of $Z(G)$	$\Lambda(G)$	G_{lin}	b_0
$\mathfrak{sl}_{2p+1}(\mathbb{R})$ $(p \geqslant 1)$	$\langle Z_1 \rangle_2$	$Z_1 = h_1 + h_{p+1}$	$\langle Z_1 \rangle_2$	$\mathrm{SL}_{2p+1}(\mathbb{R})$	0
$\mathfrak{sl}_{4p+2}(\mathbb{R})$ $(p \geqslant 1)$	$\langle Z_2 \rangle_4$	$Z_2 = (h_1 + h_3 + \cdots + h_{4p+1})/2$	$\langle 2Z_2 \rangle_2$	$\mathrm{SL}_{4p+2}(\mathbb{R})$	0
$\mathfrak{sl}_{4p}(\mathbb{R})$ $(p \geqslant 1)$	$\langle Z_2 \rangle_2 \times \langle Z_3 \rangle_2$	$Z_2 = (h_1 + h_3 + \cdots + h_{4p-1})/2$ $Z_3 = h_{2p}$	$\langle Z_3 \rangle_2$	$\mathrm{SL}_{4p}(\mathbb{R})$	0
$\mathfrak{sl}_p(\mathbb{H})$ $(p \geqslant 2)$	$\langle Z_2 \rangle_2$	$Z_2 = (h_1 + h_3 + \cdots + h_{2p-1})/2$	$\{e\}$	$\mathrm{SL}_p(\mathbb{H})$	Z_2
$\mathfrak{su}_{p,q}$ $1 \leqslant p \leqslant q$	$\langle Z_4 \rangle_\infty \times \langle Z_5 \rangle_d$, $d = \mathrm{LCD}\,(p, q)$	$Z_4 = a\pi_1^\vee + b\pi_{p+q-1}^\vee$, where $aq + bp = d$; $Z_5 = (p\pi_1^\vee - q\pi_{p+q-1}^\vee)/d$	$\langle \pi_1^\vee + \pi_{p+q-1}^\vee \rangle_\infty$	$\mathrm{SU}_{p,q}$	$\pi_p^\vee + \rho^\vee$
$\mathfrak{so}_{2,2l-1}$	$\langle Z_1 \rangle_\infty \times \langle Z_2 \rangle_2$	$Z_1 = h_1, \quad Z_2 = h_l/2$	$\langle Z_1 \rangle_\infty$	$\mathrm{Spin}_{2,2l-1}$	$(1 + l(l+1)/2)Z_2$
$\mathfrak{so}_{2p,2(l-p)-1}$ $(2 \leqslant p \leqslant l)$	$\langle Z_1 \rangle_2 \times \langle Z_2 \rangle_2$	$Z_1 = h_1, \quad Z_2 = h_l/2$	$\langle Z_1 \rangle_2$	$\mathrm{Spin}_{2p,2(l-p)+1}$	$(p + l(l+1)/2)Z_2$
$\mathfrak{sp}_{4p+2}(\mathbb{R})$ $(p \geqslant 0)$	$\langle Z_1 \rangle_\infty$	$Z_1 = (h_1 + h_3 + \cdots + h_{2p+1})/2$	$\langle 2Z_1 \rangle_\infty$	$\mathrm{Sp}_{4p+2}(\mathbb{R})$	0
$\mathfrak{sp}_{4p}(\mathbb{R})$ $(p \geqslant 1)$	$\langle Z_1 \rangle_2 \times \langle Z_2 \rangle_\infty$	$Z_1 = (h_1 + h_3 + \cdots + h_{2p-1})/2$ $Z_2 = h_{2p}$	$\langle Z_2 \rangle_\infty$	$\mathrm{Sp}_{4p}(\mathbb{R})$	0
$\mathfrak{sp}_{p,q}$ $(1 \leqslant p \leqslant q)$	$\langle Z_1 \rangle_2$	$Z_1 = (h_1 + h_3 + \cdots)/2$	$\{e\}$	$\mathrm{Sp}_{p,q}$	Z_1

Table 10 (cont.)

$\mathfrak{g}$	$Z(G)$	Generators of $Z(G)$	$\Lambda(G)$	G_{lin}	b_0
$\mathfrak{so}_{2,2l-2}$ $(l \geqslant 3)$	$\langle Z_1\rangle_\infty \times \langle Z_2\rangle_2$	$Z_1 = \pi_{l-1}^\vee$, $Z_2 = (h_{l-1}+h_l)/2$	$\langle 2Z_1\rangle_\infty (l=2p)$ $\langle 2Z_1+Z_2\rangle$ $(l=2p+1)$	$\mathrm{Spin}_{2,2l-2}$	Z_2 if $l=4q+2$ or $4q+3$, 0 otherwise
$\mathfrak{so}_{2p,2(l-p)}$ $l=2q+1$ $(2 \leqslant p \leqslant [l/2])$	$\langle Z_1\rangle_4 \times \langle Z_3\rangle_2$	$Z_1 = \pi_{l-1}^\vee$, $Z_3 = h_p$	$\langle Z_3\rangle_2$	$\mathrm{Spin}_{2p,2(l-p)}$	$(p+q)Z_2$
$\mathfrak{so}_{2p,2(l-p)}$ $(2 \leqslant p \leqslant l/2)$ $l=2q$, p odd	$\langle Z_1\rangle_4 \times \langle Z_4\rangle_2$	$Z_1 = \pi_{l-1}^\vee$, $Z_4 = h_p + (h_{l-1}+h_l)/2$	$\langle 2Z_1\rangle_2$	$\mathrm{Spin}_{2p,2(l-p)}$	$(1+q)Z_2$
$\mathfrak{so}_{2p,2(l-p)}$ $(2 \leqslant p \leqslant l/2)$ l, p even, $l=2q$	$\langle Z_1\rangle_2 \times \langle Z_4\rangle_2$ $\times \langle Z_5\rangle_2$	$Z_1 = \pi_{l-1}^\vee$, $Z_4 = h_p + (h_{l-1}+h_l)/2$ $Z_5 = \pi_l^\vee$	$\langle Z_1+Z_4+Z_5\rangle_2$	$\mathrm{Spin}_{2p,2(l-p)}$	qZ_2
$\mathfrak{so}_{2p+1,2(l-p)-1}$ $(1 \leqslant p \leqslant [(l-1)/2])$	$\langle Z_2\rangle_2 \times \langle Z_3\rangle_2$	$Z_2 = (h_{l-1}+h_l)/2$ $Z_3 = h_p$	$\langle Z_3\rangle_2$	$\mathrm{Spin}_{2p+1,2(l-p)+1}$	0
$\mathfrak{u}_l^*(\mathbb{H})$ $l=2p+1$ $(p \geqslant 1)$	$\langle Z_6\rangle_\infty$	$Z_6 = \pi_{l-1}^\vee - p(h_{l-1}+h_l)/2$	$\langle 4Z_6\rangle_\infty$	The two-sheeted covering of the group $\mathrm{U}_l^*(\mathbb{H})$	$(h_1+h_3+\cdots+h_{2p-1})/2$ $+(h_{2p}-h_{2p+1})/4$
$\mathfrak{u}_l^*(\mathbb{H})$ $l=4p+2$ $(p \geqslant 0)$	$\langle Z_7\rangle_2 \times \langle Z_8\rangle_\infty$	$Z_7 = \pi_{l-1}^\vee - p(h_{l-1}+h_l)$, $Z_8 = \pi_l^\vee - p(h_{l-1}+h_l)$	$\langle 2Z_8\rangle_\infty$		$(h_1+h_3+\cdots+h_{4p+1})/2$
$\mathfrak{u}_l^*(\mathbb{H})$ $l=4p$ $(p \geqslant 1)$	$\langle Z_9\rangle_2 \times \langle Z_{10}\rangle_\infty$	$Z_9 = \pi_l^\vee - p(h_{l-1}+h_l)$, $Z_{10} = \pi_{l-1}^\vee - p(h_{l-1}+h_l)$	$\langle 2Z_{10}\rangle_\infty$		$(h_1+h_3+\cdots+h_{4p-1})/2$

Table 10 (cont.)

$\mathfrak{g}$	$Z(G)$	Generators of $Z(G)$	$\Lambda(G)$	b_0
EI	$\langle Z_1\rangle_2$	$Z_1 = h_6$	$\langle Z_1\rangle_2$	0
EII	$\langle Z_2\rangle_6$	$Z_2 = (h_1 - h_2 + h_4 - h_5)/3$	$\langle 3Z_2\rangle_2$	0
$EIII$	$\langle Z_2\rangle_\infty$	$Z_2 = (h_1 - h_2 + h_4 - h_5)/3$	$\langle 3Z_2\rangle_\infty$	$2Z_2$
EIV	$\{e\}$		$\{e\}$	0
EV	$\langle Z_1\rangle_4$	$Z_1 = (h_1 + h_3 + h_7)/2$	$\langle 2Z_1\rangle_2$	0
EVI	$\langle Z_1\rangle_2 \times \langle Z_2\rangle_2$	$Z_1 = (h_1 + h_3 + h_7)/2$ $Z_2 = h_2$	$\langle Z_2\rangle_2$	Z_1
$EVII$	$\langle Z_1\rangle_\infty$	$Z_1 = (h_1 + h_3 + h_7)/2$	$\langle 2Z_1\rangle_\infty$	0
$EVIII$	$\langle Z_1\rangle_2$	$Z_1 = h_7$	$\langle Z_1\rangle_2$	0
EIX	$\langle Z_2\rangle_2$	$Z_2 = h_1$	$\langle Z_2\rangle_2$	0
FI	$\langle Z\rangle_2$	$Z = h_4$	$\langle Z\rangle_2$	0
FII	$\{e\}$		$\{e\}$	0
G	$\langle Z\rangle_2$	$Z = h_2$	$\langle Z\rangle_2$	0

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